Does inertia matter ?
Assessing the sustainability of optimal pollution paths when the degradation of assimilative capacity is time-dependent
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Abstract

Most formal optimal pollution control models assume a constant natural assimilative capacity, despite the biophysical evidence on feedback effects that can degrade this environmental function, as it is the case with the reduction of ocean carbon sinks in the context of climate change. The few models that do consider this degradation establish a bijective relation between the pollution stock and the assimilative capacity, thus ignoring the inertia mechanism at stake. Indeed the level of assimilative capacity is not solely determined by the current pollution stock but by the history of this stock and the time the ecosystem remains in a degradation zone. In order to determine if this ecological feature has a significant impact, we assess the efficiency and the sustainability of a myopic with respect to inertia benchmark optimal pollution path when it is injected into a dynamic system with an autonomous assimilative capacity variable allowing for the inertia ef ! fect. Our simulations show nonetheless that even with very conservative parameters, inertia has little impact on the sustainability of optimal pollution paths applied to climate change policies.

Keywords: Optimal pollution control, Assimilative Capacity, Inertia, Ecosystem Services, Climate Change

JEL classification : D62 ; H23 ; Q01 ; Q5 ; Q54.

1. Introduction

Since the seminal articles on optimal pollution control (Keeler et al., 1972 ; Plourde, 1972), most standard stylized models in partial equilibrium assume a linear and constant assimilative capacity of the environment. However this assumption ignores the degradation of this environmental function that can occur when the pollution stock crosses a critical threshold. This degradation is particularly significant in the case of greenhouse gases accumulation. Ecological science, either through direct observation (Schuster and Watson, 2007) or ocean-climate modelization (Le QuerÃl' et al., 2007), has shown that the carbon uptake rate by oceans has been decreasing over the last decade. This decrease is due to positive feedbacks such as surface

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water warming, water stratification and thermohaline currents modification. Despite the scientific uncertainty characterizing the monitoring of carbon concentration, according to the International Climate Change T ! askforce Report (2005, Chapter 10) there is "unanimous agreement among the coupled climate-carbon cycle models driven by emission scenarios run so far that future climate change would reduce the efficiency of the Earth system (land and ocean) to absorb anthropogenic CO_2". Plattner et al. (2001) forecasted a reduction in ocean carbon uptake by $7-10\%$ over the 21^{st} century compared to a constant climate scenario but the decrease might reveal much faster. According to recent observations by Schuster and Watson (2007), the North Atlantic Ocean's CO_2 assimilative capacity has already decreased by 50% over the 1995-2005 period as the atmospheric concentration of carbon increased. From an economic perspective, this alteration of the global carbon cycle might have a strong impact on the mitigation strategies as they demand increased emission reductions or sequestration in order to achieve the same stabilization objective.

This assessment of the worrying evolution of greenhouse gases assimilative capacity makes a strong case for the adaptation of the pollution control economic models to this ecological phenomenon. The standard linear representation of the natural decay activity with a constant rate of assimilation has already been questioned by different authors such as Forster (1975), Tahvonen and Salo (1996), Tahvonen and Withagen (1996), Toman and Withagen (2000), Prieur (2009) or Hediger (2009) in stylized optimal pollution control models. These contributions, referred to as *bijective models* in the rest of this paper, have tried to introduce more realistic decay function, such as a concave-convex function and to allow for the irreversible annihilation of assimilative capacity beyond a critical threshold. The main conclusion from these contributions is the existence of multiple equilibria associated with either a positive or an ir ! reversibly depleted assimilative capacity and the impossibility for the affected ecosystem to return to its initial state when it has reached an irreversible basin of attraction.

However these natural decay functions display an exclusive dependency [1] on the pollution stock variable Z that does not fit so well the ecological reality of GHG accumulation and other stock pollution problems. Indeed, according to empirical ecological evidence it seems reasonable in a wide range of cases to assume that this assimilative capacity will not be only impacted by the absolute stock level, but also by the length of time spent in the degradation zone. We define as the inertia effect the fact that the assimilative capacity available associated to the same pollution level will not be the same if the ecosystem has just reached this stock level or if it has spent a long period at this level. Consequently this assimilative capacity cannot simply increase back to a higher level if the pollution stock decreases as the models above assume. The bijective relation established in th ! e pre-cited contributions forbids to link the assimilative capacity with the pollution "history" as it attributes the same level to any stock, regardless of when this stock was reached and how long it was sustained. According to basic ecological evidence the assimilative capacity should not depend directly on the total accumulated stock of pollution but should follow dynamics on its own right that are determined by the stock of pollution relatively to a degradation threshold. The irreversibility of the degradation as well as the inertia effect should thus be reflected in the assimilative capacity autonomous dynamics. That is why we propose to test the sustainability of a benchmark optimal pollution path in a modified dynamic system with the assimilative capacity as a second state variable.

Our objective is thus to study the behaviour of a dynamic pollution system when the inertia effect is accounted for from both a theoretical and a numerical simulation perspective. We seek to determine if adding complexity to the formal models via the inertia effect is necessary in order to avoid making irreversible mistakes in the policy recommendations relying on a basic model. Hence the question "Does inertia matter ?". Since an optimal solution corresponding to the setting with inertia proved untractable, we will dedicate

1. The assimilative function $A(Z)$ is a bijective function of the pollution stock Z.

special attention to the conditions in which benchmark optimal pollution paths can lead the assimilative function to an irreversible extinction. As optimal as it may be, such a pollution path raises legitimate concerns on the sustainability of the climate policy that would be willing to implement it. The solutions can either be sustainable in the sense that they preserve indefinitely a strictly positive level of assimilative capacity or unsustainable if they lead to the extinction of the assimilative function.
What is more, our take on pollution control focusing on the dynamics of assimilative capacity can be interpreted as a case of ecosystem service management. Consequently our introduction of inertia in the degradation process of an ecosystem service might be a useful step to improve the economic analysis of other ecosystem services displaying similar dynamics.
Our analysis will follow four steps. First we recall in Section 2 the benchmark model of pollution control with constant assimilative capacity rate and study more precisely the sensitivity of the resulting optimal paths to variations in the invariant assimilative capacity level. In doing so we shall debunk a erroneous comparative static result that is often found in the literature as well as in textbooks. In Section 3 we present the modified dynamic system with time-dependent assimilative capacity degradation and we illustrate the importance of inertia on the sustainability of pollution paths through a zero-emission scenario. In Section 4 we study the importance of inertia testing the benchmark solution in the dynamical system with inertia. This technique is usually used in operational research. Our numerical results show that even with very conservative parameters the inertia effect does not affect significantly the sustainability of the optimal poll ! ution paths.

2. Standard assimilative capacity in the benchmark optimal pollution control model

2.1. Insights from the benchmark model with constant assimilative capacity

Despite the intricate dynamics at play, our model will at times come down to the benchmark optimal pollution control with constant assimilative capacity. We shall thus present here the crucial properties that we will resort to afterwards. In doing so we shall also point out an erroneous result on comparative statics that can be found in the literature as well as in textbooks.

$$\max_{y(t)} \int_0^\infty \left[f(y(t)) - D(Z(t))\right] e^{-\rho t} dt$$
$$\text{s.t.}$$
$$\dot{Z}(t) = y(t) - \alpha Z(t), \quad Z(0) > 0$$

with the standard properties : f the benefit function from pollution (increasing, concave), D the environmental damage function (increasing and convex), ρ the discount rate and α the constant assimilative capacity $(0 < \alpha < 1)$
We thus get the following Hamiltonian with λ the co-state variable associated to Z

$$\mathscr{H}_c(t) = f(y(t)) - D(Z(t)) + \lambda(t)(y(t) - \alpha Z(t)) \tag{1}$$

which yields the standard first order conditions

$$f'(y(t)) = -\lambda(t) \tag{2}$$
$$\dot{\lambda}(t) = (\rho + \alpha)\lambda(t) + D'(Z(t)) \tag{3}$$

2.1.1. Steady States

At the steady state the dynamics of Z and λ and the properties of f and D, shows that a unique Z_{ss} exists with $f'(\alpha Z_{ss}) > 0$ and that verifies

$$f'(\alpha Z_{ss})(\rho+\alpha) - D'(Z_{ss}) = 0. \tag{4}$$

Let us introduce the standard functional forms (identical to many found in the literature) that we will use thereafter to go farther than the analysis with general functions but that respect the main properties expected in the model.

$$f(y) = cy - \frac{b}{2}y^2,\ c,b>0, \quad D(Z) = \frac{d}{2}Z^2,\ d>0.$$

With these functional forms, (4) is tantamount to

$$Z_{ss} = \frac{(\rho+\alpha)c}{(\rho+\alpha)b\alpha+d}. \tag{5}$$

2.1.2. Characterization of the optimal path with functional forms

In this standard setting simplified by our specific functional forms it is possible to determine analytically the expression of $Z^*(t)$, $\lambda^*(t)$ and $y^*(t)$ along the optimal pollution path by solving the first order conditions (2, 3) for the quadratic linear functional forms.

$$Z^*(t) = \frac{(\rho+\alpha)c}{d+\rho\alpha b+\alpha^2 b} + \frac{Z(0)(d+\rho\alpha b+\alpha^2 b)-(\rho+\alpha)c}{d+\rho\alpha b+\alpha^2 b}\, e^{-\bar{\rho}t}, \tag{6}$$

$$\lambda^*(t) = (-\frac{1}{2}\,\bar{\rho}+\alpha b)\,\frac{Z(0)(d+\rho\alpha b+\alpha^2 b)-(\rho+\alpha)c}{d+\rho\alpha b+\alpha^2 b}\, e^{-\bar{\rho}t} - c + \frac{\alpha b(\rho+\alpha)c}{d+\rho\alpha b+\alpha^2 b}, \tag{7}$$

$$y^*(t) = \frac{c+\lambda}{b}. \tag{8}$$

where $\bar{\rho} = \frac{-\rho b+\sqrt{b(\rho^2 b+4d)+4b^2\alpha(\rho+\alpha b)}}{2b}$.

Equations (6 : 8) will be used in Section 5 to estimate the impact of inertia on the optimal pollution paths.

2.2. Addressing a gap in the litterature : comparative statics of α and Z_{ss}

2.2.1. Invalidating the comparative statics result

When it is mentioned in the literature (Plourde, 1972, page 124) or in textbooks (Pearman, 2010), the relation between α and Z_{ss} is assumed to be increasing : a higher assimilative capacity leading systematically to a higher pollution stock at the steady state. An economic intuition often claimed to explain this relation is that the assimilative capacity level can be interpreted as a discount factor, hence the higher α the higher Z_{ss}. However using our functional forms and the solution (6 : 8) we can invalidate this result.

Proposition 2.1 *With our functional forms respecting the standard properties we have :*

– *for a low enough d, Z_{ss} decreases with α*

– *for a higher d, Z_{ss} increases and then decreases with α (see Figure 1)*

– *We can obtain the derivative for all t*

$$\frac{\partial Z(t)}{\partial \alpha} = \left(1-e^{-\rho t}\right)\frac{\partial Z_{ss}}{\partial \alpha} + Z_0 e^{-\rho t}.$$

If $\frac{\partial Z_{ss}}{\partial \alpha} > 0$ *then* $\frac{\partial Z(t)}{\partial \alpha} > 0$ *for all* t. *If* $\frac{\partial Z_{ss}}{\partial \alpha} < 0$ *and* $Z_0 = 0$ *then* $\frac{\partial Z(t)}{\partial \alpha} < 0$ *for all* t.

Proof. The proof is evident taking into account (6) and that

$$\frac{\partial Z_{ss}}{\partial \alpha} = \left(c \frac{d - b(\alpha + \rho)^2}{(d + \rho\alpha b + \alpha^2 b)^2} \right).$$

When d is small enough, $\frac{dZ_{ss}(\alpha)}{d\alpha} < 0 \; \forall \; \alpha$. When d is high $\frac{dZ_{ss}(\alpha)}{d\alpha} > 0$ for low values of α and $\frac{dZ_{ss}(\alpha)}{d\alpha} < 0$ for high values of α. ■

It is clear that a higher given α ($d^2 = 0,2$ instead of $d^1 = 0$, in figure 1) will have respectively a negative (positive) impact on the stead state stock of pollution Z_{ss} when d is low (high). With a low d we do find the commonly accepted result enounced above but with a high d this result is not valid anymore. We have thus a proof that the simple comparative static result commonly found in the literature does not hold. With our functional forms that respect nonetheless the standard properties of the general model we find a more ambiguous result that is stated in Proposition 2.1. This ambiguity had been only noted by Forster (1973) and never dwelled upon.

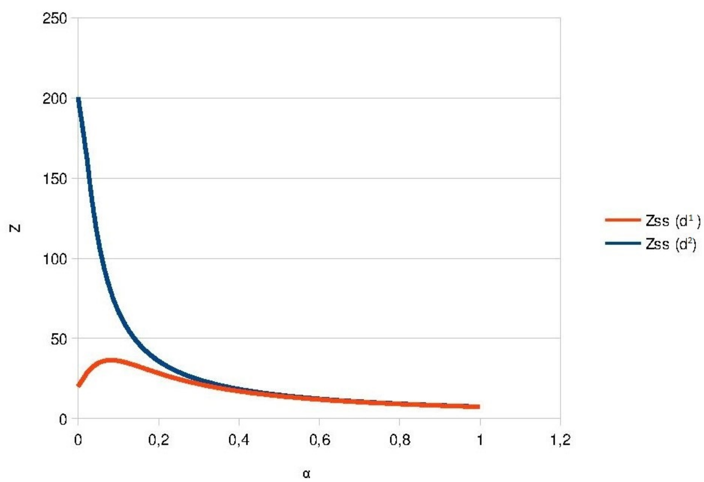

FIGURE 1: Z_{ss} as a function of α depending on the value of d

2.2.2. Assimilative capacity and social welfare at the steady state

Out of curiosity we check that the intuitive positive effect of α on the social welfare at the steady state is actually true.

$$W(\alpha) = f(\alpha Z_{ss}(\alpha)) - D(Z_{ss}(\alpha))$$

hence taking into account (4)

$$\frac{\partial W}{\partial \alpha} = \left(\alpha \frac{\partial Z_{ss}}{\partial \alpha} + Z_{ss}(\alpha) \right) f'(\alpha Z_{ss}(\alpha)) - \frac{\partial Z_{ss}}{\partial \alpha} D'(Z_{ss}(\alpha)) = \left(\rho \frac{\partial Z_{ss}}{\partial \alpha} + Z_{ss}(\alpha) \right) f'(\alpha Z_{ss}(\alpha)).$$

which is positive when $\frac{\partial Z_{ss}}{\partial \alpha} > 0$. Nevertheless with our specific forms we can prove that it is always positive, in fact :

$$W(\alpha) = \frac{1}{2}\frac{c^2(\alpha+\rho)(d\alpha+\rho\alpha^2 b+b\alpha^3-d\rho)}{(d+\rho\alpha b+b\alpha^2)^2}$$

and thus

$$\frac{dW(\alpha)}{d\alpha} = \frac{c^2 d(b(\rho+\alpha)^3+d\alpha)}{(d+\rho\alpha b+b\alpha^2)^3} > 0.$$

As expected, a greater assimilative capacity offers a greater social welfare at the steady state. However this welfare gain can be obtained through two channels : either by allowing a higher level of emissions without increasing too much the pollution stock, or by reducing the pollution stock while maintaining a given emission level. Keeping in mind this choice faced by the social planner between the most efficient of these options will help us shape a convincing economic interpretation of the second part of Proposition 2.1.

2.2.3. Economic interpretation of Proposition 2.1

In the first case (low d), since the environmental damage caused by the pollution stock has a very low impact on the total social welfare, the emissions y will be mostly determined by the private benefit they generate with little consideration for the resulting pollution accumulation. As a result, the level of emission $y^*(t)$ along the optimal path will be close to the private static optimum $\hat{y}$ that would result from the sole static optimization of the benefit function. A change in the dynamics of the pollution stock, which are barely taken into account, will thus have no significant impact on y^*_{ss} which will remain close to $\hat{y}$. That is why a greater assimilative capacity will be confronted with a similar level of steady state emissions, which will naturally yield a lower steady state pollution stock (through the dynamics that are almost ignored from the heart of the optimization problem itself). This phenomenon can be read in the expression of $y_{!ss} = \alpha Z_{ss}$: since y_{ss} remains more or less constant and close to $\hat{y}$, a higher α will be obviously compensated by a lower Z_{ss}.

In the second case, when the environmental damage is significant enough, the level of emissions will also be guided the negative externality via the accumulated stock of pollution. Given the properties of the private benefit and of the damage function, the impact of a greater assimilative capacity on the steady state pollution stock will be differentiated depending on the level of emissions at the steady state y_{ss}. Indeed as long as α, and thus y_{ss} are small enough so that the difference between the marginal benefit and the marginal damage yielded by an additional emission is positive, a higher α will induce the social planner to increase $y_{ss} = \alpha Z_{ss}$, which will in turn increase the corresponding social welfare as well as the pollution stock. Hence the first increasing part of the curve (which is often the only case envisioned in textbooks) in Figure 1. However, once α crosses the threshold $\bar{\alpha}$, y_{ss} is too ! high for an increase in emissions to add social welfare given the concavity of f and the convexity of D. On this side of the curve, a greater α will thus not be used to enjoy much higher emissions but rather to reduce the pollution stock, which is at this point the most efficient way to increase social welfare.

3. Building a dynamic pollution system with time-dependent assimilative capacity degradation

3.1. The model with α as an autonomous state variable

Most amended pollution control models (Tahvonen and Withagen, 1996, Prieur, 2009) that account for the variations in assimilative capacity do so through a bijective relation such that $\alpha(t) = \alpha(Z(t))\ \forall\ t$, with a critical threshold Z^c such that $\alpha(t) = 0 \ \forall \ t > T$ if $\exists \ T$ such that $Z(T) \geq Z^c$. The image function $\alpha(Z(t))$ can take many forms : linear (Tahvonen and Withagen, 1996), quadratic (Prieur, 2009), etc. Although these models clearly improve the benchmark dynamics, they do not reflect important properties of the assimilative function.

First, the bijective relation implies associating systematically the same set of initial conditions $(Z_0, \alpha_0 = \alpha(Z_0))$ which prevents comparative studys of pollution problem with a similar dynamic but with different initial conditions.

Second, this dynamic does not allow the possibility of artificial restoration of the assimilative capacity, which can be a relevant tool in some pollution problems (Leandri, 2009 ; El Ouardighi, 2011). The only way to increase α is to change the pollution stock which does not reflect the opportunities offered by restoration ecology to deal separately with the assimilative capacity service. However, considering the current uncertainty on geo-engineering techniques, introducing assimilative capacity restoration in pollution control models with a climate change application in mind does not seem like a priority to us.

Third, and most importantly, the dynamics in the bijective models associate identical assimilative capacity rates to the same stock pollution level, whatever time the system has spent at that stock level. Whether it has just reached this stock Z from below for the first time or it has remained at this stock or higher for a very long time, $\alpha(Z(t))$ will be the same. This property is obviously a strong simplification of the ecological dynamics at stake and does not account for the important factor of inertia in the degradation of the ecosystem service of assimilation, especially acute in the case of the variations of the ocean carbon sinks. In particular some models (Tahvonen and Withagen, 1996) do not account completely [2] for the irreversibility of the de ! gradation since returning to a lower pollution stock, even after a very long time spent a high pollution levels, will automatically increase the assimilative function back to higher levels, as if nothing happened before.

Therefore we choose to deal with $\alpha(t)$ as an autonomous state variable of its own right following its own dynamics and freed from the image function bijection with the pollution stock. Following the extensions of Pearce's intuitions (1976) carried out by Pezzey (1996) [3] and discussed by Godard (2006) and Leandri (2009), the dynamics of α are based on a degradation threshold $\bar{Z}$ (not to be confused with the prior mentioned Z^c) above which the assimilative capacity undergoes a degradation process.

With this additional state variable the model is the following :

$$\max_{y(t)} \int_0^\infty [f(y(t)) - D(Z(t))] \, e^{-\rho t} dt \tag{9}$$

$$\begin{aligned}
&\text{s.t.} \\
&\dot{Z}(t) = y(t) - \alpha(t)Z(t) \\
&\dot{\alpha}(t) = -h(Z(t)) \\
&Z(0) = Z_0 \, , \ \alpha(0) = \alpha_0 \\
&\alpha(t) \geq 0, \quad Z(t) \geq 0, \quad y(t) \geq 0, \ \forall \ t.
\end{aligned}$$

where h is the assimilative degradation function beyond the threshold $\bar{Z}$, h non decreasing and continuous in $\bar{Z}$. The endogenous regime switch at play in this model make it intractable analytically as well as through

2. It must be noted that irreversibility is present once the pollution stock reaches Z^c since it is then impossible to reduce the pollution stock with a zero assimilative capacity and no external restoration.

3. Our model displays similarities with the work of El Ouardighi et al. (2011) as they both share the same inspirations in the literature. However we do not abide by their strong assumptions on assimilative capacity restoration that make their model less fitted to tackle the crucial case of greenhouse gases accumulation.

simulations. It is important to keep in mind that in the problem we address, regime switches are not an endogenous decision taken by the agent, as it is usually the case in the literature on optimization with regime switches (Tonimiya et al, 1986). Indeed in our case the system switches from one regime to another when one of the state variables crosses a threshold, respectively $\bar{Z}$ for the pollution stock and 0 for the assimilative capacity. [4] We shall therefore focus on the dynamic system of pollution accumulation itself to address the issue of inertia in pollution c ! ontrol path.

Functional forms. In addition to the functional forms mentioned in Section 2 we will resort to the following specific dynamics for α :

$$\dot{\alpha}(t) = -h(Z(t)) = -k(\max[Z - \bar{Z}, 0]) \text{ if } \alpha > 0 \tag{10}$$

$$\begin{aligned} &\dot{\alpha}(t) = 0 \; if \; \alpha = 0 \\ &\text{if } \exists\, t_1 \text{ s.t. } \alpha(t_1) = 0 \text{ then } \alpha(t) = \dot{\alpha}(t) = 0 \; \forall t \geq t_1 \end{aligned} \tag{11}$$

3.2. Sustainable vs. unsustainable solutions

Let us discuss now the types of solution that can arise in our model, in terms of ecosystem service sustainability. Indeed dealing with the assimilative capacity as an autonomous state variable can lead us to assess an optimal path in terms of sustainability. Does the optimal path maintain indefinitely a strictly positive level of assimilative capacity ? Or in other words is it sustainable in terms of ecosystem service management ? Or does it lead this service to extinction, in which case it can be characterized as unsustainable (without questioning its economic optimality) ?

Let us try to categorize all the possible sets of solutions with the following typology. Figure 2 and 3 illustrate these types of solutions with pollution paths that are drawn arbitrarily with no reference to actual optimality nor actual monotonicity.

1. *Unsustainable solutions :* $\exists\, \tau_1$ s.t. $\forall t \geq \tau_1$, $\alpha^*(t) = 0$
   With this kind of solution the economy depletes totally the assimilative and thus deprives future generations of a crucial ecosystem service.
2. *Sustainable solutions :* $\alpha^*(t) > 0 \; \forall t$
   With this kind of solutions the economy preserves indefinitely a strictly positive level of assimilative capacity. Let us divide this category according to the traditional opposition between strong and weak sustainability.
   (a) *Strongly sustainable solutions :* $\alpha^*(t) = \alpha_0 \; \forall t$
   ie : $Z(t) \leq \bar{Z} \; \forall \; t$, $\dot{\alpha}(t) = 0 \; \forall$ t,
   This kind of solutions implies the total conservation of the assimilative ecosystem service. It is straightforward that we need to have in particular $Z_0 \leq \bar{Z}$.
   (b) *Weakly sustainable solutions :* all the sustainable solutions that are not strong.
   This kind of solutions implies the degradation of the assimilative capacity at some point but without ever leading it to its total depletion.

4. What's more, beyond the impossibility to characterize an optimal solution, the fastidious stationarity analysis we carried on for each regime of the system sheds very little light on the sustainability conditions we wish to explore and we have thus chosen not to include it here.

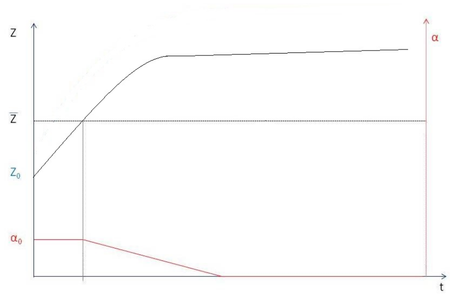

FIGURE 2: Unsustainable Path (1)

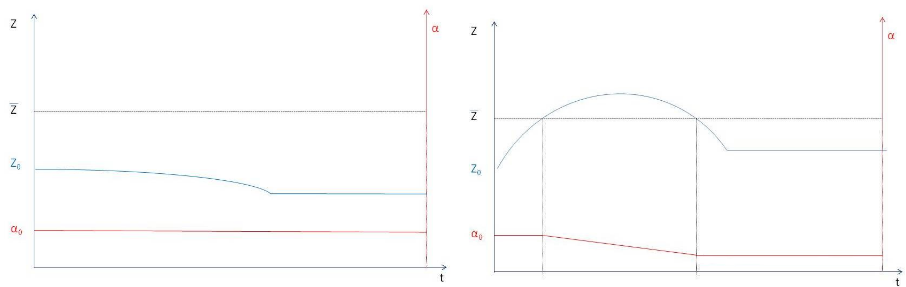

FIGURE 3: Strongly Sustainable Path (left, 2.a) and Weakly Sustainable Path (right, 2.b)

3.3. Insights on the dynamics from the emission-free scenario

The original feature of our dynamic system is best illustrated through an emission-free scenario. Observing the system's behaviour when emissions are always nil allows us to show clearly the specificity of our model. In order to emphasize the specificity of our model compared to the bijective models mentioned previously, we shall give an overview of the behaviour of the latter in a zero emission world before focusing on our dynamic system with inertia.

3.3.1. The behaviour of the bijective models with zero emissions

In order to highlight the most blatant differences between the bijective models and ours, let us present an example of a bijective image function and observe their behaviour in a zero emission world ($y(t) = 0 \; \forall \; t$). Please note that we modify the notations of the authors to make them fit the ones we used so far. One of the decay function most commonly found in the literature is the inverted U-shape function [5] (Tahvonen and Withagen, 1996, Cesar and De Zeew, 1994 Prieur, 2009). This function is obtained by a linear decreasing image function for the assimilative capacity :

5. Note that here the inverted U-shape function refers to the total absolute assimilative capacity $\alpha(t) * Z(t)$, not to the image function $\alpha(Z(t))$.

$$\dot{Z} = -\alpha(Z)Z,$$

$$\alpha(Z) = \begin{cases} a(Z^c - Z) & \text{if } Z \leq Z, \\ 0 & \text{if } Z \geq Z^c. \end{cases} \quad (12)$$

Note that Z^c, the critical threshold in these models that implies the total extinction of assimilative capacity, must not be confused with our $\bar{Z}$, the threshold at which degradation of assimilative capacity starts.

For the sake of clarity, let us rewrite (12) such that $\alpha(Z(t)) = \max(1 - aZ(t); 0)$ with $a = \frac{1}{Z^c}$. This simplified expression captures the same degradation mechanism but it also guarantees that $\alpha(t) < 1 \; \forall \, Z(t) > 0$ and that $\alpha = 0$ when $Z(t) = Z^c$.

It can be easily anticipated that with this linear decay function any pollution stock starting below Z^c will decrease down to the threshold at which α stops increasing and further on down to zero. On the contrary any system starting with $Z_0 \geq Z^c$ will remain indefinitely in Z_0.

Using the set of parameters [$a = 0,004$; $Z^c = 250$; $Z_0 = 200$], we can trace the path of a dynamic "bijective" pollution system with zero emissions when $Z_0 < Z^c$ [6]. As expected the pollution stock tends to zero while the assimilative capacity tends to its limit value 1 (Figure 4). This behaviour will be observed no matter how long the pollution stock has remained at high levels beforehand. Figure 5 illustrates this linear relation between α and Z along the path that starts in the top left hand corner and ends in the bottom right hand corner of the plane.
This "zero-emission" test reveals the optimistic assumption behind the bijective models' dynamics as the assimilative capacity will systematically be preserved and will allow the pollution stock to eventually converge towards zero, no matter the "history" of this pollution stock. We will see next that this is not the case when the dynamics allow for the inertia effect.

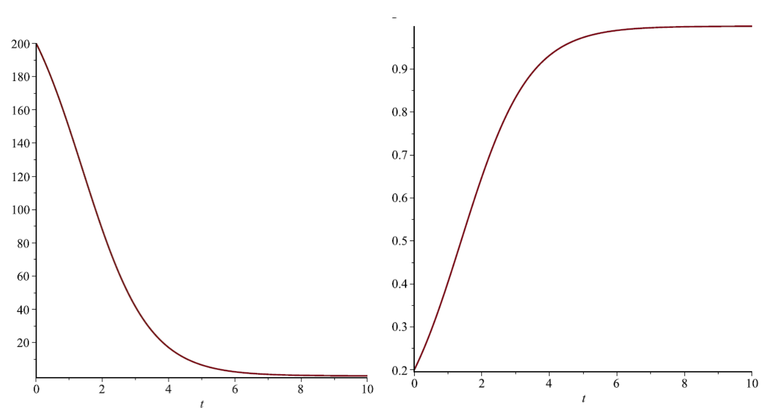

FIGURE 4: Pollution stock Z (left) and assimilative capacity α (right) along a zero-emissions bijective pollution path

6. It is straightforward that when $Z_0 \geq Z^c$, the system will remain indefinitely in Z_0 since $\alpha(t) = y(t) = 0$

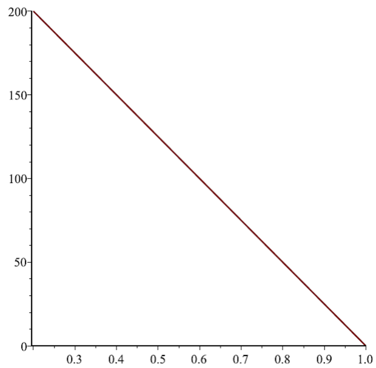

FIGURE 5: Z and α along a zero-emissions bijective pollution path

3.3.2. *Observing our model in a zero emission world*

Let us now determine the behavior of our own dynamic system in the absence of emissions, when the time paths depend exclusively on the initial conditions on Z and α.

In this case the dynamics are :

$$\dot{Z}(t) = -\alpha(t)Z(t),$$

$$\dot{\alpha}(t) = \begin{cases} -k(Z-\bar{Z}) & \text{if } Z \geq \bar{Z},\ \alpha > 0, \\ 0 & \text{if } Z < \bar{Z}, \text{ or } \alpha = 0. \end{cases}$$

And at the steady state we have :

$$\dot{Z} = 0 \iff \alpha = 0 \text{ or } Z = 0$$

$$\dot{\alpha} = 0 \iff \alpha = 0\,, \text{ or } Z < \bar{Z}\,, \text{or } \{Z = \bar{Z} \text{ and } \alpha > 0\}$$

Given the initial conditions $Z(0) = Z_0$ and $\alpha(0) = \alpha_0$:

- For all $Z_0 \leq \bar{Z}$ we have $\dot{\alpha} = 0$, and the steady state is thus $(\alpha, 0)$ for some α.
- For all $Z_0 > \bar{Z}$ we have two possibilities :

1. There exist $T \leq \infty$ such that $\alpha(T) = 0$, $Z(T) \geq \bar{Z}$, then the steady state is $(0, Z(T))$.
2. There exists T such that $Z(T) = \bar{Z}$ with $\alpha(T) > 0$, then the steady state is $(\alpha(T), 0)$.

In conclusion there exists a curve $g(\alpha, Z) = 0$ in the (α, Z) plane with the function g such that $g(0, \bar{Z}) = 0$, $g(\alpha, \bar{Z}) > 0$ when $\alpha > 0$ and g decreasing in Z and increasing in α. This curve crosses the Z-axis at a finite time T, which means that along a dynamic path following this curve there exists a time T such that $\alpha(T) = 0$ and we can assess that

- In region V (below the curve $g(\alpha, Z) = 0$) the steady state is $(\alpha(T), 0)$.
- In region DZ (above the curve $g(\alpha, Z) = 0$) the steady state is $(0, Z(T))$.

In Figure 6, the paths of the dynamic system must be read from right to left as the pollution stock will never increase in the "natural" situation without emissions. The degradation threshold is marked by the red line $Z = \bar{Z}$. The dotted black curve $g(\alpha, Z) = 0$ separates the plane in two regions. In the doomed zone "DZ", above $g(\alpha, Z) = 0$, the system will naturally converge towards an unsustainable steady state $(0, Z(T))$. This

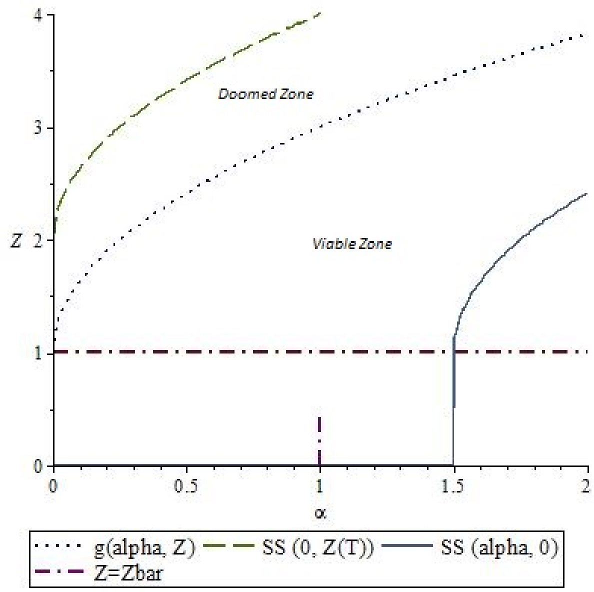

FIGURE 6: Sustainable and unsustainable paths in a world without emissions

reflects the fact that even without emissions this set of initial conditions is bound to lead to the total depletion of assimilative capacity as the pollution stock will not decrease fast enough to reach the threshold $\bar{Z}$ before α runs out. In the viable region "V", below $g(\alpha, Z) = 0$, the system starting with any set of initial conditions will converge towards a sustainable steady state $(\alpha > 0, 0)$. The level of assimilative capacity preserved will depend on how long the pollution stock remains above !Z.
For instance ; the green dotted line ($SS(0, Z(T))$) in the top left-hand corner is an example of trajectory starting in the DZ and converging towards a steady state $(0, Z(T))$ with $Z(T) > \bar{Z}$. The blue solid curve ($SS(\alpha, 0)$) on the right is an example of trajectories that start in the V zone, above the threshold $Z = \bar{Z}$. This curve corresponds to a weakly sustainable path where both the pollution stock and the assimilative capacity decrease until the $\bar{Z}$ threshold is reached and the assimilative capacity remains constant afterwards.
Note that if $\exists T$ such that $(Z(T), \alpha(T)) \in DZ$, $(Z(T), \alpha(T)) \in DZ\ \forall t \geq T$. As a consequence an economy with positive emissions located in the doomed zone at a given time will thus necessarily end up with a zero assimilative capacity. But of course the opposite is not true because an economy starting in the viable zone may also end up with a zero assimilative capacity if it chooses high levels of emissions that eventually lead it to the doomed zone but it can also remain in the viable zone and reach a sustainable solution if it wishes.

Note that the viable zone we have defined reminds of the viability kernel (Martinet & Doyen, 2007) : for any system located in the viability kernel/viable zone it is possible to find a path (not necessarily optimal) that remains in the viable zone (although it might imply a zero emission policy).
This preliminary analysis will be used subsequently to sketch the "best scenario" behavior of the economy. Indeed Figure 6 gives us the final outcome of the system for any set of initial conditions when emissions are nil. We can thus deduce *a fortiori* that if at one time along the optimal path the system is in the doomed zone, it can never recover back to a sustainable solution, even less so if the emission level is strictly positive.

4. Assessing the importance of inertia through an original simulation method

4.1. Injecting benchmark optimal emissions in our system with inertia

The complexity raised by the endogeneity of the regime switches and by the presence of two autonomous state variables prevents us from achieving a full resolution of the dynamic system. In order to answer explicitly the question at the origin of this work and to determine if inertia matters and should be made explicit in optimal pollution control models, especially in the case of climate change, we resort to an original simulation method. We shall test up to what extent an optimal emission path ($y^*(t)\ \forall$t) obtained in the benchmark model with constant assimilative capacity (and thus defined as a *myopic optimal emission path* (MOEP)) leads to the extinction of this assimilative capacity when it is injected in a system like ours accounting for degradation with inertia. Using conservative/pessimistic parameters, this simulation will enable us to determine if the degradation mechanism with inertia is crucial for climate economic policies or if it can be ignored rather safely.

The path under study will thus follow the dynamics below.

$$\dot{Z}(t) = y^*(t) - \alpha(t)Z(t) \tag{13}$$
$$\dot{\alpha}(t) = \begin{cases} -k(Z-\bar{Z}) & \text{if } Z \geq \bar{Z} \text{ and } \alpha > 0, \\ 0 & \text{if } Z < \bar{Z}, \text{ or if } \alpha = 0. \end{cases}$$

Z_0 and α_0 given and with $y^*(t) > 0\ \forall t$ and Z^*_{ss} [7] the optimal steady state level of pollution such that

$$\alpha_0 Z^*_{ss} = \lim_{t\to\infty} y^*(t) \tag{14}$$

It must be noted that when the MOEP $(y^*(t))$ is implemented in the benchmark setting it yields a monotonic path $Z(t)$, this pollution accumulation path might no longer be monotonic when we inject the MOEP in the dynamic system with inertia. This non-monotonocity can occur if the MOEP is decreasing in $Z(t)$, ie $y^*(t) < \alpha_0 Z^*(t)\ \forall t$. In that case once the MOEP is injected in the system with inertia α might decrease in such a way that at one point $y^*(t) > \alpha(t)Z(t)$, which means an increase in Z.

4.2. Steady states of a MOEP in the system with inertia

Let us now consider the outcomes in terms of steady states of a MOEP injected in the system with inertia. Once it is plugged in the dynamic system with assimilative capacity degradation, the MOEP might not lead to the steady state that was optimal in the benchmark setting since the MOEP might lead to a change in the dynamics of Z through a decrease in α.

According to (13) and the dynamics of α, at the steady state we need the following conditions to be verified :

$$\dot{Z} = 0 \iff \lim_{t\to\infty} y^*(t) = \alpha_{ss} Z_{ss} \tag{15}$$

$$\dot{\alpha} = 0 \iff \alpha = 0 \text{ or } Z \leq \bar{Z} \tag{16}$$

with Z_{ss} the pollution stock at the steady state in the system with inertia and $\alpha_{ss} = \lim_{t\to\infty} \alpha(t)$.

7. It must be noted that variables with no superscript refer to the variables in the system with inertia while variables with a * superscript refer to the variables obtained in the benchmark model without assimilative capacity degradation.

Proposition 4.1 1. *if $\exists\ T$ such that $\alpha(T) = 0$ along the path in the inertia system, then $\alpha_{ss} = 0$ and (15) cannot be verified given the strict positivity of $y^*(t)$. No steady state can exist, the system tends towards a limit point $(\infty, 0)$.*
2. *if $\alpha(t) > 0\ \forall t$, a steady state exists if $\frac{Z^*_{ss}}{\alpha_{ss}} \leq \frac{\bar{Z}}{\alpha_0}$*

Proof.

1. is straightforward.
2. if a steady state with $\alpha > 0$ exists, we have

$$\alpha_{ss} Z_{ss} = \lim_{t\to\infty} y^*(t)$$

Hence, according to (14) :

$$Z_{ss} = \frac{\alpha_0}{\alpha_{ss}} Z^*_{ss}$$

A steady state exists if (16) is verified, ie if

$$Z_{ss} \leq \bar{Z} \quad \Longleftrightarrow \frac{\alpha_0}{\alpha_{ss}} Z^*_{ss} \leq \bar{Z} \quad \Longleftrightarrow \frac{Z^*_{ss}}{\alpha_{ss}} \leq \frac{\bar{Z}}{\alpha_0}. \tag{17}$$

■ Relation (17) shows that for given ecological conditions on $\bar{Z}$ and α_0, a steady state can be reached if α_{SS} is big enough , ie if the assimilative capacity has not been too degraded along the MOEP and/or if Z^*_{ss} is small enough, ie if the benchmark optimal steady state pollution stock was not to high in the first place. Our results in Section 2 on the relation between Z^*_{ss} and α_0 can help us shed some light on the conditions of a sustainable steady state captured by (17). Indeed, if α_0 is high, α_{ss} will be preserved at a higher level even if it undergoes degradation. What's more, according to Proposition 2.1 Z_{ss} decreases in α when α is high. We can thus deduce that the higher α_0 the higher α_{ss} and the lower Z^*_{ss}.

Corollary 4.1 *If $Z^*_{ss} > \bar{Z}$, a MOEP is not sustainable.*

Proof.
– If $\exists\ T$ such that $\alpha(T) = 0$ along the MOEP, then according to Proposition 4.1 no (sustainable) steady state can be reached.
– If if $\alpha(t) > 0\ \forall t$, as $Z^*_{ss} > \bar{Z}$ and $\alpha_{ss} \leq \alpha_0$, then

$$Z^*_{ss} > \bar{Z} \Rightarrow \frac{Z^*_{ss}}{\alpha_{ss}} > \frac{\bar{Z}}{\alpha_0},$$

and since by construction $\alpha_{ss} \leq \alpha_0$, it yields

$$\frac{Z^*_{ss}}{\alpha_{ss}} > \frac{\bar{Z}}{\alpha_0}$$

Condition (17) is not verified and no (sustainable) steady state can be reached.

■

Corrolary 4.1 can be translated with our functional forms into the following condition

$$\rho \leq \frac{-c + 2\bar{Z}\sqrt{bd}}{b\bar{Z}} \tag{18}$$

that will enable us to focus our simulations on potentially sustainable MOEPs based on a desirable discount rate.
Two cases must be analyzed :

1. $Z^*_{ss} > \bar{Z}$
   We can anticipate that an optimal program $y^*(t)$ that would have ended in a myopic steady state $Z^*_{ss} > \bar{Z}$ will necessarily end up at a steady state also higher than $\bar{Z}$ since there can be at best no changes in α. It is thus straightforward that any myopic optimal emission path (MOEP) leading originally to a steady state pollution stock higher than $\bar{Z}$ will be unsustainable since we have observed in §3.3 that if the pollution stock remains higher than $\bar{Z}$ long enough, it will ineluctably drive the assimilative capacity to extinction.

   We can thus determine *a priori* as unsustainable all MOEP such that $Z^*_{ss} > \bar{Z}$ and restrict our study on to MOEPs such that $Z^*_{ss} \leq \bar{Z}$, that is to say if the set of parameters $\{\rho, \alpha_0, b, c, d\}$ are such that

   $$\frac{(\rho+\alpha_0)c}{(\rho+\alpha_0)b\alpha_0+d} \leq \bar{Z}$$

   which is tantamount to $\rho > \rho^{\Delta C}$ with our notations for Regime C (see Proposition **??**).

   Proposition 4.2 *If $\rho > \rho^{\Delta C}$, a MOEP is unsustainable*

   As a consequence we will only need to test the sustainability of MOEP within sets of parameters such that $\rho \leq \rho^{\Delta C}$, that is such that

   $$\rho \leq \frac{-c+2\bar{Z}\sqrt{bd}}{b\bar{Z}} \qquad (19)$$

2. $Z^*_{ss} \leq \bar{Z}$
   Among the MOEP with $Z^*_{ss} \leq \bar{Z}$ two cases can arise. Either the degradation along the path is too fast and the MOEP never reaches Regime C, it diverges towards $Z = \infty$ and $\alpha = 0$. Or the MOEP does reach Regime C at time $\bar{T}$ with a positive level of assimilative capacity $\alpha(\bar{T})$ and converges towards Z_{ss} such that $Z_{ss} \geq Z^*_{ss}$. We shall rely on our numerical simulations to distinguish these two possibilities.

4.3. Parameters

We shall resort to the following parameters borrowed from Schubert (2005) as far as the first four are concerned. The crucial value of k is obtained through a rough extrapolation of the estimations found in the climate and ocean science literature (Le QuerÃľ et al., 2007 ; Schuster and Watson, 2007). We are aware that the figure we use for k, like the mere concept of reducing the oceans' sink capacity to our linear degradation (10), makes little sense to biologists and climate scientist but it can nonetheless shed some light on the order of magnitude of the impact of inertia on the sustainability of economic climate policies. We will test the sustainability of the MOEP for much higher values of k to take the most conservative view on the issue. We have set Z_0 at the value of 450 (ppm) but it will be discussed later.

$$b = 1,35$$
$$c = 10$$
$$d = 0,001$$
$$\alpha_0 = 0,04$$
$$k = 8,508E^{-07}$$
$$\bar{Z} = 450$$

In accordance to (19), we restrict our analysis to values of ρ such that

$$\rho \leq 0,0379722$$

We shall thus work with $\rho = 0,03$ and run some sensibility tests.

Using these parameters and the results from section 2 (in particular (6) and (8), we can easily obtain the numerical values for $y^*(t)$ at all time t, ie the optimal emission level for a "myopic" benchmark pollution control model with constant assimilative capacity. We shall then apply this "myopic" policy in a world with assimilative capacity degradation, ie in our dynamic system with inertia.

4.4. Testing the sustainability of the "injected" MOEP

We consider two cases depending on the position of the initial pollution stock regarding the degradation threshold.

4.4.1. $Z_0 \leq \bar{Z}$*.*

In that case it is straightforward that whether Z^*_{ss} is higher or lower than Z_0, it respects Condition 19 nonetheless, and the system will always remain within Regime C, through either a decreasing (if $Z_0 > Z^*_{ss}$) or increasing (if $Z_0 < Z^*_{ss}$) monotonic path. It will thus behave exactly like the benchmark model, keeping the assimilative capacity to α_0. The MOEP will thus be sustainable and converge towards its expected steady state.

Illustrations under progress

4.4.2. $Z_0 > \bar{Z}$*.*

Since the benchmark optimal emission path considered is associated, via Condition 19, to a steady state below $\bar{Z}$ we know that when when $Z_0 > \bar{Z}$ the MOEP will be decreasing, at least until α is significantly reduced.
Figure 7-left shows, with our basic parameters and $Z_0 = 550$, that when the system starts with a pollution stock above the degradation threshold (upper part of the figure), it nevertheless reaches a steady state below $\bar{Z}$ with a strictly positive α. The initial degradation is very small : from 0,04 to 0,039 at $t = \bar{T} = 6,17$.
This sustainability is still verified when k is set at a value 50 times higher than our extrapolation. The initial degradation of α (from the upper-right part of the figure to the middle) is more remarkable as $\alpha(\bar{T})$ has been reduced to 0.023 when the system get below the degradation threshold at time $\bar{T} = 9.81$.

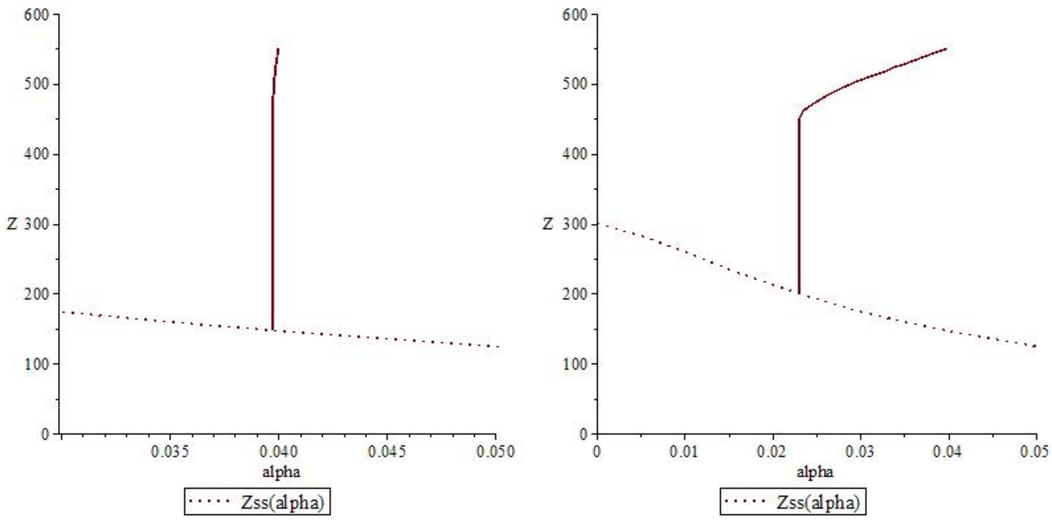

FIGURE 7: Phase portrait with the given parameters (left) ; with k times 50 (right)

Our simulations also show that the steady state pollution level Z_{ss} reached in the system with inertial by the injected MOEP is higher than the steady state $Z^*_{ss} = \frac{y^*_{ss}}{\alpha_0}$ it would have reached in the benchmark system. This result is rather intuitive since when the path starts above $\bar{Z}$, α_{ss} will be lower than α_0 following the degradation. The steady state level of pollution will thus settle at a level Z_{ss} such that

$$\begin{aligned} Z_{ss} &= \frac{y_{ss}}{\alpha_{ss}} \\ &= \frac{y^*_{ss}}{\alpha_{ss}} \\ &> \frac{y^*_{ss}}{\alpha_0} \\ &> Z^*_{ss} \end{aligned}$$

This result is confirmed in Figure 7 where the dotted line gives the steady state pollution level in the benchmark model Z^*_{ss} depending on the (constant) assimilative capacity α_0. It is clear that the new steady state reached by the MOEP in the system with inertial is higher than Z^*_{ss}.

After finding these rather reassuring results we have run simulations to identify F_{min} the smallest factor such that if $k' = k * F_{min}$ the system diverges and the MOEP is unsustainable, all things equal. We find that $F_{min} = 70$. In figure 8-left we can see that with $k' = k * 70$ α is depleted at time 15.77 before the system can get below $\bar{Z}$. The path starts in the upper-right corner with a decrease in Z and a degradation of α until alpha is so low that Z starts back up towards ∞ while α is finally depleted.
Note that if we compute the possible steady states in zone A of our problem, assuming that the control y was not myopic anymore but could be adjusted in the long run to settle the system at a steady state, we find, in the case k times 70, $Z_{ss} = 450$ and $\alpha = 0.01$ and in the case c times 3, there would never exist a steady state in regime A as the system never crosses the threshold $\bar{Z}$.

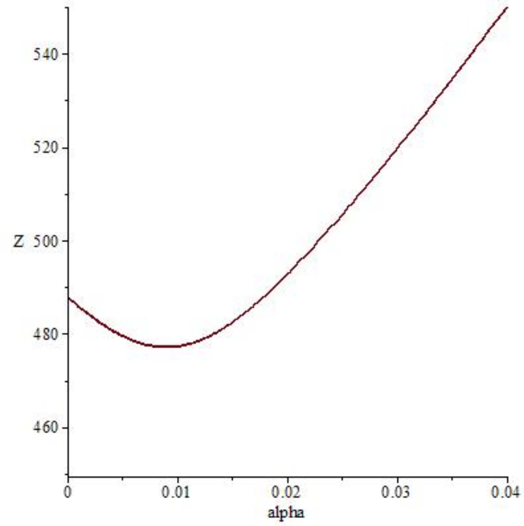

FIGURE 8: Phase portrait with k times 70

4.5. Policy interpretation

Through this original "injection" we can establish straightforwardly two important results regarding environmental policies and especially climate change programs.
Using Proposition 4.2 and Section 4.4.1 we can assert that :

Proposition 4.3 *In the presence of inertia a myopic optimal emission path starting at $Z_0 \leq \bar{Z}$ is sustainable if ρ is low enough*[8].

The contribution of Proposition 4.3 to climate policies can be questioned since climate science evidence tells us that we have already reached and crossed the degradation threshold $\bar{Z}$ (set here at 450 ppm). This proposition does however answer partly our guiding question : under these conditions, inertia does not matter.

and finally

Proposition 4.4 *In the presence of inertia a myopic optimal emission path starting at $Z_0 > \bar{Z}$ is sustainable if ρ is low enough and if the degradation coefficient k is low enough.*

Our tests show that as long as k is less than 70 times the rough approximation we provided, the MOEP will be sustainable. We can thus conservatively assert that beyond all the uncertainty characterizing carbons sinks degradation processes, it is likely that an optimal emission paths build on the benchmark model will nonetheless preserve a positive level of assimilative capacity and thus be sustainable in the sense we defined. Inertia does not matter here, unless the order of magnitude of k is very strongly underestimated

8. In our rough estimations ρ should be inferior to 0,0379722, which does not qualify as an unrealistic value.

5. Conclusion

In the light of the worrying evidence on the fast degradation of oceans' carbon sinks, we engaged in a process of complexifying the standard optimal pollution control model in order to reflect more accurately the dynamics of assimilative capacity. However our goal was not to build a more intricate model for the sake of it but to assess if this feature was a crucial one to take into consideration when it comes to actual implementation of climate economic policies.

Our numerical results tend to weigh against the use of more complex models since the benchmark optimal emission path is sustainable in the most realistic cases, as long as the discount rate is not too high. At this point we can thus answer that inertia does not matter as long as the discount rate is reasonable. If it is too high, the short sightedness of the benchmark model will belittle what takes place in the medium future, and this time dimension is precisely where inertia bites and can cause an extinction of the assimilative capacity which can lead to a catastrophic explosion of the pollution stock.

Both our analytical approach in Section 4 and our numerical simulations have underlined the crucial role of the discount rate in this issue of assimilative capacity preservation. As it is well known and much debated since the Stern Review (Stern, 2006), the choice of the discount rate prevails over the other economic or ecological parameters. The sustainability of the optimal path or of the climate policy will ultimately be determined by the discount rate.

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