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# Preferences and pollution cycles\*

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## Abstract

We consider a competitive Ramsey economy where a pollution externality affects both consumption demand and labor supply, and we assume the stock of pollution to be persistent over time. Surprisingly, when pollution jointly increases the consumption demand (*compensation effect*) and lowers the labor supply (*leisure effect*), multiple equilibria arise near the steady state (local indeterminacy) through a Hopf bifurcation (limit cycle). This result challenges the standard view of pollution as a flow to obtain local indeterminacy, and depends on the *leisure effect* which renders the pollution accumulation process more volatile.

## Résumé

Nous étudions le sentier de croissance concurrentiel d'une économie à la Ramsey où la pollution (externalité négative) affecte à la fois la demande de consommation et l'offre de travail des ménages. La pollution y est introduite comme une variable de stock avec une forte persistance au cours du temps. Dans la littérature, des situations d'indétermination locale apparaissent lorsque la pollution prend la forme d'un flux. Dans notre modèle, lorsque la pollution augmente la demande de consommation (effet de compensation), tout en réduisant l'offre de travail (effet loisir), des équilibres multiples apparaissent au voisinage de l'état stationnaire (indétermination locale) au travers d'une bifurcation de Hopf (cycle limite). Ce résultat surprenant s'explique par la présence de l'effet loisir qui rend le processus d'accumulation de la pollution plus volatile.

**Keywords:** pollution, endogenous labor supply, limit cycle, Ramsey model.

**JEL Classification:** E32, O44.

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# 1 Introduction

Since the early Seventies, theoretical growth literature has paid attention to the economic consequences of pollution. The question addressed at that time was the impact of the emergence of a global pollution on the optimal growth path<sup>1</sup> and was treated within Ramsey models. Keeler et al. (1971) and Forster (1973) pioneered a stream of theoretical literature with pollution in the utility function and focused on the optimal solution for a Ramsey economy where production activities generate a pollution externality with a negative impact on social welfare. The issue of the effects of pollution on the competitive growth path has been studied in the theoretical literature much more recently. The main question was whether pollution, as an externality in households' utility function, modifies their behaviors and thereby the economic dynamics. This is precisely the question at the heart of this paper. We develop an unified model that incorporates in a Ramsey framework the effect of pollution on households' behavior (both with regard to consumption demand and labor supply) and its impacts on the competitive dynamics.

Models that followed the seminal articles by Keeler et al. (1971) and Forster (1973) have considered various configurations. Loosely speaking, pollution externalities and depollution activities matter when they affect the fundamentals (technology and preferences). Modelling pollution as a flow or a stock has also an impact on optimal solutions. As specialists of combinatorial art, theorists revisit during the Eighties the seminal models combining different building blocks. Since, pollution is considered as an input of production or utility function<sup>2</sup> or a by-product of production or consumption activities,<sup>3</sup> and modelled as a flow or a stock.<sup>4</sup> Environmental maintenance and depollution activities are also taken in account.<sup>5</sup>

All the above papers question the dynamic properties of the optimal solution. During the Nineties, theorists pay more attention to environmental policy implications and the optimal design of market solutions.<sup>6</sup> The interest of these works does not rest only on their normative dimension but also on a positive characterization of competitive solutions. Pollution effects on preferences be-

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<sup>1</sup>The solution of the social planner's problem.

<sup>2</sup>For models with pollution in the production function, the reader is referred to Brock (1977), Stockey (1998), and Tahvonen and Kuuluvainen (1993); for models with pollution in the utility function, to Keeler et al. (1971), Forster (1973), Heal (1982), and Michel and Rotillon (1995).

<sup>3</sup>Keeler et al. (1971), Forster (1973), Selden and Song (1995), and van der Ploeg and Withagen (1991) consider pollution as a by-product of production. Heal (1982) revisits Ryder and Heal (1973) by interpreting the stock of past consumption in terms of pollution instead of habit formation.

<sup>4</sup>Among others, Forster (1973), Gradus and Smulders (1993), Selden and Song (1995), and van der Ploeg and Withagen (1991) consider pollution as a flow, while Keeler et al. (1971), Brock (1977), Heal (1982), Tahvonen and Kuuluvainen (1993), and van der Ploeg and Withagen (1991) as a stock.

<sup>5</sup>For instance, in Keeler et al. (1971), Forster (1973), Heal (1982), and van der Ploeg and Withagen (1991).

<sup>6</sup>Smith (1972) pioneered this literature but it was not until the highly-cited contribution of Tahvonen and Kuuluvainen (1993) that papers on this subject flourished.

come predominant when we look at competitive growth path. These effects are closely linked to the separable or the non-separable form of the utility function.

Additive separability implies that pollution does not affect directly the marginal utility of consumption and, therefore, the consumption demand coincides with that of a Ramsey economy without pollution externalities. The trends of pollution have no effect on the competitive growth path. Conversely, the planner internalizes the externality maximizing the welfare taking in account the trade-off between consumption and pollution: pollution and consumption lower. The cross effect of pollution on consumption was initially considered by Keeler et al. (1971). Consumption and pollution are non-separable goods and the assumption of normality ensures the uniqueness and the saddle-path stability of the steady state. The following early models focused on the impact of non-separability on the optimal growth path. The same result holds in van der Ploeg and Withagen (1991) with negative cross derivatives (decreasing marginal utility of consumption in the pollution level). The interplay between consumption and pollution in a Ramsey model was fully characterized by Heal (1982). Heal studies the optimal growth path when the marginal utility of consumption is affected by the stock of pollution without imposing the sign of the cross effect. When the stock of pollution increases the marginal utility of current consumption (adjacent complementarity), optimal growth path becomes more complex as a limit cycle arises near the optimal steady state through a Hopf bifurcation.

Michel and Rotillon (1995) generalize these models both in exogenous and endogenous growth (learning-by-doing) and for competitive and optimal growth. They study different pollution effects on households' behavior. On the one hand, pollution may stimulate the consumption demand through a (so-called) *compensation effect*: households consume more to compensate the drop in utility due to a higher pollution. On the other hand, if households like to consume in a pleasant environment, a rise in pollution lowers the consumption demand through a (so-called) *distaste effect*. The presence of negative and positive externalities (pollution and learning-by-doing) makes endogenous growth dynamics richer but more complicated: the social optimum converges to a zero growth rate in presence of distaste or weak compensation effects, while a long-run positive endogenous growth rate arises under large compensation effects.

Most of these articles focus on the effects of pollution on consumption demand. A recent empirical literature points out the negative impact of pollution on labor supply. For instance, Hanna and Oliva (2015) consider this effect in a neighborhood of a polluting refinery in Mexico City and find that a one-percent increase in air pollution results in a 0.61 percent decrease in the hours worked. Graff Zivin and Neidell (2010) and Carson et al. (2011) reach similar conclusions.

Few theoretical papers take in account the impact on labor supply through a *leisure effect* of pollution. All these works consider separable preferences in consumption and labor supply. Fernandez et al. (2012) study a competitive Ramsey economy in continuous time with endogenous labor supply. The pollution flow comes from the use of capital and reduces the household's utility. They focus on local indeterminacy and find that separability between consumption and pollution in the utility function prevents equilibrium multiplicity. Beyond the sus-

tainability issue, they raise also the question of equilibrium convergence under pollution, especially when households' preferences are non-separable. Similarly, Itaya (2008) shows that pollution effects on the household's utility promote equilibrium indeterminacy in a competitive endogenous growth model à la Romer (1986) with endogenous labor supply. As in Fernandez et al. (2012), Itaya (2008) defines pollution as a flow.

Local indeterminacy is also a known feature of environmental OLG literature. Seegmuller and Verchère (2007) show that equilibrium multiplicity arises when finite-lived households arbitrate between consumption, labor supply and environmental maintenance, but they still consider pollution as a flow. Bosi et al. (2015) build a discrete-time Ramsey economy where the stock of pollution does not affect the marginal utility of consumption but the marginal disutility of labor supply. In their model, positive or negative pollution effects on labor supply may arise, what they call, respectively, *disenchantment* or *leisure effects* in the spirit of Michel and Rotillon (1995). In the case of *disenchantment effect*, the larger pollution decreases the utility of leisure and provides an incentive to increase the worked hours. Conversely, in the case of *leisure effect*, an increase in pollution deteriorates the working conditions and urges households to work less. In Bosi et al. (2015), the competitive steady state is unique and a large leisure effect leads to persistent cycles through a flip bifurcation near the competitive steady state. Even if Bosi et al. (2015) fit the evidence, their simplified framework excludes any direct effect of pollution on marginal utility of consumption and, in turn, on consumption demand.

The added value of our paper is threefold. (1) We develop a unified model to take into account the joint effect of pollution on consumption demand (Michel and Rotillon (1995)) and labor supply (Bosi et al. (2015)) with special focus on continuous-time bifurcations. (2) Differently from the other papers on bifurcations and equilibrium indeterminacy, we consider a stock of pollution. Pollution as a flow is less pertinent in macroeconomic and empirical terms.<sup>7</sup> (3) We develop a general methodology to study local bifurcations of three-dimensional dynamic systems in continuous time with one forward and two backward-looking variables.

The most severe forms of pollution persist over time and affect future generations. Focusing on the case of a strong pollution inertia is pertinent to capture phenomena with global and macroeconomic implications such as the global warming and the nuclear waste. We show that the interplay between the *leisure* and the *compensation effect* may promote equilibrium multiplicity (indeterminacy) through a Hopf bifurcation near the steady state. This result has a twofold interest: pollution inertia and *leisure effect* empirically matter; most of the papers on local indeterminacy consider a flow and do not take in consideration the main forms of lasting pollutions. The *leisure effect* plays the key role for local indeterminacy under pollution inertia. Indeed, it neutralizes the inertial effect, promotes pollution volatility and macroeconomic fluctuations.

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<sup>7</sup>For instance, the global warming depends on the stock of greenhouse gas and represents the main environmental threat.

The outline of the paper is the following. In section 2, we present the model (technology, preferences and pollution). Sections 3 to 5 focus on the short and long-run equilibrium conditions. Section 6 provides a general methodology to study bifurcations of three-dimensional dynamic systems with two predetermined variables. In section 6, we apply the methodology to the case of isoelastic fundamentals. Section 7 concludes.

## 2 Model

We consider a continuous-time Ramsey economy with pollution and capital accumulation. The representative household supplies labor to a competitive firm and faces a consumption-leisure arbitrage. Firms produce a single commodity which is consumed by households or invested as capital. Under constant returns to scale, all these firms are equivalent to a single aggregate firm. Pollution is a by-product of production activities and, as a negative externality, it affects the utility and the consumption-leisure arbitrage.

### 2.1 Technology

At time  $t$ , the representative firm produces a single output  $Y(t)$ . Technology is represented by a constant returns to scale production function:  $Y(t) = F(K(t), L(t))$ , where  $K(t)$  and  $L(t)$  are the demands for capital and labor.

**Assumption 1** *The production function  $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is  $C^1$ , homogeneous of degree one, strictly increasing and concave. Inada conditions hold:  $f(0) = 0$ ,  $f'(0^+) = +\infty$ ,  $f'(+\infty) = 0$ , where  $f(k) \equiv F(k, 1)$  is the average productivity and  $k \equiv K/L$  denotes the capital intensity.*

The firm is price-taker and chooses the amount of capital and labor to maximize the profit:  $\max_{K,L} [F(K, L) - rK - wL]$ , where  $r$  and  $w$  are the real interest rate and the real wage. The first-order conditions write:

$$\begin{aligned} r &= f'(k) \equiv r(k) \\ w &= f(k) - kf'(k) \equiv w(k) \end{aligned}$$

Profit maximization is well-defined under Assumption 1.

Let us introduce the capital share in total income  $\alpha$  and the elasticity of capital-labor substitution  $\sigma$ :

$$\alpha(k) \equiv \frac{kf'(k)}{f(k)} \text{ and } \sigma(k) = \alpha(k) \frac{w(k)}{kw'(k)} \quad (1)$$

Thereby, the elasticities of factor prices are given by:

$$\frac{kr'(k)}{r(k)} = -\frac{1 - \alpha(k)}{\sigma(k)} \text{ and } \frac{kw'(k)}{w(k)} = \frac{\alpha(k)}{\sigma(k)} \quad (2)$$

## 2.2 Preferences

The household earns a capital income  $rh$  and a labor income  $wl$  where  $h = h(t)$  and  $l = l(t)$  denote the individual wealth and labor supply at time  $t$ . For simplicity, we will omit the time argument in the rest of the paper. Individual wealth accumulation is driven by the budget constraint

$$\dot{h} \leq (r - \delta)h + wl - c \quad (3)$$

where  $\delta$  is the capital depreciation rate and  $c$  is the consumption demand.

For the sake of simplicity, the population of consumers-workers is constant over time:  $N = 1$ . Such normalization implies  $L = Nl = l$ ,  $K = Nh = h$  and  $h = K/N = kl$ .

**Assumption 2** *Preferences are separable in consumption and labor:*

$$U(c, l, P) \equiv u(c, P) - v(l, P) \quad (4)$$

with  $u_c > 0$ ,  $u_P \leq 0$ ,  $v_l > 0$ ,  $v_P \geq 0$  as first-order restrictions,  $u_{cc} < 0$ ,  $v_{ll} > 0$  as second-order restrictions, and  $\lim_{c \rightarrow 0^+} u_c = \infty$ ,  $\lim_{l \rightarrow 0^+} v_l = 0$  as a limit conditions.

We do not impose any restriction on the sign of the cross-derivatives  $u_{cP}$  and  $v_{lP}$ . Even if preferences are separable in consumption and labor supply, pollution affects both their marginal utilities and the consumption-labor arbitrage through a general equilibrium effect.

According to Michel and Rotillon (1995), pollution has a *distaste effect* on consumption if  $U_{cP} < 0$ : an increase in pollution reduces the marginal utility of consumption. The opposite effect ( $U_{cP} > 0$ ) is called *compensation effect*: an increase in pollution raises the propensity to consume. This terminology has been extended by Bosi et al. (2015) to the effects of pollution on labor supply. Pollution is said to have a *leisure effect* in the case of a positive effect of pollution on labor disutility ( $U_{lP} < 0$ ): an increase in pollution decreases labor supply by increasing the leisure demand. Pollution worsens working conditions (for example, the negative impact of global warming rests on a positive correlation between heat and work painfulness) and gives an incentive to substitute leisure to work. The opposite effect ( $U_{lP} > 0$ ) is called *disenchantment effect*. In this case, leisure time decreases with pollution. Households like to enjoy leisure in a healthy and pleasant environment (for instance, pollution may dissuade people from going outdoor and encourage them to work more).

The agent maximizes the intertemporal utility function  $\int_0^\infty e^{-\rho t} U(c, l, P) dt$  under the budget constraint (3).  $\rho > 0$  is the rate of time preference. This program is well-defined under Assumption 2.

**Proposition 1** *The first-order conditions result in a static consumption-leisure arbitrage*

$$U_c = \lambda = -U_l/w \quad (5)$$

a dynamic Euler equation  $\dot{\lambda} = \lambda(\rho + \delta - r)$  and the budget constraint (3) now binding  $\dot{h} = (r - \delta)h + wl - c$ . The optimal path satisfies also the transversality condition:  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) h(t) = 0$ .

**Proof.** See the Appendix. ■

## 2.3 Pollution

The aggregate stock of pollution  $P$  is a pure negative externality. Technology is dirty and pollution persists. We assume a simple linear accumulation process:

$$\dot{P} = -aP + bY \quad (6)$$

where  $a \geq 0$  captures the rate of pollution absorption by nature and  $b \geq 0$  the environmental impact of production. Since, under Assumption 1,  $Y = Lf(k) = lf(k)$ , the process of pollution accumulation (6) writes:

$$\dot{P} = -aP + blf(k) \quad (7)$$

This formulation is adopted by many authors (Keeler et al. (1971), Heal (1982), Michel and Rotillon (1995)) but, to the best of our knowledge, all the papers on local indeterminacy consider only a pollution flow (Seegmuller and Verchère (2007), Itaya (2008), Fernandez et al. (2012)). In contrast, we assume pollution as a stock whose natural absorption is captured by  $a$ . We will see that local indeterminacy requires strong inertia under a *leisure effect*.

## 3 Equilibrium

In equilibrium, good and labor markets clear. Applying the Implicit Function Theorem to the consumption-labor arbitrage (5), we obtain  $(c, l)$  as a function of  $(\lambda, k, P)$ , that is  $c = c(\lambda, k, P)$  and  $l = l(\lambda, k, P)$ . Let us introduce the following second-order elasticities of the utility function  $U(c, l, P)$ :<sup>8</sup>

$$E \equiv \begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cl} & \varepsilon_{cP} \\ \varepsilon_{lc} & \varepsilon_{ll} & \varepsilon_{lP} \\ \varepsilon_{Pc} & \varepsilon_{Pl} & \varepsilon_{PP} \end{bmatrix} \equiv \begin{bmatrix} \frac{cU_{cc}}{U_c} & \frac{cU_{cl}}{U_l} & \frac{cU_{cP}}{U_P} \\ \frac{lU_{lc}}{U_c} & \frac{lU_{ll}}{U_l} & \frac{lU_{lP}}{U_P} \\ \frac{PU_{Pc}}{U_c} & \frac{PU_{Pl}}{U_l} & \frac{PU_{PP}}{U_P} \end{bmatrix} \quad (8)$$

The different effects of pollution on preferences can be captured through these elasticities. Pollution has a *distaste effect* on consumption if  $\varepsilon_{Pc} < 0$  and a *compensation effect* on consumption if  $\varepsilon_{Pc} > 0$ . According to Assumption 2,  $U_l < 0$  and, thus, pollution has a *leisure effect* if  $\varepsilon_{Pl} > 0$  and a *disenchantment effect* if  $\varepsilon_{Pl} < 0$ .

<sup>8</sup>In the case of isoelastic utility functions, the first and second-order elasticities are related because the same fundamental parameters appear in both of them.



**Proposition 2** In the separable case (4), the elasticities matrix of consumption demand  $c = c(\lambda, k, P)$  and labor supply  $l = l(\lambda, k, P)$  is given by

$$\begin{bmatrix} \frac{\lambda}{c} \frac{\partial c}{\partial \lambda} & \frac{k}{c} \frac{\partial c}{\partial k} & \frac{P}{c} \frac{\partial c}{\partial P} \\ \frac{\lambda}{l} \frac{\partial l}{\partial \lambda} & \frac{k}{l} \frac{\partial l}{\partial k} & \frac{P}{l} \frac{\partial l}{\partial P} \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon_{cc}} & 0 & -\frac{\varepsilon_{Pc}}{\varepsilon_{cc}} \\ \frac{1}{\varepsilon_{ll}} & \frac{\alpha}{\sigma} \frac{1}{\varepsilon_{ll}} & -\frac{\varepsilon_{Pl}}{\varepsilon_{ll}} \end{bmatrix} \quad (9)$$

**Proof.** See the Appendix. ■

In our model, dynamics are represented by a three-dimensional system with two predetermined variables ( $k$  and  $P$ ) and one non-predetermined ( $\lambda$ ).

**Proposition 3** The equilibrium transition is driven by the following dynamic system:

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= \rho + \delta - r(k) \\ \frac{\dot{k}}{k} &= \frac{r(k) - \delta + \frac{w(k)}{k} - \frac{c(\lambda, k, P)}{kl(\lambda, k, P)} - \frac{\lambda}{l} \frac{\partial l}{\partial \lambda} [\rho + \delta - r(k)] - \frac{P}{l} \frac{\partial l}{\partial P} \left[ b \frac{l(\lambda, k, P) f(k)}{P} - a \right]}{1 + \frac{k}{l} \frac{\partial l}{\partial k}} \\ \frac{\dot{P}}{P} &= b \frac{l(\lambda, k, P) f(k)}{P} - a \end{aligned} \quad (10)$$

**Proof.** See the Appendix. ■

Considering pollution as a stock instead of a flow adds a third dimension to the basic Ramsey model and dynamics turns out to be more complicated.

## 4 Long run

Long-run dynamics are captured by attractors such as a steady state or a limit cycle. Let us focus on the steady state and the impact of the main fundamental parameters on the stationary solution.

At the steady state,  $\dot{\lambda} = \dot{k} = \dot{P} = 0$  and system (10) becomes

$$r(k) = \rho + \delta \quad (11)$$

$$c(\lambda, k, P) = [\rho k + w(k)] l(\lambda, k, P) = \frac{a}{b} P - \delta k l(\lambda, k, P) \quad (12)$$

$$l(\lambda, k, P) f(k) = \frac{a}{b} P \quad (13)$$

because  $f(k) = kr(k) + w(k)$ .

We observe that the capital intensity  $k = r^{-1}(\rho + \delta)$  is not affected by pollution and remains that of Modified Golden Rule (MGR).

Given  $k$ , system (11)-(13) allows us to compute the other variables ( $\lambda$ ,  $P$ ,  $c$  and  $l$ ). Even if the capital intensity is unique and coincides with its MGR value, the multiplicity of steady states depend on the multiplicity of solutions ( $\lambda, P$ ). The following assumption is a sufficient condition to ensure the solution uniqueness of system (11)-(13).

**Assumption 3**

$$\frac{1 + \frac{\varepsilon_{Pc}}{\varepsilon_{cc}}}{1 + \frac{\varepsilon_{Pl}}{\varepsilon_{ll}}} > \frac{\varepsilon_{ll}}{\varepsilon_{cc}} \quad (14)$$

where the elasticities are evaluated at the steady state.

Assumption 3 is not very demanding. The inequality holds for instance if  $\varepsilon_{Pc} < -\varepsilon_{cc}$  (*distaste effect* ( $\varepsilon_{Pc} < 0$ ) or weak *compensation effect* ( $0 < \varepsilon_{Pc} < -\varepsilon_{cc}$ )) jointly with  $\varepsilon_{Pl} > -\varepsilon_{ll}$  (*leisure effect* ( $\varepsilon_{Pl} > 0$ ) or weak *disenchantment effect* ( $-\varepsilon_{ll} < \varepsilon_{Pl} < 0$ )). We will provide an explicit inequality in terms of the exogenous parameters in the case of separable and isoelastic preferences.

**Proposition 4 (uniqueness of the steady state)** *Let Assumptions 1 and 2 hold. The stationary capital intensity  $k$  is unique. In addition, under Assumption 3, the steady state  $(\lambda, k, P)$  is unique (sufficient condition).*

**Proof.** See the Appendix. ■

## 5 Short run

The equilibrium path may converge to an attractor such as a steady state or a limit cycle. Convergence to a long-run attractor takes place in the short run. If the economy is sufficiently close to an attractor, the linearization of the dynamic system is a good approximation and informs us about the nature of the underlying nonlinear dynamics.

Let us study the local dynamics, that is linearize the three-dimensional dynamic system (10):

$$\begin{aligned} \dot{\lambda} &= f_1(\lambda, k, P) \\ \dot{k} &= f_2(\lambda, k, P) \\ \dot{P} &= f_3(\lambda, k, P) \end{aligned}$$

around the steady state . We obtain a Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial \lambda} & \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial P} \\ \frac{\partial f_3}{\partial \lambda} & \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial P} \end{bmatrix} \quad (15)$$

Most of the contributions focusing on the local dynamics of a polluted economy (Fernandez et al. (2012) and Itaya (2008)) consider pollution as a flow. In this case, dynamics are represented by a more tractable two-dimensional system. Instead, we consider pollution as a stock. From a mathematical point of view, pollution as a stock is less tractable, because the Ramsey model becomes higher-dimensional. Characterizing three-dimensional dynamics with two predetermined and one jump variables requires some additional skill.

In this section, we present a general methodology to characterize the occurrence of local bifurcations and local indeterminacy of three-dimensional dynamic systems in continuous time. We will apply this methodology later, in the case of isoelastic functional forms.

## 5.1 Bifurcations

In continuous time, a local bifurcation generically arises when the real part of an eigenvalue  $\lambda(p)$  of the Jacobian matrix crosses zero in response to a change of parameter  $p$ . Denoting by  $p^*$  the critical parameter value of bifurcation, we get generically two cases.

(1) When a real eigenvalue crosses zero:  $\lambda(p^*) = 0$ , the system undergoes a saddle-node bifurcation (either an elementary saddle-node or a transcritical or a pitchfork bifurcation) depending upon the number of steady states.

(2) When the real part of two complex and conjugate eigenvalues  $\lambda(p) = \tilde{a}(p) \pm i\tilde{b}(p)$  crosses zero, the system undergoes a Hopf bifurcation. More precisely, in this case, we require  $\tilde{a}(p^*) = 0$  and  $\tilde{b}(p) \neq 0$  in a neighborhood of  $p^*$  (see Bosi and Ragot (2011) p. 76).

System (10) is three-dimensional with two predetermined variables ( $k$  and  $P$ ) and one jump variable ( $\lambda$ ). Thus, multiple equilibria (local indeterminacy) arise when the three eigenvalues of the Jacobian matrix (15) evaluated at the steady state have negative real parts: either  $\lambda_1, \lambda_2, \lambda_3 < 0$  or  $\text{Re } \lambda_1, \text{Re } \lambda_2 < 0$  and  $\lambda_3 < 0$ .

Consider the Jacobian matrix  $J$  and focus on the expressions of determinant, sum of minors of order two and trace in terms of eigenvalues:

$$\begin{aligned} D &= \lambda_1 \lambda_2 \lambda_3 \\ S &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = \sum_{i=1}^3 \det J_{ii} \\ T &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

where  $J_{ii}$  is the sub-matrix of  $J$  obtained canceling out the  $i$ th row and column.

A saddle-node bifurcation is associated to a multiplicity of steady states exchanging their stability properties. It occurs when a real eigenvalue crosses zero.

**Proposition 5 (saddle-node bifurcation)** *Under Assumption 3, saddle-node bifurcations are ruled out.*

**Proof.** Under Assumption 3, the steady state is unique. The class of saddle-node bifurcations (elementary saddle node, transcritical and pitchfork) always involves multiple steady states (Bosi and Ragot (2011)). ■

A Hopf bifurcation occurs when the real part of two complex and conjugate eigenvalues  $\lambda(p) = \tilde{a}(p) \pm i\tilde{b}(p)$  crosses zero. More precisely, we require  $\tilde{a}(0) = 0$  and  $\tilde{b}(p) \neq 0$  in a neighborhood of  $p = 0$ , where  $p = 0$  is the normalized

bifurcation value of parameter. The following proposition characterizes the occurrence of limit cycles through a Hopf bifurcation.

**Proposition 6 (Hopf bifurcation)** *In the case of a three-dimensional system, a Hopf bifurcation generically arises if and only if  $D = ST$  and  $S > 0$ .*

**Proof.** See the Appendix. ■

We will provide explicit conditions for the occurrence of a Hopf bifurcation in the case of isoelastic fundamentals (section 6).

## 5.2 Indeterminacy

In our model, dynamics involves two predetermined variables ( $k$  and  $P$ ) and a jump variable ( $\lambda$ ). As seen above, indeterminacy requires the three eigenvalues with negative real parts: either  $\lambda_1, \lambda_2, \lambda_3 < 0$  or  $\text{Re } \lambda_1, \text{Re } \lambda_2 < 0$  and  $\lambda_3 < 0$ .

**Proposition 7 (local indeterminacy)** *In the case of system (10), if all the eigenvalues are real, the equilibrium is locally indeterminate if and only if  $D, T < 0$  and  $S > 0$ .*

**Proof.** See the Appendix. ■

Consider the possibility of local indeterminacy through a Hopf bifurcation. Unfortunately, Proposition 7 is of little use because, it is difficult to know whether the eigenvalues are real. In the non-real case, the necessary condition of Proposition 7 still holds. Indeed, indeterminacy ( $\text{Re } \lambda_1 = \text{Re } \lambda_2 < 0$  and  $\lambda_3 < 0$ ) implies

$$\begin{aligned} D &= \lambda_1 \lambda_2 \lambda_3 = \left[ (\text{Re } \lambda_1)^2 + (\text{Im } \lambda_1)^2 \right] \lambda_3 < 0 \\ S &= \lambda_1 \lambda_2 + (\lambda_1 + \lambda_2) \lambda_3 = (\text{Re } \lambda_1)^2 + (\text{Im } \lambda_1)^2 + 2 \text{Re } \lambda_1 \lambda_3 > 0 \\ T &= \lambda_1 + \lambda_2 + \lambda_3 = 2 \text{Re } \lambda_1 + \lambda_3 < 0 \end{aligned}$$

However, the sufficient condition fails: even if

$$D = \lambda_1 \lambda_2 \lambda_3 = \left[ (\text{Re } \lambda_1)^2 + (\text{Im } \lambda_1)^2 \right] \lambda_3 < 0$$

still implies  $\lambda_3 < 0$ , conditions  $D, T < 0$  and  $S > 0$  do not rule out the unpleasant case  $\text{Re } \lambda_1 = \text{Re } \lambda_2 > 0$ .

Instead, we provide another sufficient condition for local indeterminacy, that is more restrictive.

**Proposition 8 (local indeterminacy through a Hopf bifurcation)** *Let  $p_H$  be the Hopf bifurcation value of a parameter  $p$  such that  $D(p_H) = S(p_H)T(p_H)$  and  $S(p_H) > 0$ . If  $D(p_H) < 0$ , the equilibrium is locally indeterminate for some value of  $p$  around  $p_H$ .*

**Proof.** See the Appendix. ■

## 6 Isoelastic case

In order to provide explicit conditions for local bifurcations and indeterminacy in terms of fundamental parameters and relevant economic interpretations, we introduce isoelastic functional forms in the general model presented above.

The separable case (Assumption 2) is suitable for our local analysis because of the lack of direct cross effects between the marginal utility of consumption and labor. However, we need to introduce more structure for the purpose of economic analysis. In the isoelastic case, the elasticities of matrix (9) are constant by definition and have an easy economic interpretation. Thus, we consider isoelastic separable preferences:

$$u(c, P) \equiv \frac{(cP^{-\eta})^{1-\varepsilon}}{1-\varepsilon} \text{ and } v(l, P) \equiv \omega \frac{(lP^\psi)^{1+\varphi}}{1+\varphi} \quad (16)$$

where  $1/\varepsilon \geq 0$  is the consumption elasticity of intertemporal substitution,  $1/\varphi \geq 0$  is the Frisch elasticity of intertemporal substitution and  $\omega > 0$  is the weight of disutility of labor in total utility. In addition, we require  $\eta, \psi \geq 0$  (Assumption 2).

Moreover, we focus on a Cobb-Douglas production function giving the following intensive output:

$$f(k) = Ak^\alpha \quad (17)$$

We observe that, in this case,  $\alpha$  becomes constant and  $\sigma = 1$ .

The elasticities on the RHS of matrix (9) appear only in the first two columns of the elasticities matrix (8)

$$\tilde{E} \equiv \begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cl} \\ \varepsilon_{lc} & \varepsilon_{ll} \\ \varepsilon_{Pc} & \varepsilon_{Pl} \end{bmatrix} = \begin{bmatrix} -\varepsilon & 0 \\ 0 & \varphi \\ (\varepsilon - 1)\eta & (1 + \varphi)\psi \end{bmatrix}$$

and depends directly on the fundamental parameters.

The elasticities in the third column of  $E$  (see (8)) are more complicated because they are not directly parametric but involve the endogenous variables:  $\lambda, k, P$ . Fortunately, we no longer need them in the following. Hence, matrix (9) simplifies:

$$\begin{bmatrix} \frac{\lambda c_\lambda}{c} & \frac{k c_k}{c} & \frac{P c_P}{c} \\ \frac{\lambda l_\lambda}{l} & \frac{k l_k}{l} & \frac{P l_P}{l} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\varepsilon} & 0 & e_c \\ \frac{1}{\varphi} & \frac{\alpha}{\varphi} & e_l \end{bmatrix} \quad (18)$$

where now the more compact expression  $y_x$  denotes the derivative  $\partial y / \partial x$  and

$$e_c \equiv \frac{P c_P}{c} = \eta \frac{\varepsilon - 1}{\varepsilon} \text{ and } e_l \equiv \frac{P l_P}{l} = -\psi \frac{1 + \varphi}{\varphi} \quad (19)$$

represent the pollution impacts on consumption demand and labor supply.

We observe that, if  $\eta > 0$ , a *distaste effect* holds when  $0 < \varepsilon < 1$ , while a *compensation effect* arises when  $\varepsilon > 1$ . Our isoelastic specification rules out any *disenchantment effect* if  $\psi > 0$ , but captures the *leisure effect*.

The dynamic system (10) writes:

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - r(k) \quad (20)$$

$$\frac{\dot{k}}{k} = \frac{\rho + \frac{w(k)}{k} - \frac{c(\lambda, k, P)}{kl(\lambda, k, P)} - \frac{1+\varphi}{\varphi} \left( \rho + \delta - r(k) + \psi \left[ a - b \frac{l(\lambda, k, P) f(k)}{P} \right] \right)}{1 + \frac{\alpha}{\varphi}} \quad (21)$$

$$\frac{\dot{P}}{P} = b \frac{l(\lambda, k, P) f(k)}{P} - a \quad (22)$$

## 6.1 Long run

In the isoelastic case, the steady state values depend explicitly on the fundamental parameters and the comparative statics leads to unambiguous results.

**Proposition 9** *In the isoelastic case, there exists a unique steady state:*

$$\lambda = \left( \frac{B}{\rho k + w} \right)^{\frac{\varepsilon \varphi (1 - e_l)}{\varepsilon (1 - e_c) + \varphi (1 - e_l)}} \quad (23)$$

$$k = \left( \frac{\alpha A}{\rho + \delta} \right)^{\frac{1}{1 - \alpha}} \quad (24)$$

$$P = C \lambda^{\frac{1}{\varphi (1 - e_l)}} \quad (25)$$

where  $w = (1 - \alpha) A k^\alpha$ ,

$$B \equiv \frac{C e_c - e_l}{(w/\omega)^{\frac{1}{\varphi}}} \text{ and } C \equiv \left[ A k^\alpha \frac{b}{a} \left( \frac{w}{\omega} \right)^{\frac{1}{\varphi}} \right]^{\frac{1}{1 - e_l}}$$

The elasticities  $e_1$  and  $e_2$  are given by (19).

**Proof.** See the Appendix. ■

It is interesting to note that we do not need to impose restriction (14) to ensure the uniqueness of steady state. (14) represents a sufficient condition for uniqueness (Proposition 4)<sup>9</sup>. The only restriction we need in the isoelastic case is  $\varepsilon(1 - e_c) + \varphi(1 - e_l) \neq 0$ . Otherwise, the steady state ( $\lambda$ ) fails to exist (see (23)). In other terms, the steady state vanishes for a particular value of *compensation effect*. In the case of a distaste effect ( $e_c < 0$ , that is  $\varepsilon < 1$ ), the existence of steady state is ensured.

Focus now on the comparative statics.

As seen in the general case, the capital intensity remains that of MGR as in the Ramsey model and technology ( $A$ ,  $\alpha$ ,  $\delta$ ) as well the time preference ( $\rho$ )

<sup>9</sup>In the isoelastic case, inequality (14) is written as  $\varepsilon(1 - e_c) + \varphi(1 - e_l) > 0$ .

have the usual effects on  $k$ , while the felicity  $(\varepsilon, \eta, \varphi, \psi)$  as well as the pollution process  $(a, b)$  have no impact on  $k$  but affect the consumption demand and the labor supply through  $\lambda$  and  $P$ . We are not surprised: the pollution externality is not internalized by a market economy (differently from a planner) and has no marginal effect on the Euler equation  $\dot{\lambda}/\lambda = \rho + \delta - r(k)$  evaluated at the steady state.

We recall that the shadow price  $\lambda$  is the marginal utility of consumption  $u_c$ : as a first approximation, when  $\lambda$  increases, the consumption demand decreases under Assumption 2.

In the following, we leave aside the effects of technology  $(A, \alpha, \delta)$  and the pollution process  $(a, b)$  on  $\lambda$  and  $P$  and we consider only the impact of preferences, that is of felicity  $(\varepsilon, \eta, \varphi, \psi)$  and time preference  $(\rho)$ .<sup>10</sup>

Felicity is composed by two sub-felicities:  $u(c, P)$  and  $v(l, P)$ .

(i) The key parameters for  $u$  are the consumption elasticity of intertemporal substitution  $(1/\varepsilon)$  and the consumption sensitivity to pollution  $(\eta)$ .

(ii) The key parameters for  $v$  are the Frisch elasticity of intertemporal substitution  $(1/\varphi)$  and the labor supply sensitivity to pollution  $(\psi)$ .

Consider the point (i). The following proposition focuses on the role of parameters  $(\varepsilon, \eta)$  in the pollution elasticity of consumption demand  $(e_c)$ .

**Proposition 10** (1) *The impacts of the consumption elasticity of intertemporal substitution  $(1/\varepsilon)$  on  $\lambda$  and  $P$  have the same sign. When  $\eta = 1$ , they are negative if and only if the natural absorption is sufficiently small ( $a < b(\rho + \delta) / [\rho + (1 - \alpha)\delta]$ ).*

(2) *The impacts of the consumption sensitivity to pollution  $(\eta)$  on  $\lambda$  and  $P$  have the same sign. If  $\varepsilon(1 - e_c) + \varphi(1 - e_l) > 0$ , they are positive under dominant income effects ( $\varepsilon > 1$ ) and high pollution ( $P > 1$ ) or dominant substitution effects ( $\varepsilon < 1$ ) and low pollution ( $P < 1$ ).*

**Proof.** See the Appendix. ■

Notice that  $\varepsilon(1 - e_c) + \varphi(1 - e_l)$  is a global measure of the pollution effects on preferences. It is positive under the joint assumption of *distaste* ( $e_c < 0$ ) and *leisure effect* ( $e_l < 0$ ).

Focus for instance on the second case (substitution effects and low pollution). According to (18), a higher  $\eta$  implies a stronger *distaste effect*. Then, for a given pollution level, individuals consume less and save more which increases the production level and the pollution stock in turn.

Consider now the point (ii). The following proposition focuses on the role of parameters  $(\varphi, \psi)$  in the pollution elasticity of labor supply  $(e_l)$ . For simplicity, we focus on the case of a logarithmic felicity of consumption ( $\varepsilon = 1$ ).

<sup>10</sup>It is possible to show that, in the case of a logarithmic felicity of consumption ( $\varepsilon = 1$ ), the impacts of parameters  $A, \delta, a, b$  and  $\omega$  on  $\lambda$  and  $P$  have the following signs:

	$A$	$\delta$	$a$	$b$	$\omega$
$\lambda$	-	+	-	+	+
$P$	+	-	-	+	-

The effects of  $\alpha$  are complicated and ambiguous. Computations are available upon request.

**Proposition 11** *Let  $\varepsilon = 1$ .*

*The impacts of the Frisch elasticity of intertemporal substitution ( $1/\varphi$ ) on  $\lambda$  and  $P$  have opposite sign. The impact on  $\lambda$  is positive (on  $P$  is negative) if and only if*

$$\omega > \frac{(1-\alpha)(\rho+\delta)}{\rho+(1-\alpha)\delta} (< 1)$$

*The impacts of the labor supply sensitivity to pollution ( $\psi$ ) on  $\lambda$  and  $P$  have opposite sign. The impact on  $\lambda$  is positive (on  $P$  is negative) if and only if*

$$A > \left(\frac{a}{b}\right)^{1-\alpha} \left(\frac{\rho+\delta}{\alpha}\right)^\alpha \left[\omega \frac{\rho+(1-\alpha)\delta}{(1-\alpha)(\rho+\delta)}\right]^{\frac{1-\alpha}{1+\varphi}}$$

**Proof.** See the Appendix. ■

Let us interpret this proposition. A higher  $\psi$  implies a stronger leisure effect and, thus, a lower labor supply, which reduces the production level and the pollution level in turn. Such a relation is magnified under a large environmental effect of production ( $b$ ).

Following the MGR, such variations of  $\psi$ ,  $l$  and  $P$  have no effect on the stationary value of capital intensity ( $k$ ). In addition, at the steady state,  $c = \gamma kl$  (see the proof of Lemma 13): the decrease of labor supply ( $l$ ) induced by a higher  $\psi$  entails a lower consumption level ( $c$ ) and a higher marginal utility of consumption, that is  $\lambda$  (see (5)).

We observe that the impact of  $\varphi$  on pollution is positive if the exogenous and constant TFP ( $A$ ) is low, the natural rate of pollution absorption ( $a$ ) is high or the environmental impact of production ( $b$ ) is low.

A higher  $\varphi$  means a lower *leisure effect*. Then, for a given pollution level, the representative household works more which enhances the production level and the pollution stock in turn. Under a *distaste effect*, a higher pollution level implies that the household reduces his consumption demand. This increases the marginal utility of consumption and, according to (5),  $\lambda$  as well.

Finally, we consider the effects of time preference ( $\rho$ ) on the main variables ( $\lambda$  and  $P$ ).

**Proposition 12** *Let  $\varepsilon = 1$ . The impact of agents' impatience ( $\rho$ ) is positive on the shadow price ( $\lambda$ ) if and only if*

$$\rho > \left(\psi - \frac{1}{1+\varphi}\right) (1-\alpha)\delta \quad (26)$$

*while it is always negative on the pollution level ( $P$ ).*

**Proof.** See the Appendix. ■

While, in basic Ramsey models,  $\rho$  has always a positive impact on  $\lambda$  (and, thus, negative on  $c$ ), interestingly and surprisingly, in our model, the impact of  $\rho$  on  $\lambda$  is ambiguous. If, for instance,  $\psi = 0$  (pollution has no effect on labor supply), we recover the positive impact on  $\lambda$  as in the basic Ramsey.



Because of the MGR, a higher  $\rho$  implies a lower capital stock, production level and pollution stock at the end. Conversely, since a higher  $\rho$  induces a lower pollution level, the representative household increases his labor supply (*leisure effect*), the marginal disutility of labor supply and, eventually,  $\lambda$ .

## 6.2 Short run

Let us apply the methodology presented in Section 5 to characterize the occurrence of local bifurcations and indeterminacy. In order to disentangle the *compensation* and the *leisure effects* on the equilibrium dynamics, we will consider three cases where pollution affects : (1) both consumption demand and labor supply, (2) only consumption demand and (3) only labor supply.

### 6.2.1 Pollution affects both consumption demand and labor supply

The novelty of our paper rests on considering the joint effect of pollution on consumption demand and labor supply when pollution is a persistent stock.

System (20)-(22) writes:

$$\begin{aligned}\dot{\lambda} &= f_1(\lambda, k, P) \equiv \lambda[\rho + \delta - r(k)] \\ \dot{k} &= f_2(\lambda, k, P) \equiv \frac{\varphi}{\alpha + \varphi} \\ &\quad \left[ \rho k + w(k) - \frac{c(\lambda, k, P)}{l(\lambda, k, P)} - k \frac{1 + \varphi}{\varphi} \left( \rho + \delta - r(k) + \psi \left[ a - b \frac{l(\lambda, k, P) f(k)}{P} \right] \right) \right] \\ \dot{P} &= f_3(\lambda, k, P) \equiv bl(\lambda, k, P) f(k) - aP\end{aligned}\tag{27}$$

We linearize (27) around the steady state to obtain the Jacobian matrix.

To simplify the notation, we introduce some reduced parameters:

$$\begin{aligned}\theta &\equiv \alpha \frac{1 + \varphi}{\varphi} \text{ and } \tau \equiv \frac{\alpha + \varphi}{\varphi} > \theta \\ \mu &\equiv \psi \frac{1 + \varphi}{\varphi}, \gamma \equiv \frac{r}{\alpha} - \delta, s \equiv (1 - \alpha) r \\ n &\equiv \mu \frac{a}{\varphi} + \gamma \left( \frac{1}{\varepsilon} + \frac{1}{\varphi} \right) \text{ and } \xi \equiv \gamma \left( \mu + \eta \frac{\varepsilon - 1}{\varepsilon} \right)\end{aligned}\tag{28}$$

with  $r = \rho + \delta$ .

**Lemma 13** *Let  $D$ ,  $S$  and  $T$  be the determinant, the sum of diagonal minors of order two and the trace of the Jacobian matrix evaluated at the steady state. Thus,*

$$\begin{aligned}D &= \frac{as}{\tau} \left[ (1 + \mu) \left( n - \frac{a\mu}{\varphi} \right) - \frac{\xi}{\varphi} \right] \\ S &= \frac{a\theta\xi - ns}{\tau} - a\rho(1 + \mu) \\ T &= \rho - a + a\mu \frac{\theta - \tau}{\tau}\end{aligned}\tag{29}$$

**Proof.** See the Appendix. ■

Let

$$\eta_H \equiv \frac{\varepsilon}{\varepsilon - 1} \left( \frac{\xi_H}{\gamma} - \mu \right) \quad (30)$$

with

$$\xi_H \equiv \frac{s(1 + \mu) \left( n - \frac{a\mu}{\varphi} \right) + [\rho\tau(1 + \mu) + \frac{ns}{a}] (\rho - a - a\mu \frac{\tau - \theta}{\tau})}{\frac{s}{\varphi} + \theta(\rho - a - a\mu \frac{\tau - \theta}{\tau})} \quad (31)$$

and  $S > 0$ , that is

$$\xi_H > \frac{ns + a\rho\tau(1 + \mu)}{a\theta} \quad (32)$$

**Proposition 14 (limit cycles)** *There exists a parameter region such that, when  $\eta$  goes through  $\eta_H$ , the system undergoes a Hopf bifurcation.*

**Proof.** See the Appendix. ■

It is interesting to see that  $\lim_{\alpha \rightarrow 1} \eta_H = \varepsilon/(\varepsilon - 1)$ . Then,  $\eta_H > 0$  if and only if  $\varepsilon > 1$ , that is the occurrence of a Hopf bifurcation requires dominant income effects. Since

$$\frac{P}{c} \frac{\partial c}{\partial P} = \eta \frac{\varepsilon - 1}{\varepsilon}$$

in this limit case, a Hopf bifurcation occurs only under a compensation effect ( $\partial c/\partial P > 0$  or  $\varepsilon_{Pc} > 0$ ) as in Michel and Rotillon (1995).

Assume a rise of  $P$  near the steady state. Since  $\partial c/\partial P > 0$  and  $\partial l/\partial P < 0$  (matrix (18)), this entails an increase in  $c$  jointly with a decrease in  $k$  and in  $l$ . These two effects imply a fall in the production level and, in turn, a decrease in pollution. By this channel, deterministic endogenous fluctuations occur near the steady state.

**Proposition 15** *Let  $\varepsilon > 1$  and  $\rho > \alpha a$ . With no capital depreciation ( $\delta = 0$ ),*

$$\frac{\partial \eta_H}{\partial \psi} < 0 \text{ iff } \varepsilon > \alpha \frac{1 + \varphi(2 + \varphi)}{1 - \alpha(2 - \alpha)}$$

**Proof.** See the Appendix. ■

Proposition 15 means that, in the case of strong income effects (high  $\varepsilon$ ) and weak natural absorption of pollution ( $\rho > \alpha a$ ), the greater is the sensitivity of labor supply to pollution ( $\psi$ ), the lower is the critical sensitivity of consumption demand to pollution ( $\eta_H$ ) for which a limit cycle occurs.

**Proposition 16 (local indeterminacy through a Hopf bifurcation)** *If*

$$(1 + \mu) \left( n - \frac{a\mu}{\varphi} \right) - \frac{\xi_H}{\varphi} < 0 \quad (33)$$

*then there exists a parameter region where indeterminacy occurs.*

**Proof.** See the Appendix. ■

**Corollary 17** *In the case of compensation effects ( $\varepsilon > 1$ ), local indeterminacy through a Hopf bifurcation arises if*

$$\eta_H > \frac{\varepsilon + \varphi(1 + \mu)}{\varepsilon - 1}$$

**Proof.** Replace  $n$  and  $\xi_H$  from (30) in (33), and solve the inequality for  $\eta_H$ . ■

Focus on relation (33):

$$\lim_{a \rightarrow 0^+} \left[ (1 + \mu) \left( n - \frac{a\mu}{\varphi} \right) - \frac{\xi_H}{\varphi} \right] = -\infty \quad (34)$$

From (34), it appears that local indeterminacy is more likely when the rate of pollution absorption ( $a$ ) is low, that is pollution is more persistent and the negative effects of production as well.

The possibility of self-fulfilling prophecies rests on equilibrium indeterminacy. We provide an intuition for these prophecies in our economy. Let the economy be at the steady state and assume that consumers expect today an increase in the pollution level tomorrow. Since  $\partial c / \partial P > 0$  and  $\partial l / \partial P < 0$ , any consumer wants a higher consumption demand tomorrow jointly with a lower labor supply. She needs to save more today to finance a larger consumption tomorrow under a lower labor income. Higher savings today are possible only if she works more today (the capital stock being predetermined). The higher production today increases the pollution tomorrow. The expectation of higher pollution tomorrow ends up to be self-fulfilling.

In contrast to the existing literature where pollution is defined as a flow, we find local indeterminacy with pollution as a stock but under a strong inertia. More intuition about the consequences of a strong inertia will be provided in the next two sections.

### 6.2.2 Pollution affects only consumption demand

We assume now that pollution affects only the consumption demand:  $e_l = 0$ , that is,  $\psi = \mu = 0$ .

**Proposition 18 (Hopf bifurcation)** *If  $\varepsilon > 1$  (compensation effect) and  $a < \rho$  (pollution inertia), a Hopf bifurcation generically occurs at*

$$\eta = \eta_H \equiv \frac{\varepsilon}{\varepsilon - 1} \frac{\rho\varphi ns + a\tau(\rho - a)}{a\gamma s + \varphi\theta(\rho - a)} > 0$$

*provided that  $S > 0$ , that is  $\rho < (\varphi\theta - s/a)n/\tau$ .*

**Proof.** Simply reconsider (30), (31) and (32) with  $\mu = 0$ . ■

Interestingly, we recover Heal (1982) and Michel and Rotillon (1995) in the sense that a *compensation effect* implies deterministic cycles through a Hopf bifurcation, even if labor supply is endogenous in our case.

**Proposition 19 (pollution inertia)** *If  $a < \rho$ , there is no room for local indeterminacy through a Hopf bifurcation.*<sup>11</sup>

**Proof.** If  $\psi = 0$ ,  $T = \rho - a > 0$ . Therefore, there exists at least one unstable eigenvalue. ■

This proposition highlights the impossibility of local indeterminacy under pollution inertia ( $a < \rho$ ). Conversely, local indeterminacy may occur when pollution affects also the labor supply. Indeed, when pollution affects both consumption and labor through *compensation* and *leisure effects*, if the representative household expects a higher pollution level tomorrow, she saves and works more today. Thus, the production increases and the variation of pollution as well: prophecies become self-fulfilling.

### 6.2.3 Pollution affects only labor supply

We assume that pollution has no effects on consumption demand, namely  $\eta = 0$  and then  $\xi = \gamma\mu$ . From (28) and (29):

$$D = as\gamma \frac{\varepsilon + \varphi(1 + \mu)}{\tau\varphi\varepsilon} > 0 \quad (35)$$

**Proposition 20** *There is no room for local indeterminacy.*

**Proof.**  $D > 0$  implies that there always exists, at least, one unstable eigenvalue. ■

Notice that, by continuity, the determinant remains positive, that is the equilibrium is locally unique, also for sufficiently small  $\eta$ 's.

The main motivation of this paper is to focus on a strong pollution inertia when  $a$  is sufficiently small. This inertia is magnified when  $a$  tends to 0.

**Proposition 21** *There is no room for a Hopf bifurcation under strong pollution inertia.*

**Proof.** Simply consider proposition 6, when  $\eta = 0$ :

$$\lim_{a \rightarrow 0} S = -\gamma \left( \frac{1}{\varepsilon} + \frac{1}{\varphi} \right) \frac{s}{\tau} < 0$$

■

Of course, by continuity,  $S < 0$  also when  $a$  and  $\eta$  are strictly positive but sufficiently small.

In an analogous discrete time context, Bosi et al. (2015) have shown that deterministic cycles arise through a flip bifurcation. Since this type of bifurcation occurs only in discrete time, the fact that cycles are impossible in our economy when pollution affects only labor supply, is not surprising.

<sup>11</sup>Assumption  $a < \rho$  fits the evidence. Indeed, the atmosphere contains 800 gigatons of carbon (GtC) while nature (lands and oceans) absorbs between 2 and 3 GtC per year (Mélières and Maréchal (2015), p. 333). Thereby, according to yearly data, we find  $a \in (0.25, 0.375)$  % while the yearly benchmark for time preference is  $\rho = 1\%$ .

### 6.2.4 Synopsis

The following table summarizes our results when pollution exhibits a strong inertia.

Pollution affects	Hopf	indeterminacy
<i>consumption demand and labor supply</i>	<i>YES</i>	<i>YES</i>
<i>consumption demand</i>	<i>YES</i>	<i>NO</i>
<i>labor supply</i>	<i>NO</i>	<i>NO</i>

When pollution is characterized by a strong inertia, local indeterminacy through a limit cycle emerges if and only if pollution affects both consumption demand (*compensation effect*) and labor supply (*leisure effect*). In this sense, the limit cycle arising near the steady state when pollution affects only consumption demand is preserved when pollution affects both consumption demand and labor supply.

The reader may wonder why local indeterminacy arises when pollution affects both consumption demand and labor supply, while there is no room for local indeterminacy when pollution affects only consumption demand or labor supply. This is due to the *leisure effect*. Indeed, at the steady state:

$$\lim_{\psi \rightarrow \infty} \frac{a}{P} \frac{\partial P}{\partial a} = - \lim_{\psi \rightarrow \infty} \frac{1}{1 + \psi} = 0 \quad (36)$$

From (36), it appears that a strong *leisure effect* ( $\psi \rightarrow \infty$ ) neutralizes the effect of the pollution inertia ( $a$ ) on the pollution of steady state. That is, the *leisure effect* renders the pollution stock more volatile and promotes the emergence of local indeterminacy even if pollution is a stock variable with a very strong inertia. This striking result can also be analyzed simply considering equation (7) with  $a = 0$  (very strong pollution inertia), namely,  $\dot{P} = blf(k)$ . Without *leisure effect*, a higher pollution level at time  $t$  implies a higher pollution level tomorrow (pollution inertia). Under a *leisure effect*, a higher pollution level today implies: (1) a higher pollution level tomorrow because of the pollution inertia and (2) a lower pollution tomorrow because of the lower labor supply. That is, the *leisure effect* makes the resulting effect of a higher pollution today on pollution tomorrow ambiguous. In this sense, the *leisure effect* renders the pollution accumulation process more volatile. This explains the occurrence of local indeterminacy through a local bifurcation.

## 7 Conclusion

We have considered a unified model to study the joint effect of pollution on consumption demand and labor supply. We have provided sufficient conditions to ensure the uniqueness of the steady state and introduced a general

method to address the issue of local bifurcations and indeterminacy in the case of continuous-time three-dimensional dynamic systems with two predetermined variables. Applying the general method to the case of separable isoelastic preferences, we have found that a *compensation effect* coupled with a *leisure effect* leads to local indeterminacy through a Hopf bifurcation. Equilibrium multiplicity is obtained under the assumption of persistent pollution stock (strong inertia), an empirically convincing case neglected by the literature on local indeterminacy, more focused on pollution flows. This literature has left aside the inertial forms of pollution more relevant to represent severe macroeconomic consequences such as the global warming. The occurrence of equilibrium multiplicity under pollution inertia rests on the role of the *leisure effect*. Neutralizing the inertial impact on pollution, the latter promotes pollution fluctuations and macroeconomic volatility at the end.

## 8 Appendix

**Proof of Proposition 1.** The Hamiltonian writes  $\tilde{H} = e^{-\rho t}U(c, l, P) + \tilde{\lambda}[(r - \delta)h + wl - c]$  and the first-order conditions

$$\begin{aligned}\partial\tilde{H}/\partial\tilde{\lambda} &= (r - \delta)h + wl - c = \dot{h} \\ \partial\tilde{H}/\partial h &= \tilde{\lambda}(r - \delta) = -\tilde{\lambda}' \\ \partial\tilde{H}/\partial c &= e^{-\rho t}U_c - \tilde{\lambda} = 0 \\ \partial\tilde{H}/\partial l &= e^{-\rho t}U_l + \tilde{\lambda}w = 0\end{aligned}$$

jointly with the transversality condition  $\lim_{t \rightarrow \infty} \tilde{\lambda}(t)h(t) = 0$ . Setting  $\lambda \equiv e^{\rho t}\tilde{\lambda}$ , we find  $\dot{\lambda} - \rho\lambda = e^{\rho t}\tilde{\lambda}'$  and equations in Proposition 1. The discounted Hamiltonian  $H \equiv e^{\rho t}\tilde{H}$  becomes  $H = U(c, l, P) + \lambda[(r - \delta)h + wl - c]$ . ■

**Proof of Proposition 2.** Differentiating the system

$$\begin{aligned}\lambda - U_c(c, l, P) &= 0 \\ \lambda w(k) + U_l(c, l, P) &= 0\end{aligned}$$

we get

$$\begin{aligned}\varepsilon_{cc}\frac{dc}{c} + \varepsilon_{lc}\frac{dl}{l} &= \frac{d\lambda}{\lambda} - \varepsilon_{Pc}\frac{dP}{P} \\ \varepsilon_{cl}\frac{dc}{c} + \varepsilon_{ll}\frac{dl}{l} &= \frac{d\lambda}{\lambda} + \alpha\frac{dk}{k} - \varepsilon_{Pl}\frac{dP}{P}\end{aligned}$$

that is

$$\begin{bmatrix} \frac{dc}{c} \\ \frac{dl}{l} \end{bmatrix} = \frac{M}{\varepsilon_{cc}\varepsilon_{ll} - \varepsilon_{lc}\varepsilon_{cl}} \begin{bmatrix} \frac{d\lambda}{\lambda} \\ \frac{dk}{k} \\ \frac{dP}{P} \end{bmatrix}$$

where  $M$  is given by

$$M \equiv \begin{bmatrix} \varepsilon_{ll} - \varepsilon_{lc} & -\frac{\alpha}{\sigma} \varepsilon_{lc} & \varepsilon_{lc} \varepsilon_{Pl} - \varepsilon_{ll} \varepsilon_{Pc} \\ \varepsilon_{cc} - \varepsilon_{cl} & \frac{\alpha}{\sigma} \varepsilon_{cc} & \varepsilon_{cl} \varepsilon_{Pc} - \varepsilon_{cc} \varepsilon_{Pl} \end{bmatrix}$$

Thus, we obtain the matrix of partial elasticities

$$\begin{bmatrix} \frac{\lambda}{c} \frac{\partial c}{\partial \lambda} & \frac{k}{c} \frac{\partial c}{\partial k} & \frac{P}{c} \frac{\partial c}{\partial P} \\ \frac{\lambda}{l} \frac{\partial l}{\partial \lambda} & \frac{k}{l} \frac{\partial l}{\partial k} & \frac{P}{l} \frac{\partial l}{\partial P} \end{bmatrix} = \frac{M}{\varepsilon_{cc} \varepsilon_{ll} - \varepsilon_{lc} \varepsilon_{cl}}$$

In the separable case (4), the elasticities matrix (8) simplifies:

$$E \equiv \begin{bmatrix} \varepsilon_{cc} & 0 & \varepsilon_{cP} \\ 0 & \varepsilon_{ll} & \varepsilon_{lP} \\ \varepsilon_{Pc} & \varepsilon_{Pl} & \varepsilon_{PP} \end{bmatrix}$$

and we get (9). ■

**Proof of Proposition 3.** Let us reconsider the dynamic system:

$$\begin{aligned} \dot{\lambda} &= \lambda [\rho + \delta - r(k)] \\ \dot{h} &= (r - \delta) h + w l - c \\ \dot{P} &= -aP + blf(k) \end{aligned}$$

We observe that  $h = kl$  and, thus,  $\dot{h}/h = \dot{k}/k + \dot{l}/l$ . In addition,  $l = l(\lambda, k, P)$ . Thus,

$$\frac{\dot{l}}{l} = \left( \frac{\lambda}{l} \frac{\partial l}{\partial \lambda} \right) \frac{\dot{\lambda}}{\lambda} + \left( \frac{k}{l} \frac{\partial l}{\partial k} \right) \frac{\dot{k}}{k} + \left( \frac{P}{l} \frac{\partial l}{\partial P} \right) \frac{\dot{P}}{P}$$

where the elasticities within the parentheses are given by (9).

We obtain the following three-dimensional dynamic system:

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= \rho + \delta - r(k) \\ \frac{\dot{k}}{k} &= r(k) - \delta + \frac{w(k)}{k} - \frac{c(\lambda, k, P)}{kl(\lambda, k, P)} - \frac{\dot{l}}{l} \\ &= r(k) - \delta + \frac{w(k)}{k} - \frac{c(\lambda, k, P)}{kl(\lambda, k, P)} - \frac{\lambda}{l} \frac{\partial l}{\partial \lambda} \frac{\dot{\lambda}}{\lambda} - \frac{k}{l} \frac{\partial l}{\partial k} \frac{\dot{k}}{k} - \frac{P}{l} \frac{\partial l}{\partial P} \frac{\dot{P}}{P} \\ \frac{\dot{P}}{P} &= b \frac{l(\lambda, k, P) f(k)}{P} - a \end{aligned}$$

that is system (10). ■

**Proof of Proposition 4.** Assumption 1 ensures that a stationary level of capital  $k$  exists according to equation (11). The concavity of  $f$  ensures also that there is a unique stationary level of capital.

$P(\lambda)$  is implicitly defined by (13). Applying the Implicit Function Theorem to equation (13), we obtain the slope of  $P(\lambda)$ :

$$P'(\lambda) = \frac{\frac{\lambda l}{l} \frac{l}{\lambda}}{\frac{a}{bf(k)} - \frac{Pl_P}{l} \frac{l}{P}}$$

Noticing that, at the steady state,  $l/P = a/(bf)$ , we get the multiplier elasticity of pollution:

$$\zeta \equiv \frac{\lambda P'(\lambda)}{P(\lambda)} = \frac{\frac{\lambda l}{l}}{1 - \frac{Pl_P}{l}}$$

Let

$$\varsigma(\lambda) \equiv \frac{c(\lambda, k, P(\lambda))}{l(\lambda, k, P(\lambda))} > 0 \text{ and } \varepsilon_\varsigma(\lambda) \equiv \frac{\lambda \varsigma'(\lambda)}{\varsigma(\lambda)}$$

Replacing  $k$  from (11) in

$$\varsigma(\lambda) = \rho k + w(k) \quad (37)$$

we compute  $\lambda$  and, eventually, the pollution  $P = P(\lambda)$  of steady state.

The continuity of  $\varsigma$  implies that, if there are multiple steady state, the slope  $\varsigma'(\lambda)$  changes its sign from a steady state to another. Conversely, if  $\varsigma'(\lambda)$  is always negative at the steady state  $\lambda$ , then the steady state is unique.

The sign of  $\varsigma'(\lambda)$  is the same of  $\varepsilon_\varsigma(\lambda)$ . Under Assumption 2 (separability), the elasticity  $\varepsilon_\varsigma$  writes

$$\varepsilon_\varsigma(\lambda) = \frac{\lambda c_\lambda}{c} - \frac{\lambda l_\lambda}{l} + \zeta \left( \frac{Pc_P}{c} - \frac{Pl_P}{l} \right) = \frac{\lambda c_\lambda}{c} - \frac{\lambda l_\lambda}{l} \frac{1 - \frac{Pc_P}{c}}{1 - \frac{Pl_P}{l}} = \frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{ll}} \frac{1 + \frac{\varepsilon_{Pc}}{\varepsilon_{cc}}}{1 + \frac{\varepsilon_{Pl}}{\varepsilon_{ll}}} \quad (38)$$

with  $\varepsilon_{cc} < 0$  and  $\varepsilon_{ll} > 0$ . Thus,  $\varepsilon_\varsigma(\lambda) < 0$  if and only if (14) holds. ■

**Proof of Proposition 6. Necessity.** In a three-dimensional dynamic system, we require at the bifurcation value:  $\lambda_1 = i\tilde{b} = -\lambda_2$  with no generic restriction on  $\lambda_3$  (see Bosi and Ragot (2011) or Kuznetsov (1998) among others). The characteristic polynomial of  $J$  is given by:  $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - T\lambda^2 + S\lambda - D$ . Using  $\lambda_1 = i\tilde{b} = -\lambda_2$ , we find  $D = \tilde{b}^2\lambda_3$ ,  $S = \tilde{b}^2$ ,  $T = \lambda_3$ . Thus,  $D = ST$  and  $S > 0$ .

*Sufficiency.* In the case of a three-dimensional system, one eigenvalue is always real, the others two are either real or non-real and conjugated. Let us show that, if  $D = ST$  and  $S > 0$ , these eigenvalues are non-real with zero real part and, hence, a Hopf bifurcation generically occurs.

We observe that  $D = ST$  implies

$$\lambda_1\lambda_2\lambda_3 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)$$

or, equivalently,

$$(\lambda_1 + \lambda_2) [\lambda_3^2 + (\lambda_1 + \lambda_2)\lambda_3 + \lambda_1\lambda_2] = 0 \quad (39)$$



This equation holds if and only if  $\lambda_1 + \lambda_2 = 0$  or  $\lambda_3^2 + (\lambda_1 + \lambda_2)\lambda_3 + \lambda_1\lambda_2 = 0$ . Solving this second-degree equation for  $\lambda_3$ , we find  $\lambda_3 = -\lambda_1$  or  $-\lambda_2$ . Thus, (39) holds if and only if  $\lambda_1 + \lambda_2 = 0$  or  $\lambda_1 + \lambda_3 = 0$  or  $\lambda_2 + \lambda_3 = 0$ . Without loss of generality, let  $\lambda_1 + \lambda_2 = 0$  with, generically,  $\lambda_3 \neq 0$  a real eigenvalue. Since  $S > 0$ , we have also  $\lambda_1 = -\lambda_2 \neq 0$ . We obtain  $T = \lambda_3 \neq 0$  and  $S = D/T = \lambda_1\lambda_2 = -\lambda_1^2 > 0$ . This is possible only if  $\lambda_1$  is non-real. If  $\lambda_1$  is non-real,  $\lambda_2$  is conjugated, and, since  $\lambda_1 = -\lambda_2$ , they have a zero real part. ■

**Proof of Proposition 7. Necessity.** In the real case, we obtain  $D = \lambda_1\lambda_2\lambda_3 < 0$ ,  $S = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$  and  $T = \lambda_1 + \lambda_2 + \lambda_3 < 0$ .

*Sufficiency.* We want to prove that, if  $D, T < 0$  and  $S > 0$ , then  $\lambda_1, \lambda_2, \lambda_3 < 0$ . Notice that  $D < 0$  implies  $\lambda_1, \lambda_2, \lambda_3 \neq 0$ .

$D < 0$  implies that at least one eigenvalue is negative. Let, without loss of generality,  $\lambda_3 < 0$ . Since  $\lambda_3 < 0$  and  $D = \lambda_1\lambda_2\lambda_3 < 0$ , we have  $\lambda_1\lambda_2 > 0$ . Thus, there are two sub-cases: (1)  $\lambda_1, \lambda_2 < 0$ , (2)  $\lambda_1, \lambda_2 > 0$ . If  $\lambda_1, \lambda_2 > 0$ ,  $T < 0$  implies  $\lambda_3 < -(\lambda_1 + \lambda_2)$  and, hence,

$$S = \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\lambda_3 < \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)^2 = -\lambda_1^2 - \lambda_2^2 - \lambda_1\lambda_2 < 0$$

a contradiction. Then,  $\lambda_1, \lambda_2 < 0$ . ■

**Proof of Proposition 8.** From Proposition 6, we have  $\text{Re } \lambda_1(p_H) = \text{Re } \lambda_2(p_H) = 0$ .  $\lambda_3(p_H) < 0$  is implied by  $D(p_H) = [\text{Im } \lambda_1(p_H)]^2 \lambda_3(p_H) < 0$ . Thus, there exists  $\varepsilon > 0$  such that, generically, we have  $\text{Re } \lambda_1(p), \text{Re } \lambda_2(p), \lambda_3(p) < 0$  (local indeterminacy) for any  $p \in (p_H - \varepsilon, p_H)$  or, alternatively, for any  $p \in (p_H, p_H + \varepsilon)$ . ■

**Proof of Proposition 9.** From (4) and (16), (5) writes

$$c = \left[ \lambda P^{\eta(1-\varepsilon)} \right]^{-1/\varepsilon} \text{ and } l = \left[ w \lambda P^{-\psi(1+\varphi)} / \omega \right]^{1/\varphi} \quad (40)$$

(11) gives (24). Equation (13) yields (25). Replacing (24) and (25) in (40) and (40) in (37), we find (23). ■

**Proof of Proposition 10.** Consider

$$\ln \lambda = \frac{\varepsilon(\varphi z - y)(1 - e_l) + \varepsilon(\varphi x + y)(e_c - e_l)}{\varepsilon(1 - e_c) + \varphi(1 - e_l)} \quad (41)$$

$$\ln P = \frac{\varphi x + y + \ln \lambda}{\varphi(1 - e_l)} \quad (42)$$

where  $(x, y, z) \equiv (\ln(Ak^\alpha b/a), \ln(w/\omega), -\ln(\rho k + w))$ , that is

$$\begin{aligned} x &= \frac{\alpha}{1-\alpha} \ln \alpha + \frac{1}{1-\alpha} \ln A - \frac{\alpha}{1-\alpha} \ln(\rho + \delta) - \ln \frac{a}{b} \\ y &= \frac{\alpha}{1-\alpha} \ln \alpha + \frac{1}{1-\alpha} \ln A - \frac{\alpha}{1-\alpha} \ln(\rho + \delta) + \ln(1-\alpha) - \ln \omega \\ z &= -\frac{\alpha}{1-\alpha} \ln \alpha - \frac{1}{1-\alpha} \ln A + \frac{1}{1-\alpha} \ln(\rho + \delta) - \ln[\rho + (1-\alpha)\delta] \end{aligned}$$

Computations give

$$\frac{\partial P/\partial \varepsilon}{P/\varepsilon} = \frac{\ln \lambda + \eta \ln P}{\varepsilon(1 - e_c) + \varphi(1 - e_l)} \text{ and } \frac{\partial \lambda/\partial \varepsilon}{\lambda/\varepsilon} = \varphi(1 - e_l) \frac{\partial P/\partial \varepsilon}{P/\varepsilon} \quad (43)$$

$$\frac{\partial P/\partial \eta}{P/\eta} = \frac{\varepsilon e_c \ln P}{\varepsilon(1 - e_c) + \varphi(1 - e_l)} \text{ and } \frac{\partial \lambda/\partial \eta}{\lambda/\eta} = \varphi(1 - e_l) \frac{\partial P/\partial \eta}{P/\eta} \quad (44)$$

$$\varepsilon(1 - e_c) + \varphi(1 - e_l)$$

with  $\ln(\lambda P^\eta) = \ln(cP^{-\eta})^{-\varepsilon}$ .

Using (41) and (42) we get more explicitly

$$\frac{\partial P/\partial \varepsilon}{P/\varepsilon} = \frac{(1 + \varphi)(\eta + \psi)(x + z) + (\eta - 1)(y - \varphi z)}{[\varepsilon(1 - e_c) + \varphi(1 - e_l)]^2} \varepsilon \quad (45)$$

$$\frac{\partial P/\partial \eta}{P/\eta} = \frac{\varphi x + y + \varepsilon(x + z)}{[\varepsilon(1 - e_c) + \varphi(1 - e_l)]^2} \varepsilon e_c$$

The qualitative impacts of  $\varepsilon$  on  $\lambda$  and  $P$  are the same because of (43). If  $\eta = 1$ , (45) becomes

$$\frac{\partial P/\partial \varepsilon}{P/\varepsilon} = \frac{\varepsilon(x + z)}{(1 + \varphi)(1 + \psi)} \text{ with } x + z = \ln \left[ \frac{b}{a} \frac{\rho + \delta}{\rho + (1 - \alpha)\delta} \right]$$

and the first part of proposition follows.

The qualitative impacts of  $\eta$  on  $\lambda$  and  $P$  are the same because of (44). We observe that, under Assumption 3,  $(\partial P/\partial \eta)/(P/\eta) > 0$  iff  $(\varepsilon - 1) \ln P > 0$ . ■

**Proof of Proposition 11.** We have

$$\ln \lambda = \frac{x\psi + x\psi\varphi - y + z(\varphi + \psi + \varphi\psi)}{(1 + \varphi)(1 + \psi)}$$

$$\ln P = \frac{\varphi x + y + \ln \lambda}{\varphi + \psi + \varphi\psi}$$

and, thus,

$$\frac{\partial P/\partial \varphi}{P/\varphi} = -\frac{\varphi}{1 + \varphi} \frac{y + z}{(1 + \varphi)(1 + \psi)} \text{ and } \frac{\partial \lambda/\partial \varphi}{\lambda/\varphi} = -\frac{\partial P/\partial \varphi}{P/\varphi}$$

$$\frac{\partial P/\partial \psi}{P/\psi} = -\frac{\psi}{1 + \psi} \frac{(1 + \varphi)x + y + z}{(1 + \varphi)(1 + \psi)} \text{ and } \frac{\partial \lambda/\partial \psi}{\lambda/\psi} = -\frac{\partial P/\partial \psi}{P/\psi}$$

■

**Proof of Proposition 12.** Laborious computations give:

$$\frac{\partial \lambda/\partial \rho}{\lambda/\rho} = M \frac{\alpha \rho (\rho(1 + \varphi) + \delta(1 - \alpha)[1 - \psi(1 + \varphi)])}{[(1 - \alpha)(\rho + \delta)(\rho + (1 - \alpha)\delta)(1 + \varphi)(1 + \psi)]} > 0$$

if and only if (26) holds, and

$$\frac{\partial P/\partial \rho}{P/\rho} = -\frac{\alpha \rho [\rho(1+\varphi) + \delta(1-\alpha)(2+\varphi)]}{[(1-\alpha)(\rho+\delta)(\rho+(1-\alpha)\delta)(1+\varphi)(1+\psi)]} < 0$$

■

**Proof of Lemma 13.** The Jacobian matrix (15) becomes:

$$J = \begin{bmatrix} 0 & s \frac{\lambda}{k} & 0 \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial P} \\ a \frac{\lambda \lambda}{l} \frac{P}{\lambda} & a \left( \alpha + \frac{kl_k}{l} \right) \frac{P}{k} & a \left( \frac{Pl_P}{l} - 1 \right) \end{bmatrix}$$

with

$$\begin{aligned} \frac{\partial f_2}{\partial \lambda} &= \frac{1}{\tau} \frac{k}{\lambda} \left[ a\mu \frac{\lambda \lambda}{l} + \gamma \left( \frac{\lambda \lambda}{l} - \frac{\lambda c \lambda}{c} \right) \right] \\ \frac{\partial f_2}{\partial k} &= \frac{1}{\tau} \left[ a\mu \left( \alpha + \frac{kl_k}{l} \right) + \gamma \left( \frac{kl_k}{l} - \frac{kc_k}{c} \right) + \rho - \frac{s}{\varphi} \right] \\ \frac{\partial f_2}{\partial P} &= \frac{1}{\tau} \frac{k}{P} \left[ a\mu \left( \frac{Pl_P}{l} - 1 \right) + \gamma \left( \frac{Pl_P}{l} - \frac{Pc_P}{c} \right) \right] \end{aligned}$$

because, at the steady state,

$$\frac{c}{kl} = \gamma > 0, \quad \frac{w}{k} = r \frac{1-\alpha}{\alpha} \quad \text{and} \quad b \frac{lf(k)}{P} = a$$

Using (2) and (18), we find

$$J = \begin{bmatrix} 0 & s \frac{\lambda}{k} & 0 \\ \frac{n}{\tau} \frac{k}{\lambda} & \rho + a\mu \frac{\theta}{\tau} & -\frac{\xi + a\mu(1+\mu)}{\tau} \frac{k}{P} \\ \frac{a}{\varphi} \frac{P}{\lambda} & a\theta \frac{P}{k} & -a(1+\mu) \end{bmatrix}$$

■

**Proof of Proposition 14.** Focus on Proposition 6 and expressions (29) for  $D$ ,  $S$  and  $T$ . We know that a Hopf bifurcation arises if and only if  $D = ST$  and  $S > 0$ , that is if and only if

$$\begin{aligned} \frac{as}{\tau} \left[ (1+\mu) \left( n - \frac{a\mu}{\varphi} \right) - \frac{\xi}{\varphi} \right] &= \left[ \frac{a\theta\xi - ns}{\tau} - a\rho(1+\mu) \right] \left( \rho - a + a\mu \frac{\theta - \tau}{\tau} \right) \\ \frac{a\theta\xi - ns}{\tau} - a\rho(1+\mu) &> 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} \xi_H &\equiv \frac{s(1+\mu) \left( n - \frac{a\mu}{\varphi} \right) + [\rho\tau(1+\mu) + \frac{ns}{a}] \left( \rho - a - a\mu \frac{\tau - \theta}{\tau} \right)}{\frac{s}{\varphi} + \theta \left( \rho - a - a\mu \frac{\tau - \theta}{\tau} \right)} \\ \xi_H &> \frac{ns + a\rho\tau(1+\mu)}{a\theta} (> 0) \end{aligned}$$

A Hopf bifurcation generically occurs if the following restriction is satisfied:

$$\frac{s(1+\mu)\left(n - \frac{a\mu}{\varphi}\right) + [\rho\tau(1+\mu) + \frac{ns}{a}]\left(\rho - a - a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi} + \theta\left(\rho - a - a\mu\frac{\tau-\theta}{\tau}\right)} > \frac{ns + a\rho\tau(1+\mu)}{a\theta} \quad (46)$$

If

$$\frac{s}{\varphi} + \theta\left(\rho - a - a\mu\frac{\tau-\theta}{\tau}\right) > 0 \quad (47)$$

(46) becomes equivalent to

$$a(1+\mu)(\varphi n\theta - \tau\rho - a\mu\theta) - ns > 0 \quad (48)$$

Let us show that inequalities (47) and (48) are satisfied for some parametric values. Consider the case  $a < \rho$  and  $\alpha \approx 1$ . Inequalities (47) and (48) become

$$\lim_{\alpha \rightarrow 1} \left[ \frac{s}{\varphi} + \theta\left(\rho - a - a\mu\frac{\tau-\theta}{\tau}\right) \right] = \theta(\rho - a) > 0$$

and

$$\lim_{\alpha \rightarrow 1} [a(1+\mu)(\varphi n\theta - \tau\rho - a\mu\theta) - ns] = a\rho\tau(1+\mu)\frac{\varphi}{\varepsilon} > 0$$

because

$$\lim_{\alpha \rightarrow 1} n \equiv \mu\frac{a}{\varphi} + \rho\left(\frac{1}{\varepsilon} + \frac{1}{\varphi}\right)$$

■

**Proof of Proposition 15.** Using (31), we compute the derivative with  $\delta = 0$ .

$$\frac{\partial\eta_H}{\partial\psi} = \frac{1+\varphi}{\varphi} \frac{\rho\varphi(\rho - a\alpha)(1-\alpha)(\alpha+\varphi)[\alpha - \varepsilon + \alpha\varepsilon(2-\alpha) + \alpha\varphi(2+\varphi)]}{(\varepsilon-1)(\rho(\alpha\varphi+1)(\alpha+\varphi) - a\alpha(1+\varphi)[(\alpha+\varphi) + \mu\varphi(1-\alpha)]^2}$$

■

**Proof of Proposition 16.** Notice that

$$D(p_H) = \frac{as}{\tau} \left[ (1+\mu)\left(n - \frac{a\mu}{\varphi}\right) - \frac{\xi_H}{\varphi} \right] < 0$$

and apply Proposition 8. ■

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