A dynamic model of recycling with endogenous technological breakthrough*

(Preliminary draft – Please do not quote)

Gilles Lafforgue[†] and Luc Rouge[‡]

September 6, 2016

Abstract

We study an endogenous growth model in which the use of a non-renewable resource yields waste that can be recycled. The recycling activity only starts after the quality of recycled waste has reached a minimal threshold - which is not initially the case. The economy therefore has to invest in a dedicated R&D sector so as to improve this quality. We study the optimal trajectories of the economy; in particular, we analyze the discontinuity occurring at the date of the technological breakthrough, which is endogenous. We also discuss the environmental implications of the availability of the recycling technology. We show that the recycling option may have unexpected negative impacts on the environment, at least in the short run.

Keywords: Recycling; Non-renewable resource; Technical change; Growth.

JEL classifications: C61, O41, O44, Q32, Q53.

*These researches have received funding from the French Research Agency under grant agreement number ANR-13-ECOT-0005-04.

[†]University of Toulouse, Toulouse Business School. E-mail address: g.lafforgue@tbs-education.fr

[‡]University of Toulouse, Toulouse Business School. E-mail address: l.rouge@tbs-education.fr

1 Introduction

"Recycling is defined as any reprocessing of waste material in a production process that diverts it from the waste stream, except reuse as fuel. Both reprocessing as the same type of product, and for different purposes should be included. Recycling within industrial plants i.e. at the place of generation should be excluded." (United Nations[1]). A recycling activity presents two main benefits. First, by using waste as an input in the production process, it alleviates the scarcity of other resources. Second, it reduces the accumulated stock of waste, and, by doing so, reduces environmental damages.

Even if current levels of recycling activity greatly vary from sector to sector, recycling activity in the world is however low today. For instance, a UNEP report states that "many metal recycling rates are discouragingly low, and a "recycling society" appears no more than a distant hope" (UNEP, 2011). The main reasons behind this low-level activity are first that it remains comparatively expensive - i.e. non-recycled materials remain relatively cheap. Regarding municipal solid waste, for example, "in some cases the value of recyclables are less than the extra costs associated with collecting the disturbed waste" (World Bank, 2012). Second, the recycled materials are not always perfect substitutes to the virgin materials, which entails reduced marketing possibilities. If recycling pulp allows producing good-quality paper (ADEME, 2009), the recycling of carbon fiber reinforced polymer (CFRP) waste yields materials that cannot yet have the same industrial use as the virgin materials, particularly in advanced technology sectors such as aeronautics (Oliveux, 2015). This means that recycling technologies need to be improved.

The aim of the present paper is to understand how an economy will invest in research so that recycled waste can be used at a large scale. To do so, we consider an economy in which a recycling technology is available, but the current quality of recycled materials makes them non-usable by the production process. The only way to trigger the recycling activity is thus to improve the quality of these materials. This can be done by investing in a specific type of R&D. After a certain threshold quality level has been reached, a technological breakthrough occurs in the sense that the production of consumption goods starts using both the virgin (primary) resource and the recycled (secondary) waste. We characterize the optimal trajectories of the economy and their properties; in particular, we study the discontinuity occurring at the technological breakthrough. We also consider the

[1] United Nations – Environmental indicators: http://unstats.un.org/unsd/environment/wastetreatment.htm

impact of the recycling activity on the environment.

The economic literature has already considered the issue of recycling in growth contexts. Hoel (1978) analyzes the long-term path of an economy that consumes a non-renewable resource and a recycled resource. He shows how the impact of the use of these resources on the environment affects the optimal trajectories of the economy. Di Vita (2001) also studies, in an endogenous growth growth context, an economy that uses both a non-renewable resource and recycled materials, the use of which harming the environment by producing waste. The model endogenizes the degree of recyclability of the accumulated waste: investing in a dedicated R&D sector allows improving recyclability. Di Vita (2007) focuses on the degree of substitutability between the non-renewable resource and recycled waste in the production process and analyzes its impact on the economy's growth path and the time profile of resource extraction. Amigues et al. (2010) also use an endogenous growth model with non-renewable and recycled resources; they consider that the waste flow resulting from the use of these resources depends on the level of economic activity. They show how a market for waste and subsidies to resource extraction and recycling allows restoring the social optimum. In all these studies, the recycling technology is immediately available and used by the economy. Conversely, in the present paper, we consider that the recycling technology initially produces materials of poor quality, that cannot be used in the production process (i.e. at a large scale). Therefore, as in Di Vita (2001), we consider a sector of R&D devoted to the recycled resource, but here technological improvements are needed so that the recycled resource reaches a minimal quality threshold. When this quality level is attained, the secondary material can be used and the recycling activity starts.

The endogenous growth model we develop can be sketched as follows. The production of a consumption good requires two types of inputs: labor and a non-renewable resource, in its virgin or recycled form. The use of virgin (primary) resource yields waste flows that add to an overall stock. This waste can be recycled but the quality of the recycled waste is initially too low to allow its use within the production process. We consider that the economy can invest in a research sector dedicated to improving this quality; recycled waste starts being used as an input as soon as its quality meets a certain (exogenous) threshold. The date at which the recycling activity starts is therefore endogenous. The main trade-offs faced by the economy are the following: the intertemporal management of the stock of non-renewable resource, the intertemporal management of the stock of waste, the use of the virgin vs. recycled resource and the allocation of efforts between the three competing activities that are output production, R&D and recycling.

By using standard functional forms, we characterize the socially optimal trajectories of the economy and we study their properties. In particular, we analyze the discontinuity occurring at the date at which the quality of recycled waste meets the minimum standard and starts being used within the production process. We show for instance that resource extraction is less intensive and economic growth stronger once the technological breakthrough has occurred.

We then analyze the impact of the recycling activity on the environment. If the availability of the recycling option is unambiguously beneficial for the environment in the long term, it is detrimental in the short term. Indeed, it increases waste flows over the period during which the recycled resource has not yet reached the minimal quality threshold. Furthermore, since we consider that the recycled resources are non-renewable resources, i.e. derived from fossil fuels, the impact of the recycling activity is bad for climate since it increases greenhouse gas emissions in the short (and possibly long) term.

The general model is exposed in Section 2, and Section 3 presents the optimal program of the economy. Then, we characterize the socially optimal trajectories and we study their properties in Section 4. In Section 5, we analyze the environmental impacts of the recycling activity. Section 6 concludes.

2 The model

Final good sector

At each time t, a quantity $Y(t)$ of consumption good is produced from virgin resource, secondary material and labor according to the following technology:

$$Y(t) = \begin{cases} \tilde{f}(A_X(t)X(t), L_Y(t)), & A_R(t) < \bar{A}_R \\ f(A_X(t)X(t), A_R(t)R(t), L_Y(t)), & A_R(t) \geq \bar{A}_R \end{cases}$$

where $X(t)$, $R(t)$ and $L_Y(t)$ denote the quantity of virgin resource, recycled resource and labor that enter into the production process at time t, respectively. Each resource input – virgin and recycled – is characterized by an index of quality, respectively $A_X(t)$ and $A_R(t)$. Then, $A_X(t)X(t)$ and $A_R(t)R(t)$ must be viewed as augmented inputs. We assume that the recycled material can substitute the virgin resource only once its quality index has reached an exogenous minimal threshold $\bar{A}_R$. Last, we assume that production functions $\tilde{f}(\cdot)$ and $f(\cdot)$ have the standard properties (increasing, concave...).

R&D sectors

Each quality index can be improved through a dedicated R&D process. For simplicity and since the papers focuses on the recycling activity, we assume that R&D in the virgin resource quality is exogenous. Then, starting from the initial level $A_{X0} > 0$, the stock of knowledge $A_X(t)$ dedicated to this sector grows at a given constant rate $g_{A_X} > 0$. R&D in the recycled resource is endogenous and the dynamics of A_R is given by:

$$\dot{A}_R(t) = \delta L_A(t) A_R(t), \tag{1}$$

where $A_R(0) \equiv A_{R0}$ is given, $\delta > 0$ is a parameter of productivity and $L_A(t)$ denotes the quantity of labor (effort) invested in this R&D activity at time t. For the problem to be meaningful, we must assume that $0 < A_{R0} < \bar{A}_R$.

Resource extraction sector

The virgin material is obtained from a non-renewable resource according to a one-to-one technology: one unit of extracted resource yields one unit of virgin material. We assume that extraction cost is negligible. Denoting by $S(t)$ the stock of resource at time t, with $S(0) \equiv S_0$, we thus have:

$$\dot{S}(t) = -X(t). \tag{2}$$

Secondary material sector

A flow $R(t)$ of recycled resource can be obtained from a flow $Z(t)$ of waste and a flow $L_R(t)$ of labor according to the technology h: $R(t) = h\left(Z(t), L_R(t)\right)$. The production function $h(\cdot)$ is assumed to exhibit all the "good" properties (increasing, concave...).

Output dismantling and waste management sector

We do not take explicitly into account the useful life of the consumption good but we assume that its consumption and its dismantling occur instantaneously at the same time. When the final output is separated after consumption, which is considered as cost-free, only the primary physical inputs generate wastes. We assume that the waste rates of the virgin and recycled materials are, respectively, α and β, with $\alpha, \beta \in (0, 1)$. At any time, the incoming flow of wastes is then $\alpha X(t) + \beta Z(t)$. As a flow $Z(t)$ of waste is used by the recycling sector and then removed from the waste stock $W(t)$, we have:

$$\dot{W}(t) = \alpha X(t) - (1 - \beta)Z(t), \tag{3}$$

where the stock $W(0)$ inherited from the past is given and equal to W_0.

Labor supply

Labor supply L is assumed to be perfectly rigid and constant. At any point in time, the capacity constraint $L \geq L_A(t) + L_R(t) + L_Y(t)$ must be verified. As usual, this constraint will be binding along any optimal path.

Final consumers

The consumption of $C(t)$ units of final good generates an instantaneous utility $u(C(t))$. We assume that $u(\cdot)$ has the standard properties (increasing, concave, Inada conditions). Consumers discount their flow of instantaneous surplus at the social discount rate ρ, supposed to be positive and constant.

Conventional notations

In the following developments, we denote by f_x the partial derivative of any function $f(\cdot)$ with respect to variable x when this function contains more than one argument: $f_x \equiv \partial f(\cdot)/\partial x$. As usual, g_x characterizes the growth rate of variable x: $g_x(t) \equiv \dot{x}(t)/x(t)$. Last, for simplicity, we drop the time index when this causes no confusion.

3 The optimal program

The social planner program consists in determining the trajectories of X, Z, L_A, L_R and L_Y that maximize the discounted sum of utility flows subject to the set of constraints described in the previous section. However, the final output production function has two different expressions depending on whether A_R is larger or smaller than $\bar{A}_R$. Then we have to split the program into two parts.

T is the (endogenous) time at which A_R is equal to $\bar{A}_R$. As $g_{A_R} = \delta L_A \geq 0$, the trajectory of A_R is always non-decreasing. Henceforth if T exists, it is unique. We define respectively as $\mathcal{P}_1$ and $\mathcal{P}_2$ the social planner programs before and after time T, i.e. the date at which recycling starts. Following the standard optimal control methodology, we solve these two programs backward.

3.1 Solution of program $\mathcal{P}_2$

The optimal program along the second part of the planning horizon is:

$$(\mathcal{P}_2): \max_{\{X,Z,L_A,L_R,L_Y\}} \int_T^{\infty} u(C) e^{-\rho(t-T)} dt,$$

subject to the static constraints $C = f(A_X X, A_R R, L_Y)$, $R = h(Z, L_R)$ and $L = L_A + L_R + L_Y$, the dynamic constraints (**??**), (**??**) and (**??**), the initial condition $A_R(T) = \bar{A}_R$, and the following non-negativity constraints on the control variables:

$$X(t) \geq 0 \tag{4}$$

$$Z(t) \geq 0 \tag{5}$$

$$L_j(t) \in [0, L], \quad j = \{A, R, Y\}. \tag{6}$$

For the moment, we omit these non-negativity constraints which will be verified ex-post.

Denoting by λ_A, λ_S and λ_W the co-state variables associated with the state variables A_R, S and W respectively, an optimal interior solution must satisfy the following first-order conditions:

$$u'(C) f_X = \lambda_S - \alpha \lambda_W \tag{7}$$

$$u'(C) f_R h_Z = (1-\beta)\lambda_W \tag{8}$$

$$u'(C) f_{L_Y} = \delta A_R \lambda_A \tag{9}$$

$$u'(C) f_R h_{L_R} = \delta A_R \lambda_A \tag{10}$$

$$\dot{\lambda}_S = \rho \lambda_S \tag{11}$$

$$\dot{\lambda}_W = \rho \lambda_W \tag{12}$$

$$\dot{\lambda}_A = (\rho - \delta L_A)\lambda_A - u'(C) f_{A_R}. \tag{13}$$

The transversality conditions are:

$$\lim_{t \to \infty} e^{-\rho(t-T)} \lambda_\kappa(t) \kappa(t) = 0, \quad \kappa = \{A_R, S, W\}. \tag{14}$$

Conditions (**??**)-(**??**) state that the marginal social gain (in terms of utility) of one unit of input must be equal to its corresponding social marginal cost. More precisely, in (**??**), the marginal social gain of one unit of primary material equals the shadow value λ_S of the non-renewable resource, reduced by $\alpha\lambda_W$ to take into account that this unit generates wastes up to $\alpha\%$, which can be valuable through recycling. The same interpretation applies in (??) for the recycled resource, except that it does not involve the stock of natural resource but directly the stock of recycled resource. Last, equations (??) and (??) are static arbitrage conditions relative to the labor allocation. Their left-hand-sides read as the marginal social gain (in terms of utility) of increasing by one unit labor in production (cf. (??)) or in recycling activity (cf. (??)) while their common right-hand-side represents the marginal social cost (in terms of knowledge value) of these labor reallocation resulting from a diminution of the effort devoted to R&D.

Conditions (??) and (??) imply that $\lambda_S(t) = \lambda_S(T)e^{\rho(t-T)}$ and $\lambda_W(t) = \lambda_W(T)e^{\rho(t-T)}$. Consequently, and conditionally on the fact that both resource stocks have a positive value at time T (i.e. $\lambda_S(T) > 0$ and $\lambda_W(T) > 0$), the transversality conditions (??) associated with S and W reduce to $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} W(t) = 0$. The stock of natural resource and the stock of waste must be asymptotically exhausted: $S(T) = \int_T^\infty X(t)dt$ and $W(T) = \int_T^\infty [(1-\beta)Z(t) - \alpha X(t)]dt$.

Denoting by $\sigma(C)$ the inverse of the elasticity of intertemporal substitution, i.e. $\sigma(C) \equiv -Cu''(C)/u'(C)$, the growth rate of the marginal utility can be simply expressed as $-\sigma(C)g_C$. Log-differentiating (??)-(??) with respect to time and using (??)-(??), we obtain:

$$\rho + \sigma(C)g_C = \frac{\dot{f}_X}{f_X} \tag{15}$$

$$\rho + \sigma(C)g_C = \frac{\dot{f}_R}{f_R} + \frac{\dot{h}_Z}{h_Z} \tag{16}$$

$$\rho + \sigma(C)g_C = \frac{\dot{f}_{L_Y}}{f_{L_Y}} + \frac{\delta A_R f_{A_R}}{f_{L_Y}} \tag{17}$$

$$\rho + \sigma(C)g_C = \frac{\dot{f}_R}{f_R} + \frac{\dot{h}_{L_R}}{h_{L_R}} + \frac{\delta A_R f_{A_R}}{f_R h_{L_R}}. \tag{18}$$

However, taking into account the recycling specificity of our model, we must proceed slightly differently to obtain more informative dynamic conditions characterizing the intertemporal optimal arbitrages. Log-differentiating (??) and (??) with respect to time and using (??) and (??), we obtain a first condition as follows:

$$\rho + \sigma(C)g_C = \frac{d\left[f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z\right]/dt}{\left[f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z\right]}, \tag{19}$$

where the term $f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z$ is the total marginal productivity of one unit of extracted resource. This unit is used a first time under its virgin state, which increases production by f_X. However its generates $\alpha\%$ of wastes from which $(1-\beta)\%$ can be recycled. Then $\alpha/(1-\beta)$ reads as the recyclability rate of the virgin resource and the recycled resource generates an increase in production by $f_R h_Z$. Condition (**??**) is the Ramsey-Keynes condition in the specific context of our economy. The standard Ramsey-Keynes condition characterizes the socially optimal arbitrage made between consumption and capital accumulation. Here, the arbitrage is made between consumption and the use of the virgin resource. Assume, at date t, a marginal reduction of the production of consumption good through the diminution of resource use. At date $t + dt$, the economy uses this amount of resource whose total marginal productivity has increased while it was kept in situ. The extra amount of consumtion good accordingly produced, represented by the right handside of (**??**) must be the amount of consumption that compensates households from the loss of one unit of consumption at date t, represented by the left handside of (**??**). What is new here is that, as previously mentioned, the total marginal productivity of the virgin resource features the term $\left(\frac{\alpha}{1-\beta}\right) f_R h_Z$, which accounts for the fact that the waste Z induced by the use of the virgin resource is recycled and used as an input for consumption good in production.

The second condition is:

$$\frac{d\left[f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z\right]/dt}{\left[f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z\right]} = \frac{\dot{f}_{L_Y}}{f_{L_Y}} + \frac{\delta A_R f_{A_R}}{f_{L_Y}}. \tag{20}$$

This is an efficiency condition that characterizes the socially opimal intertemporal use of the virgin resource. The economic intuition behind this condition is the following. We first consider a given set of time profiles of all variables of the economy: we will refer to this benchmark as situation 1. We then assume a modification of the time profile of certain variables that keep the level of consumption good production unchanged at each date. We refer to this new set of time profiles as situation 2. We thus have $Y^1(t) = Y^2(t)$ or equivalently $C^1(t) = C^2(t)$ for all t. At date t, the economy reallocates one unit of labor from production to research: $L_Y^2(t) - L_Y^1(t) \equiv \Delta L_Y(t) = -\Delta L_{A_R}(t) = -1$; in other words, $L_Y^2(t) = L_Y^1(t) - 1$ and $L_{A_R}^2(t) = L_{A_R}^1(t) + 1$. In order to maintain the level of consumption good production, the economy increases its virgin resource use by $\Delta X(t)$: $X^2(t) = X^1(t) + \Delta X(t)$, where $\Delta X(t) = \frac{f_{L_Y}}{f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z}$, that is, the marginal productivity of labor in output production expressed in terms of resource.

At date $t+\epsilon$ (with $\epsilon \to 0$), the economy reallocates the unit of labor from research to production: $L_Y^2(t+\epsilon) = L_Y^1(t+\epsilon)$ and $L_{A_R}^2(t+\epsilon) = L_{A_R}^1(t\epsilon)$. Between t and $t+\epsilon$, the stock of knowledge A_R has increased due to the additional unit of labor in research within this interval and due to the (accordingly) increased level of knowledge A_R (through intertemporal knowledge spillovers). This allows the economy to maintain the level of output production ($Y^2(t+\epsilon) = Y^1(t+\epsilon)$) while saving a certain amount of virgin resource, $\Delta X(t+\epsilon)$. To express $\Delta X(t+\epsilon)$, one has to consider the extra amount of output produced through the increased effort in research: $d[\delta A_R f_{A_R}]/dt + \delta A_R f_{A_R}$, where δA_R is simply the marginal productivity of labor in research (see equation (**??**)). Expressed in terms of resource, this amount is: $\frac{d[\delta A_R f_{A_R}]/dt + \delta A_R f_{A_R}}{f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z}$. Then, by dividing this expression by the rate of growth of the global marginal productivity of the virgin resource, we take into account that the productivity of the resource evolves over the interval $(t; t+\epsilon)$. We thus have:

$$\Delta X(t+\epsilon) = \frac{f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z}{d\left[f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z\right]/dt} \times \frac{d[\delta A_R f_{A_R}]/dt + \delta A_R f_{A_R}}{f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z}.$$

This expression can be rewritten by eliminating the numerator of the first ratio and the denominator of the second and by taking into account the fact that $\delta A_R f_{A_R} = f_{L_Y}$, that is, the marginal productivity is the same in production and research. We obtain:

$$\Delta X(t+\epsilon) = \frac{\dot{f}_{L_Y} + \delta A_R f_{A_R}}{d\left[f_X + \left(\frac{\alpha}{1-\beta}\right) f_R h_Z\right]/dt}.$$

Condition (**??**) states that $\Delta X(t)$ and $\Delta X(t+\epsilon)$ must be equal at the social optimum; in other words, the labor transfer from output production to research does not allow to save virgin resource. This can be seen as a Hotelling condition in the context of a dynamic general equilibrium framework, though modified in two ways. First, there is no physical capital but knowledge (intellectual capital) accumulation, with intertemporal spillovers. Second, there is a recycling sector, that is, the virgin-resource use yields waste flows that can be used to produce the recycled resource.

3.2 Solution of program $\mathcal{P}_1$

In what follows, variables with an upper tilde refer to the optimal solution of program $\mathcal{P}_1$. Before time T, as the secondary material cannot be used as an input, we clearly must have $Z = L_R = 0$, which implies $R = 0$. Denoting by $V_2(A_R(T), S(T), W(T))$ the value function of program $\mathcal{P}_2$ at time T, we can write the initial program $\mathcal{P}_1$ as follows:

$$(\mathcal{P}_1): \max_{\{X, L_A, L_Y\}} \int_0^T u(C) e^{-\rho t} dt + e^{-\rho T} V_2(A_R(T), S(T), W(T)),$$

subject to the new static constraints $C = \tilde{f}(A_X X, L_Y)$ and $L = L_A + L_Y$, the dynamic constraints (**??**) and (**??**), the non-negativity constraints (**??**) and (**??**) and, since there is no recycling possibility before T, the following new law of motion of the waste stock:

$$\dot{W}(t) = \alpha X(t). \tag{21}$$

Comparing this last equation with (**??**), we note that the resource and waste stocks are linked through the following relation: $W(t) = W_0 + \alpha(S_0 - S(t))$, $\forall t \in [0, T)$.

The first-order conditions of $\mathcal{P}_1$ are very closed to those of $\mathcal{P}_2$. Conditions (**??**) and (**??**) are the same, except that the production function is now $\tilde{f}(\cdot)$. Conditions (**??**) and (**??**) are no longer valid since $Z = L_R = 0$ for $t < T$. Last, conditions (**??**) and (**??**) are preserved, whereas (**??**) becomes:

$$\dot{\tilde{\lambda}}_A = (\rho - \delta \tilde{L}_A) \tilde{\lambda}_A. \tag{22}$$

The transversality conditions are now:

$$\tilde{\lambda}_A(T) = \frac{\partial}{\partial A_R(T)} V_2(A_R(T), S(T), W(T)) \tag{23}$$

$$\tilde{\lambda}_S(T) = \frac{\partial}{\partial S(T)} V_2(A_R(T), S(T), W(T)) \tag{24}$$

$$\tilde{\lambda}_W(T) = \frac{\partial}{\partial W(T)} V_2(A_R(T), S(T), W(T)). \tag{25}$$

Last, the new intertemporal trade-off conditions write:

$$\rho + \sigma(\tilde{C}) g_{\tilde{C}} = \frac{\dot{\tilde{f}}_X}{\tilde{f}_X} = \frac{\dot{\tilde{f}}_{L_Y}}{\tilde{f}_{L_Y}}, \tag{26}$$

meaning that resource and labor productivity must grow at the same rate along the first branch of the optimal path.

4 Optimal trajectories

4.1 Analytical characterization of the optimal trajectories

Getting closed-form solutions requires to assign some specific functional forms to the technology and utility functions. We consider the following specifications:

$$\begin{aligned} u(C) &= \frac{C^{1-\sigma}}{1-\sigma} \\ h(Z, L_R) &= Z^{\phi} L_R^{1-\phi} \\ f(A_X X, A_R R, L_Y) &= (A_X X + A_R R)^{\epsilon} L_Y^{1-\epsilon} \\ \tilde{f}(A_X X, L_Y) &= (A_X X)^{\epsilon} L_Y^{1-\epsilon} \end{aligned}$$

The utility function is a standard CES form, with $\sigma = -cu''(c)/u'(c) > 0$ denoting the inverse of the intertemporal elasticity of substitution. The recycling technology is characterized by a CRS Cobb-Douglas function of parameter $\phi \in (0, 1)$. Last we consider a Cobb-Douglas function of parameter $\epsilon \in (0, 1)$ for the final good production and we assume that the two types of resources are inessential inputs (as taken in isolation) and perfect substitutes as long as the quality threshold has been reached by the recycled material.

The following proposition characterizes the optimal trajectories of the model associated with these specific analytical forms.

Proposition 1 *The optimal solution is characterized by the following trajectories:*

- *Virgin and recycled resource use:*

$$X(t) = \begin{cases} \tilde{k} S_0 e^{-\tilde{k}t} & , \quad t < T \\ k S_0 e^{(k-\tilde{k})T - kt} & , \quad t \geq T \end{cases} \tag{27}$$

$$Z(t) = \begin{cases} 0 & , \quad t < T \\ \frac{k(W_0 + \alpha S_0)}{(1-\beta)} e^{-k(t-T)} & , \quad t \geq T \end{cases} \tag{28}$$

where $\tilde{k} \equiv [\rho - \epsilon(1-\sigma) g_{A_X}]/\sigma$ *and* $k \equiv [\rho - \epsilon(1-\sigma) g_{A_X}]/[1 - \epsilon(1-\sigma)]$*;*

- *Virgin resource and waste stocks:*

$$S(t) = \begin{cases} S_0 e^{-\tilde{k}t} & , \quad t < T \\ S_0 e^{(k-\tilde{k})T - kt} & , \quad t \geq T \end{cases} \tag{29}$$

$$W(t) = \begin{cases} W_0 + \alpha S_0 (1 - e^{-\tilde{k}t}) & , \quad t < T \\ \left[W_0 + \alpha S_0 (1 - e^{-\tilde{k}T}) \right] e^{-k(t-T)} & , \quad t \geq T \end{cases} \tag{30}$$

- *Effort intensities in recycling, production and R&D:*

$$L_R(t) = \begin{cases} 0 & , \quad t < T \\ (1-\phi)k/\delta & , \quad t \geq T \end{cases} \tag{31}$$

$$L_Y(t) = \begin{cases} L_{Y0}e^{-\tilde{k}t} & , \quad t < T \\ L - g_{A_X}/\delta & , \quad t \geq T \end{cases} \tag{32}$$

$$L_A(t) = \begin{cases} L - L_{Y0}e^{-\tilde{k}t} & , \quad t < T \\ \left[g_{A_X} - (1-\phi)k\right]/\delta & , \quad t \geq T \end{cases} \tag{33}$$

  where the initial level of productive labor is defined as:

$$L_{Y0} \equiv \Phi e^{g_{A_X}T}, \text{ with } \Phi \equiv \left(\frac{1-\beta}{W_0 + \alpha S_0}\right)^{\phi} \left(\frac{\delta}{1-\phi}\right)^{1-\phi} \frac{\tilde{k}(1-\epsilon)S_0 A_{X0}}{\delta\epsilon\bar{A}_R}; \tag{34}$$

- *Consumption:*

$$C(t) = \begin{cases} C(0)e^{g_{\tilde{C}}t} & , \quad t < T \\ C(T)e^{g_C(t-T)} & , \quad t \geq T \end{cases} \tag{35}$$

  where the growth rates before and after time T are, respectively, $g_{\tilde{C}} = (\epsilon g_{A_X} - \rho)/\sigma$ and $g_C = \epsilon(g_{A_X} - \rho)/[1 - \epsilon(1-\sigma)]$;

- *R&D level in recycling:*

$$A_R(t) = \begin{cases} A_{R0} \exp\left[\delta L t - \frac{\delta L_{Y0}}{\tilde{k}}(1 - e^{-\tilde{k}t})\right] & , \quad t < T \\ \bar{A}_R e^{[g_{A_X} - (1-\phi)k](t-T)} & , \quad t \geq T \end{cases} \tag{36}$$

Last, the optimal switching time T is defined as the solution of the following equation:

$$\delta L T - \ln\left(\frac{\bar{A}_R}{A_{R0}}\right) = \frac{\delta\Phi}{\tilde{k}} e^{g_{A_X}T}\left(1 - e^{-\tilde{k}T}\right). \tag{37}$$

Proof: See the proof in Appendix A.1. ■

4.2 Existence conditions

The existence conditions can be summarized by inequalities contained in the following proposition.

Proposition 2 *An analytical solution of the optimal program exists if the parameters of the model satisfy the following conditions:*

$$\epsilon(1-\sigma)g_{A_X} \leq \rho \leq \frac{[1 - \epsilon\phi(1-\sigma)]}{(1-\phi)} g_{A_X} \tag{38}$$

$$g_{A_X} \leq \delta L \tag{39}$$

$$\Phi e^{g_{A_X}T} \leq L, \tag{40}$$

*where Φ is given by (**??**) and T^o is determined from (**??**).*

Proof: First, resource extraction X and waste recycling Z are always non-negative if both $\tilde{k}$ and k are non-negative. Next, labor L_Y dedicated to recycling is always non-negative for $k \geq 0$, and it is smaller then L for $(1-\phi)k \leq \delta L$. Labor L_Y dedicated to production is non-negative if i) $L_{Y0} \geq 0$, which is always true given (**??**), and ii) $\delta L \geq g_{A_X}$. It is smaller than L if $L_{Y0} \leq L$. This last condition implies that L_A is non-negative for $t < T$. R&D effort is also non-negative for $t \geq T$ if $(1-\phi)k \leq g_{A_X}$. Last, L_A is smaller than L if $(1-\phi)k \geq (g_{A_X} - \delta L)$, which is always satisfied provided that $\delta L \geq g_{A_X}$ and $k \geq 0$. Given all these existence conditions, the various parameters of the model must satisfy:

$$0 \leq (1-\phi)k \leq g_{A_X} \leq \delta L, \quad 0 \leq \tilde{k} \text{ and } L_{Y0} \leq L,$$

which, by developing the expression of k and $\tilde{k}$, yields the inequalities contained in Proposition **??**. ■

These existence conditions depend on the set of all exogenous parameters. Identifying specific inequality constraints for each one is almost impossible given the complexity of the model. However, these conditions have some significant implications. For instance, the social discount rate must take intermediate values as compared with the exogenous trend parameter of technical progress, which is related to our condition (**??**). Condition (**??**) says that the total amount of effort (i.e. labor) must be large enough and (**??**) states that the optimal switching date T must be early enough to justify a first phase of R&D investment in a sector which is not competitive yet.

4.3 Growth sustainability conditions

First, from their respective expressions (**??**) and (**??**) and from the existence condition (**??**), one can easily show that $g_C > g_{\tilde{C}}$. Then, consumption grows faster after time T once the recycled material becomes a substitute for the virgin resource. Next, the consumption growth rates along the two branches of the optimal path are either positive or negative depending on the level of the growth rate g_{A_X}, relative to the social discount rate. We then have three cases:

$$\begin{cases} \rho < \epsilon g_{A_X} & \Rightarrow \quad g_{\tilde{C}} > 0 \text{ and } g_C > 0 \\ \epsilon g_{A_X} \leq \rho < g_{A_X} & \Rightarrow \quad \tilde{g_C} \leq 0 \text{ and } g_C > 0 \\ g_{A_X} \leq \rho & \Rightarrow \quad g_{\tilde{C}} < 0 \text{ and } g_C \leq 0 \end{cases}$$

As usual, the larger the social discount rate, the weaker the consumption growth rate, with negative values below a given threshold. For intermediate values of ρ, the optimal growth path may be U-shaped: decreasing through time during the first phase without recycling, and then increasing once recycling starts. For simplicity, and to reduce the number of scenarios under study, we will focus on the first case where both g_C and $g_{\tilde{C}}$ are positive.

4.4 Qualitative dynamic properties

First of all, we note that a large part of the forthcoming analysis is based on the comparison between k and $\tilde{k}$. By using their respective expressions as given in Proposition **??**, we have $\tilde{k} - k = (1-\sigma)(1-\epsilon)k/\sigma > 0$.

Resource extraction is always decreasing through time. As $X(T) - \tilde{X}(T) = (k - \tilde{k})S_0 e^{-\tilde{k}T} < 0$ from (**??**), it jumps down at time T and then follows a less sloping declining path that converges towards zero, as illustrated in Figure 1-a. The resource stock is continuously declining until its full exhaustion. Its trajectory is less sloping after T (cf. Figure 1-b).

The flow of recycled material is nil before T. Then it jumps upwards and follows a declining trajectory that asymptotically converges towards zero, as depicted in Figure 1-c. The stock of waste resulting from the combination between resource use and recycling is first increasing before T and next declining until exhaustion (cf. Figure 1-d). Hence, in the long run, there is no more waste accumulation: the flows of waste are fully recycled.

As stated by (**??**), the optimal effort devoted to recycling is nil during the first phase (not depicted). Then it jumps upwards at time T and follows a constant level forever. Consequently, as labor supply is fixed, labor devoted to both production and R&D must jump downwards at the time the recycling technology becomes mature. Labor dedicated to production is declining through time and during the first phase, and then it can jump upwards or downwards, as illustrated in Figure 2. Labor dedicated to the R&D sector dedicated to the recycled material quality is first increasing before jumping, up or down, to its steady state level (cf. Figure 3). Three scenarios can thus occur depending on the nature of the jumps made by each of these two trajectories at time T: either L_Y jumps upwards and L_A jumps downwards (Scenario L1), or both L_Y and L_A jump downwards (Scenario L2), or L_y jumps downwards and L_A jumps upwards (Scenario L3).

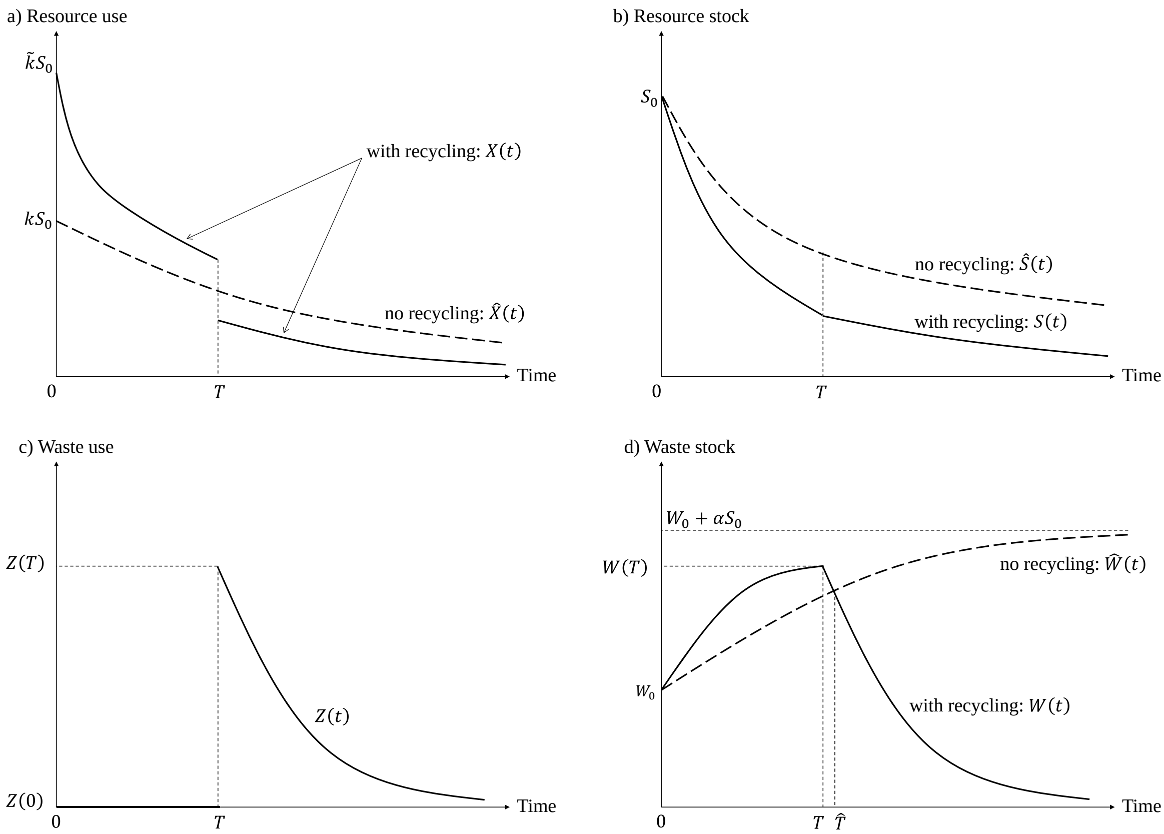

Figure 1: Resource extraction and waste recycling trajectories

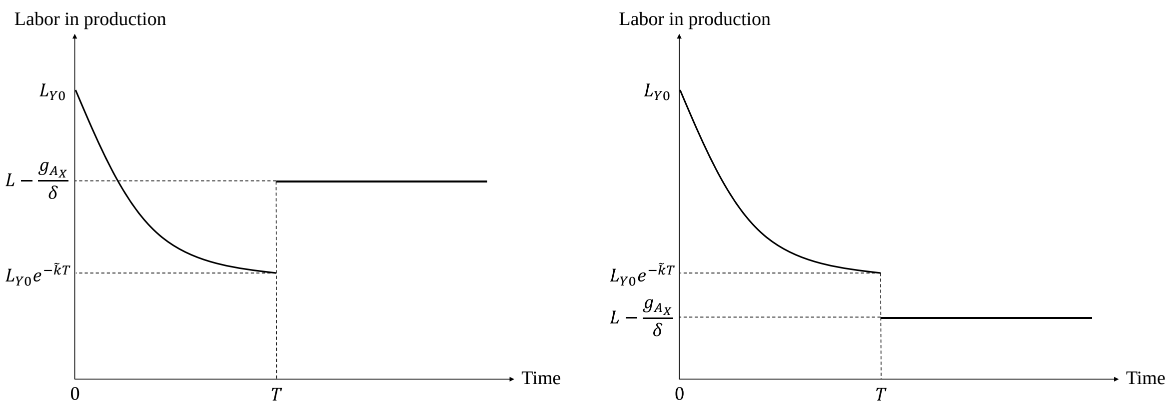

Figure 2: Labor devoted to the production sector

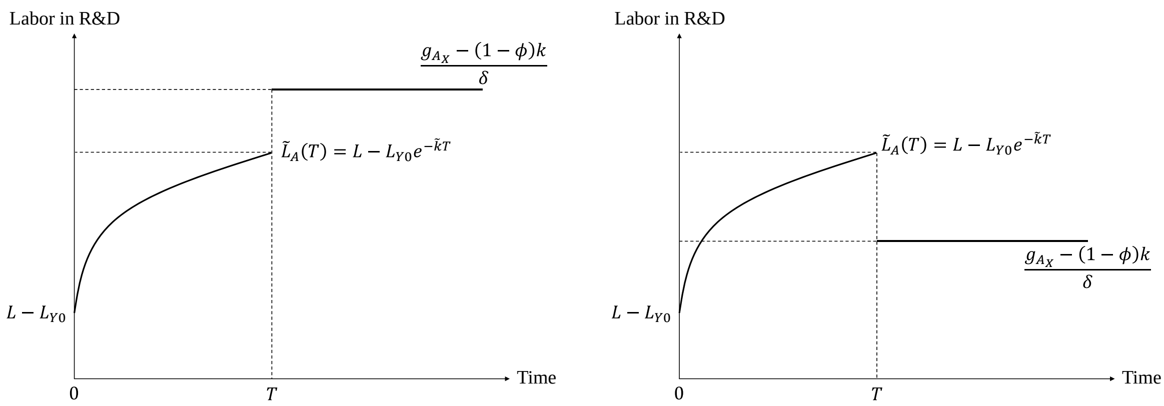

Figure 3: Labor devoted to R&D in the recycling technology

5 Environmental impact of the recycling option

We consider here two negative environmental impacts of the use of resources (recycled or not): first, the negative impact of the accumulated stock of waste $W(t)$; second, the impact of resource use in terms of greenhouse gas (hereafter GHG) emissions and subsequent accumulated stock. In order to analyze the environmental impact of the recycling option, we first characterize the trajectory of the economy if this option is not available.

5.1 The non-recycling economy

Here we study the optimal trajectory of an economy in which no recycling technology is available at each date t, hereafter referred to as "the non-recycling economy". This is obviously a particular case of the economy studied in the last sections, in which date T never occurs. This corresponds formally to program $(\mathcal{P}_1)$ when T tends towards infinity. In this case, we can easily show that the marginal social value of knowledge is always nil and that it is never optimal to invest in the R&D activity devoted to the quality of the recycled waste. Consequently, the total amount of available labor is allocated at any point in time to production and the set of control variables reduces to resource extraction only. We thus obtain a simple "cake-eating" problem where the final consumption good is produced according to the following technological form: $C(t) = L^{1-\epsilon}\left[A_{X0}e^{g_{A_X}t}X(t)\right]^{\epsilon}$. The optimal trajectories of such a program are:

$$\hat{X}(t) = \hat{X}_0 e^{-kt}, \quad \hat{X}_0 = kS_0 \tag{41}$$

$$\hat{C}(t) = \hat{C}_0 e^{g_C t}, \quad \hat{C}_0 = (A_{X0}kS_0)^{\epsilon} L^{1-\epsilon} \tag{42}$$

$$\hat{S}(t) = S_0 e^{-kt} \tag{43}$$

$$\hat{W}(t) = W_0 + \alpha S_0 \left(1 - e^{-kt}\right), \tag{44}$$

where $k \equiv [\rho - \epsilon(1-\sigma)g_{A_X}]/[1-\epsilon(1-\sigma)]$ and $g_C = \epsilon(g_{A_X} - \rho)/[1-\epsilon(1-\sigma)]$.

5.2 Impact of recycling on the stock of waste

As shown in the preceding section, in an economy where recycling is not possible and will not be in the future, the stock of waste $\hat{W}(t)$ grows over time and asymptotically tends to its upper limit level $W(0) + \alpha S(0)$. In other words, as the economy uses the non-renewable resource, the associated waste adds to the existing stock until the whole resource stock is exhausted.

Consider now that the recycling option is available. We first assume that it is available at date 0. In other words, the quality threshold at which the recycled resource starts to be used is instantaneously met: $T = 0$, and thus $A_R(0) = \bar{A}_R$. We will refer to this case as "the immediately-recycling economy". In this case, from (**??**), the expression of the stock of wastes would be $W(t) = W_0 e^{-kt}$ for any t, implying an exponentially decline from the initial stock W_0 down to 0. This means that the economy tends to a long-run situation in which the amount of waste produced is instantaneously re-used by the production process: there is no more waste accumulation. Comparing this last expression with $\hat{W}(t)$ as given by (**??**), the environmental impact of the immediate availability of a recycling technology is therefore unambiguously positive: the stock of waste is lower at each date $t > 0$ and progressively vanishes.

The impact of recycling on consumption when the technology is immediately available is more ambiguous. By comparing Equations (**??**) and (**??**)-post T, one can immediately see that the growth rate of consumption is identical in the non-recycling and the immediately-recycling economies. The initial level of consumption in the non-recycling economy is $\hat{C}_0 = (A_{X0}kS_0)^\epsilon L^{1-\epsilon}$, while this initial level in the immediately-recycling economy is $C(0) = [A_{X0}kS_0 + A_{R0}R(0)]^\epsilon (L - g_{A_X}/\delta)^{1-\epsilon}$. By comparing these two expressions, one can see that two effects – a "resource effect" and a "labor effect" – go in opposite directions. One the one hand, we have a positive resource effect that comes from the fact that recycling yields another input for the production of the consumption good: the recycled material R. For given levels of non-renewable resource use and labor, production is higher. On the other hand, the recycling activity and research in the quality of recycled materials reduce the flow of labor devoted to production: this is the negative labor effect. The overall effect of the immediate availability of a recycling technology on consumption thus depends on the relative magnitude of these two effects, which obviously depends on the output elasticity of (virgin and recycled) resources ϵ. The higher ϵ, the likelier is the resource effect to dominate the labor effect. It is easy to show that $C(0) > \hat{C}_0$ if and only if:

$$\epsilon > \frac{\ln\left(\frac{L}{L - g_{A_X}/\delta}\right)}{\ln\left(1 + \frac{A_{R0}R(0)}{A_{X0}X(0)}\right) + \ln\left(\frac{L}{L - g_{A_X}/\delta}\right)},$$

where $X(0) = kS_0$ and $R(0) = Z_0^\phi L_R^{1-\phi} = k\left(\frac{W_0 + \alpha S_0}{1-\beta}\right)^\phi \left(\frac{1-\phi}{\delta}\right)^{1-\phi}$. In other words, if ϵ is high enough, the recycling technology fosters consumption. Conversely, if ϵ is lower than this threshold, the labor effect dominates and recycling is therefore detrimental to consumption.

One can thus conclude that if the productivity of the resources is high (as compared to labor), the immediate availability of a recycling technology has both a positive impact on the environment and on consumption. However, if the productivity of resources is relatively low (as compared to labor), the positive impact of immediate recycling on the environment comes at the expense of consumption: the immediately-recycling economy faces a consumption-environment trade-off.

We assume now more generally that the recycled resource is used only when its quality level $A_R(t)$ reaches the threshold level $\bar{A}_R$ at date T. This is the general case presented in Sections 2-4. We will simply refer to this case as "the recycling economy". Equation (**??**) presents the trajectory of the stock of waste in this case. This trajectory and the trajectory of the stock of waste $\hat{W}(t)$ in the non-recycling economy are depicted in Figure 1-d. Before date T, $W(t)$ increases and is higher than $\hat{W}(t)$ at each date. At date T, the stock reaches its maximum level $W(0) + \alpha S(0)(1 - e^{-\tilde{k}T})$ and then starts to steadily decline, to asymptotically converge towards 0. In the long-run, the recycling economy thus also re-uses the whole amount of waste that it yields so that there is no more waste accumulation. Conversely, the non-recycling economy keeps accumulating waste after date T, and $\hat{W}(t)$ gets higher than $W(t)$ at date $\hat{T} > T$. In other words, the recycling option is bad for the environment until date $\hat{T}$. After this date, the environmental benefit becomes positive.

Regardless of its date of occurrence, the maximum level reached by a stock of pollutant is a serious concern for many - since irreversible damages may occur after certain thresholds. Here, we show that it is reached earlier by the recycling economy; however, this maximum level $W(T)$ is lower in the recycling economy than in the non-recycling economy. Indeed, the difference between the two is equal to $\left[W(0) + \alpha S(0)(1 - e^{-\tilde{k}T})\right] - [W(0) + \alpha S(0)]$, which is negative since $e^{-\tilde{k}T} > 0$.

5.3 Impact of recycling on climate

As is well known, the use of non-renewable resources in the production process yields GHG emissions. If we consider that emissions at date t are proportional to resource use at the same date, and that there is no decay in the GHG stock (for simplicity), the accumulated stock of GHG is the sum of the flows of resource use over the interval $[0; +\infty[$ multiplied by a constant. For this reason, in what follows, we will approximate stock of GHG at any date t by the sum of all resource flows from date 0 to t.

Using virgin (primary) material to produce output does not yield the same amount of GHG than using recycled materials. "Producing new products with secondary materials can save significant energy. For example, producing aluminum from recycled aluminum requires 95% less energy than producing it from virgin materials." (World Bank, 2012). Suppose thus that only using the virgin resource $X(t)$ yields GHG: here, the recycled resource $Z(t)$ does not pollute. Figure 1-a shows that the availability of a recycling technology accelerates GHG emissions in the sense that emissions before date T are higher and they are lower after this date. A well-known result of the climate-change literature is that emissions should be postponed (Withagen, 1994). Thus, while alleviating the burden of resource scarcity, the recycling technology has a negative impact on climate.

As concerns the stock of GHG itself, before date T, it can approximated by the stock of waste $W(t)$; Figure 1-d makes clear that this stock is unambiguously higher in the recycling economy. After date T, the stock of GHG increases in both the recycling and the non-recycling economies; in the non-recycling economy this stock may become higher at a certain date after T if virgin resource use is sufficiently higher than in the recycling economy. This means that recycling may have a positive impact on climate in the sense that it reduces the stock of GHG, but only in the long run.

Suppose now that both resource uses (X and Z) yield GHG emissions. Here, the negative impact of recycling on climate is reinforced. Before date T, the preceding conclusions apply. After this date, the emissions of the recycling economy are caused by the combined effect of both resource uses ($(X(t) + Z(t))$), while in the non-recycling economy they stem from the non-renewable resource use $\hat{X}(t)$ only. This means that, depending on the level of $Z(t)$, emissions may be higher in the recycling economy. The possibility that recycling induces a lower stock of GHG in the long-run is thus lower.

References

[1] ADEME (2009). Matières premières et recyclage, Bilan du recyclage 1998-2007 – Synthèse. *Agence de l'Environnement et de la Maîtrise de l'Energie.*

[2] Ayres R. (1998). Eco-thermodynamics: Economics and the second law. *Ecological Economics*, 26, 189-209.

[3] Ayres R. (1999). The second law, the fourth law, recycling and limits to growth. *Ecological Economics*, 29, 473-483.

[4] De Beir J., Fodha M., Girmens G. (2007). Recyclage et externalités environnementales: Faut-il subventionner les activités de récupération recyclage? *Revue Economique*, 58(3), 609-618.

[5] Di Vita G. (2001). Technological change, growth and waste recycling. *Energy Economics*, 23, 549-567.

[6] Di Vita G. (2007). Exhaustible resources and secondary materials: A macroeconomic analysis. *Ecological Economics*, 63, 138-148.

[7] Fullerton D., Kinnaman T. (1995). Garbage, recycling, and illicit burning or dumping. *Journal of Environmental Economics and Management*, 29, 78-91.

[8] Fullerton D., Wu W. (1998). Policies for green design. *Journal of Environmental Economics and Management*, 36, 131-148.

[9] Hoel M. (1978). Resource extraction and recycling with environmental costs. *Journal of Environmental Economics and Management*, 5, 220-235.

[10] Oliveux G., Dandy L.O., Leeke G.A. (2015). Current status of recycling of fibre reinforced polymers: Review of technologies, reuse and resulting properties. *Progress in Materials Science*, 72, 61-99.

[11] Pittel K., Amigues J-P., Kuhn T. (2010). Recycling under a material balance constraint. *Resource and Energy Economics*, 32, 379-394.

[12] Romer P. (1990). Endogenous Technological Change. *Journal of Political Economy*, 98, 71-102.

[13] Withagen C.A. (1994). Pollution and Exhaustibility of Fossil Fuels. *Resource and Energy Economics*, 16, 235-242.

[14] UNEP (2011). Recycling rate of metals: A status report. UNEP.

[15] World Bank (2012). What a Waste: A Global Review of Solid Waste Management, *The World Bank*.

APPENDIX

A.1 Proof of Proposition 1

Analytical solution of program $\mathcal{P}_2$

In case where both the virgin and the recycled resources are used simultaneously, i.e. for $t \geq T$, we define as $M \equiv A_X X + A_R R$ the instantaneous quantity of materials (augmented inputs) used to produce the final consumption good so that $Y = M^{\epsilon} L_Y^{1-\epsilon}$. Given our specific analytical forms, we can compute the following derivatives:

$$\begin{aligned}
h_Z = \phi R/Z &\;\Rightarrow\; \dot{h}_Z/h_Z = g_R - g_Z \\
h_{L_R} = (1-\phi)R/L_R &\;\Rightarrow\; \dot{h}_{L_R}/h_{L_R} = g_R - g_{L_R} \\
f_X = \epsilon A_X Y/M &\;\Rightarrow\; \dot{f}_X/f_X = g_{A_X} + g_Y - g_M \\
f_R = \epsilon A_R Y/M &\;\Rightarrow\; \dot{f}_R/f_R = g_{A_R} + g_Y - g_M \\
f_{L_Y} = (1-\epsilon)Y/L_Y &\;\Rightarrow\; \dot{f}_{L_Y}/f_{L_Y} = g_Y - g_{L_Y},
\end{aligned}$$

where $g_Y = \epsilon g_M + (1-\epsilon) g_{L_Y} = g_C$ as $Y = C$, and $g_R = \phi g_Z + (1-\phi) g_{L_R}$. The optimal dynamic system (**??**)-(**??**) can thus be rewritten as:

$$g_M - g_{A_X} = (1-\sigma)g_C - \rho \tag{A.1}$$

$$g_M - g_{A_R} + (1-\phi)(g_Z - g_{L_R}) = (1-\sigma)g_C - \rho \tag{A.2}$$

$$g_{L_Y} - \frac{\delta\epsilon}{(1-\epsilon)}\left(\frac{A_R R L_Y}{M}\right) = (1-\sigma)g_C - \rho \tag{A.3}$$

$$g_M - g_{A_R} - \phi(g_Z - g_{L_R}) - \frac{\delta L_R}{(1-\phi)} = (1-\sigma)g_C - \rho\,. \tag{A.4}$$

Moreover, equations (**??**) and (**??**) imply:

$$\left(\frac{L_R}{L_Y}\right)\left(\frac{M}{A_R R}\right) = \frac{\epsilon(1-\phi)}{(1-\epsilon)} \quad\Rightarrow\quad g_M - g_{A_R} - \phi(g_Z - g_{L_R}) - g_{L_Y} = 0\,. \tag{A.5}$$

Combining (**??**) with (**??**) and (**??**)with (**??**) yields:

$$(1-\phi)(g_Z - g_{L_R}) = g_{A_R} - g_{A_X} = -\delta L_R\,. \tag{A.6}$$

Since $g_{A_R} = \delta L_A$, equation (**??**) implies that $\delta(L_A + L_R) = g_{A_X}$ and, from $L = L_A + L_R + L_Y$, we obtain $L_Y = L - g_{A_X}/\delta$. As L_Y is proved to be constant, $g_{L_Y} = 0$ and $g_C = \epsilon g_M$. From (**??**), this implies:

$$g_C = \frac{\epsilon(g_{A_X} - \rho)}{[1-\epsilon(1-\sigma)]}\,. \tag{A.7}$$

Next, we obtain L_R from (**??**) and (**??**), and L_A from the labor allotment condition: $L_R = (1-\phi)[\rho-(1-\sigma)g_C]/\delta$ and $L_A = L - L_R - L_Y$. Since g_C and L_Y are constant, L_A and L_R are constant too: $g_{L_A} = g_{L_R} = 0$.

From (**??**), we have $g_Z = -\delta L_R/(1-\phi)$, which is non-positive provided that L_R is non-negative. To determine g_X, let go back to equation (**??**), which can be equivalently expressed as follows:

$$\frac{A_X X}{A_R R} = \frac{\epsilon(1-\phi)}{(1-\epsilon)}\left(\frac{L_Y}{L_R}\right) - 1\,.$$

As both L_R and L_Y are constant, log-differentiating this expression through time yields $g_X = g_{A_R} - g_{A_X} + g_R = g_{A_R} - g_{A_X} + \phi g_Z$. Using the first equality in (**??**), this last expression simply reduces to $g_X = g_Z$. Last, we deduce the values of X and Z at time T as functions of $S(T)$ and $W(T)$ by solving the differential equation system (**??**)-(**??**). The following lemma summarizes the expressions of the optimal trajectories of program $\mathcal{P}_2$.

Lemma 1 *For $t \in [T,\infty)$, the optimal trajectories of the model are:*

$$\begin{aligned}
X(t) &= X(T)e^{-k(t-T)}; \quad X(T) = kS(T)\\
Z(t) &= Z(T)e^{-k(t-T)}; \quad Z(T) = \frac{k[W(T)+\alpha S(T)]}{(1-\beta)}\\
L_R(t) &= L_R = \frac{(1-\phi)k}{\delta}\\
L_Y(t) &= L_Y = L - \frac{g_{A_X}}{\delta}\\
L_A(t) &= L_A = \frac{g_{A_X}-(1-\phi)k}{\delta}\\
S(t) &= S(T)e^{-k(t-T)}\\
W(t) &= W(T)e^{-k(t-T)}\\
A_R(t) &= A_R(T)e^{\delta L_A(t-T)}, A_R(T) = \bar{A}_R\,,
\end{aligned}$$

where $k \equiv [\rho - \epsilon(1-\sigma)g_{A_X}]/[1-\epsilon(1-\sigma)]$.

Determination of the value function V_2

As the optimal consumption trajectory is characterized by $C(t) = C(T)e^{g_C(t-T)}$ for $t \geq T$ and given that $k \equiv [\rho-(1-\sigma)g_C] > 0$, the optimal value of program $\mathcal{P}_2$ can be simply expressed as:

$$\begin{aligned}
V_2(A_R(T),S(T),W(T)) &= \int_T^\infty u(C)e^{-\rho(t-T)}dt\\
&= \frac{C(T)^{1-\sigma}}{(1-\sigma)}\int_T^\infty e^{-k(t-T)}dt = \frac{C(T)^{1-\sigma}}{(1-\sigma)k}\,, \qquad \text{(A.8)}
\end{aligned}$$

where $C(T) = \left[A_X(T)X(T) + A_R(T)Z(T)^\phi L_R^{1-\phi}\right]^\epsilon L_Y^{1-\epsilon}$ and $A_R(T) = \bar{A}_R$. We can then compute the following derivatives:

$$\frac{\partial V_2}{\partial A_R(T)} = \frac{\epsilon R(T)}{kM(T)} C(T)^{1-\sigma} \tag{A.9}$$

$$\frac{\partial V_2}{\partial S(T)} = \left[A_X(T) + \frac{\alpha\phi\bar{A}_R}{(1-\beta)}\frac{R(T)}{Z(T)}\right]\frac{\epsilon}{M(T)} C(T)^{1-\sigma} \tag{A.10}$$

$$\frac{\partial V_2}{\partial W(T)} = \frac{\epsilon\phi\bar{A}_R}{(1-\beta)}\frac{R(T)}{Z(T)M(T)} C(T)^{1-\sigma}. \tag{A.11}$$

Analytical solution of program $\mathcal{P}_1$

As long as $t < T$, consumption is obtained from $\tilde{C} = \tilde{f}(A_X X, L_Y) = (A_X X)^\epsilon L_Y^{1-\epsilon}$. Then we can write: $\dot{\tilde{f}}_X/\tilde{f}_X = g_{\tilde{C}} - g_{\tilde{X}}$ and $\dot{\tilde{f}}_{L_Y}/\tilde{f}_{L_Y} = g_{\tilde{C}} - g_{\tilde{L_Y}}$. The intertemporal trade-off conditions (**??**) directly imply that $g_{\tilde{X}} = g_{\tilde{L_Y}} = (1-\sigma)g_{\tilde{C}} - \rho$. Since $g_{\tilde{C}} = \epsilon(g_{A_X} + g_{\tilde{X}}) + (1-\epsilon)g_{\tilde{L_Y}}$, we can deduce the optimal growth rate of consumption for $t \in [0, T)$:

$$g_{\tilde{C}} = \frac{\epsilon g_{A_X} - \rho}{\sigma}. \tag{A.12}$$

The next lemma characterizes the optimal trajectories that solve program $\mathcal{P}_1$.

Lemma 2 *For $t \in [0, T)$, the optimal trajectories of the model are:*

$$\begin{aligned}
\tilde{X}(t) &= \tilde{k}S_0 e^{-\tilde{k}t}; \quad \tilde{Z}(t) = 0 \\
\tilde{L}_Y(t) &= \tilde{L}_Y(0)e^{-\tilde{k}t}; \quad \tilde{L}_A(t) = L - \tilde{L}_Y(t); \quad \tilde{L}_R(t) = 0 \\
\tilde{S}(t) &= S_0 e^{-\tilde{k}t} \\
\tilde{W}(t) &= W_0 + \alpha S_0(1 - e^{-\tilde{k}t}) \\
\tilde{A}_R(t) &= A_{R0}\exp\left[\delta L t - \frac{\delta\tilde{L}_Y(0)}{\tilde{k}}(1 - e^{-\tilde{k}t})\right],
\end{aligned}$$

where $\tilde{k} \equiv [\rho - \epsilon(1-\sigma)g_{A_X}]/\sigma$. Variables $\tilde{L}_Y(0)$ and T are endogenous and must be determined from the set of continuity and transversality conditions.

The continuity conditions on the state variables at time T are:

$$\tilde{S}(T) = S(T) \quad \Leftrightarrow \quad S_0 e^{-\tilde{k}T} = S(T) \tag{A.13}$$

$$\tilde{W}(T) = W(T) \quad \Leftrightarrow \quad W_0 + \alpha S_0(1 - e^{-\tilde{k}T}) = W(T) \tag{A.14}$$

$$\tilde{A}_R(T) = \bar{A}_R \quad \Leftrightarrow \quad \delta L T - \frac{\delta\tilde{L}_Y(0)}{\tilde{k}}\left(1 - e^{-\tilde{k}T}\right) = \ln\left(\frac{\bar{A}_R}{A_{R0}}\right). \tag{A.15}$$

To investigate the transversality conditions, we need first to identify the optimal trajectories of the co-state variables. Solving the differential equations (??), (??) and (??) for $t \in [0, T)$ results respectively, in:

$$\tilde{\lambda}_S(t) = \tilde{\lambda}_S(0)e^{\rho t} \quad \text{(A.16)}$$

$$\tilde{\lambda}_W(t) = \tilde{\lambda}_W(0)e^{\rho t} \quad \text{(A.17)}$$

$$\tilde{\lambda}_A(t) = \frac{\tilde{\lambda}_A(0)A_{R0}}{\tilde{A}_R(t)}e^{\rho t}, \quad \text{(A.18)}$$

where $\tilde{\lambda}_S(0)$, $\tilde{\lambda}_W(0)$ and $\tilde{\lambda}_A(0)$ are endogenous variables that must satisfy the first-order conditions (??) and (??) at time $t = 0$:

$$\tilde{\lambda}_S(0) - \alpha\tilde{\lambda}_W(0) = \frac{\epsilon\tilde{C}(0)^{1-\sigma}}{\tilde{k}S_0} \quad \text{(A.19)}$$

$$\tilde{\lambda}_A(0) = \frac{(1-\epsilon)\tilde{C}(0)^{1-\sigma}}{\delta A_{R0}\tilde{L}_Y(0)}. \quad \text{(A.20)}$$

Using (??)-(??) and (??)-(??), the transversality conditions (??)-(??) can be rewritten as follows:

$$\frac{\tilde{\lambda}_A(0)A_{R0}}{\bar{A}_R}e^{\rho T} = \frac{\epsilon R(T)}{kM(T)}C(T)^{1-\sigma} \quad \text{(A.21)}$$

$$\tilde{\lambda}_S(0)e^{\rho T} = \left[A_X(T) + \frac{\alpha\phi\bar{A}_R}{(1-\beta)}\frac{R(T)}{Z(T)}\right]\frac{\epsilon}{M(T)}C(T)^{1-\sigma} \quad \text{(A.22)}$$

$$\tilde{\lambda}_W(0)e^{\rho T} = \frac{1}{\alpha}\left[\tilde{\lambda}_S(0)e^{\rho T} - \frac{\epsilon A_X(T)}{M(T)}C(T)^{1-\sigma}\right], \quad \text{(A.23)}$$

where $A_X(T) = A_{X0}e^{g_{A_X}T}$ and given the continuity conditions (??)-(??) at time T. Together with the initial first-order conditions (??), (??), these three last equations form a system of five equations with the five variables $\tilde{\lambda}_A(0)$, $\tilde{\lambda}_S(0)$, $\tilde{\lambda}_W(0)$, $\tilde{L}_Y(0)$ and T. To entirely characterize the optimal trajectories, only the determination of $\tilde{L}_Y(0)$ and T is relevant.

Replacing $\left[\tilde{\lambda}_S(0) - \alpha\tilde{\lambda}_W(0)\right]$ in (??) by its expression (??) allows to write:

$$M(T)\left[\frac{\tilde{C}(0)}{C(T)}\right]^{1-\sigma} = \tilde{k}S_0A_X(T)e^{-\rho T}. \quad \text{(A.24)}$$

Next, using (??) and (??), the transversality condition (??) implies:

$$\tilde{L}_Y(0) = \frac{(1-\epsilon)k\tilde{k}S_0A_X(T)}{\delta\epsilon\bar{A}_R R(T)},$$

where $A_X(T) = A_{X0}e^{g_{A_X}T}$ and $R(T) = Z(T)^{\phi}L_R^{1-\phi}$. From the continuity conditions (??) and (??), and given that $Z(T) = k[W(T) + \alpha S(T)]/(1-\beta)$, we have $Z(T) = k(W_0 +$

$\alpha S_0)/(1-\beta)$. Since $L_R = (1-\phi)k/\delta$ (cf. Lemma 1), we thus get the following expression of $\tilde{L}_Y(0)$ as a function of T:

$$\tilde{L}_Y(0) = \left(\frac{1-\beta}{W_0 + \alpha S_0}\right)^{\phi} \left(\frac{\delta}{1-\phi}\right)^{1-\phi} \frac{\tilde{k}(1-\epsilon)S_0 A_{X0} e^{g_{A_X} T}}{\delta \epsilon \bar{A}_R} \,. \tag{A.25}$$

Finally, replacing $\tilde{L}_Y(0)$ in (**??**) by its expression (**??**), we get an equation with the single variable T:

$$\delta L T - \left(\frac{1-\beta}{W_0 + \alpha S_0}\right)^{\phi} \left(\frac{\delta}{1-\phi}\right)^{1-\phi} \frac{(1-\epsilon)S_0 A_{X0} e^{g_{A_X} T}}{\epsilon \bar{A}_R} \left(1 - e^{-\tilde{k}T}\right) = \ln\left(\frac{\bar{A}_R}{A_{R0}}\right) .$$