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The Social Cost of Carbon and the Ramsey Rule.

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Abstract

The objective of this paper is to critically assess the use of simple rules for the social cost of carbon (SCC) that employ a rudimentary form of the Ramsey Rule. Two interrelated caveats apply. First, if climate change poses a serious problem, it is hard to justify an exogenous constant growth rate of consumption and *GDP*, as is done in several contributions by prominent scholars. Second, to derive the optimal SCC one needs full knowledge of the entire future, in spite of the use of popular ways to try to get around this. Moreover, it is shown that some simple rules suffer from inconsistencies in their derivation.

Keywords: Social Cost of Carbon; Ramsey Rule.

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1. Introduction

In a rudimentary form the Ramsey Rule reads $r = \rho + \eta g$. Here, r is the discount rate, ρ is the pure rate of time preference, η is the elasticity of marginal utility, and g is the (per capita) consumption growth rate. The rule was derived by Ramsey (1928) as a necessary condition for optimality in his seminal model of economic growth.[2] The Ramsey Rule is often used to calculate the so-called Social Cost of Carbon (SCC), defined as the (monetized) current and future damages from emitting a marginal unit of CO2 into the atmosphere. The reason for invoking the Ramsey Rule for this purpose is that future damages need to be discounted, and the discount rate r is typically a candidate to do so. Oftentimes the variables in the Rule are taken as exogenous constant parameters, which makes the discount rate a constant, which is easy to work with. The policy relevance of the SCC lies in the fact that it can be interpreted as the Pigouvian tax to be imposed on CO2 emissions. Since the level of the carbon tax and its development over time is a central issue in the debate on how to fight climate change, the importance of the Ramsey Rule is evident. This is confirmed by a study by Tol (2008) who surveys 211 estimates of the SCC, developed over the period 1982-2006, and who argues that the vast majority of these estimates is based on the Ramsey Rule. Drupp et al. (2018) in their survey among experts of the economics profession mention "….the prominence of the simple Ramsey Rule in public policy….".[3]

Each component of the Ramsey Rule has been subject to studies, critiques, and debates. A clear example is the pure rate of time preference, which according to Ramsey himself should be zero for a social planner. Many agree with Ramsey. Also IPCC (2014) claims that there is "a broad consensus for a zero or near-zero pure rate of time preference". However, Drupp et al. (2018) find from their survey that the mean is 1.1%. Groom and Maddison (2018) look into the elasticity of marginal utility for the UK and find a central estimate of 1.5. But they also mention studies that yield other estimates, ranging from 0.5 to 10. Gollier (2012) considers the case of uncertainty with regard to the consumption growth rate and shows how the discount rate should be corrected for this, resulting in a lower (risk-free) constant rate for a

[2] An alternative interpretation is that it is an equilibrium condition in a market economy, where consumers maximize their dynamic welfare subject to the intertemporal budget constraint and producers equalize marginal product of capital to the interest rate. It can be argued that in the absence of externalities the market economy is Pareto-efficient and minimizes expenditures to reach a certain welfare level.

[3] A fuller quote will be given at the end of this paper.

prudent planner (i.e., with a positive third derivative of the utility function). In these studies the Ramsey Rule *as such* is still assumed to be helpful in determining the discount rate.

Of course, there are other ways to determine the appropriate social discount rate. Weitzman (1998 and 2001), Gollier (2004), Gollier and Weitzman (2010) and Newell and Pizer (2003) all consider uncertainty in the effective future discount rate. Weitzman (2001) and Drupp et al. (2018) are examples of studies where experts are asked for their view on the social discount rate, and the constituent parts of the Ramsey Rule, respectively. So, in the latter the Ramsey Rule plays a role, in the former it does not. Experts appear as well in a proposal made by Pindyck (2016) based on his critical evaluation of Integrated Assessment Models. So does an expert panel in Arrow et al. (2012, 2013) who advocate a declining discount rate without referring to the Ramsey Rule. Finally, there are studies where the SCC is endogenously derived from an Integrated Assessment Model. See e.g., Tsur and Zemel (2008) and Nordhaus and Sztorc (2013). Nevertheless, although it is to be acknowledged that there are valuable alternatives to applying the Ramsey Rule, in economic practice many empirical studies are performed to calculate the 'right' interest rate as well as the SCC, based on just postulating a growth rate and an elasticity of marginal utility (see Werkgroep Discontovoet (2015) and Centraal Planbureau (2015) for the Netherlands, Cropper (2012) for the US, and the Committee for an Official Shadow Price of Carbon (2018) for France).

It is the purpose of the present contribution to investigate when the use of the Ramsey Rule is justified and when not, with a focus on climate change policy. There are two interrelated problems with this application.

First of all, if, as is broadly agreed upon, climate change poses a serious problem, then the use of the Ramsey Rule to derive the appropriate interest rate may lead to errors, because as a consequence of the required non-marginal climate change projects, the growth rate of consumption is likely to change: Climate change policy is a non-marginal phenomenon. Dietz and Hepburn (2013) provide a nice overview of the literature, showing that the economics profession has long been aware of the consequences of large projects for cost-benefit analysis (see also Dasgupta et al. (1972) and Starrett (1988)). Dietz and Hepburn present a convincing example where valuing the reduction of global carbon emissions as if it were a marginal project, leads to serious errors. Referring to the study by Drupp et al. once more, their results show a large variation in the projected per capita growth rates, the elasticities of marginal utility of consumption and in the rates of pure time preference, although there is some agreement among the experts regarding the social discount rate to be used (over 75% find a discount rate of 2% acceptable).

Second, another objective is to show that the by now popular custom of deriving ‘simple’ rules for the SCC (see Golosov et al. (2014), Van den Bijgaart et al. (2016), Rezai and Van der Ploeg (2016)), Li (2019) and Dietz and Venmans (2019)) is hard to justify.[4] In an optimal decentralized economy, the social cost of carbon can only be derived with full knowledge of the complete optimal future path of the relevant variables, including the interest rate, which is endogenous in optimal growth models. This has been argued before by Smulders (2012) in a general context where he argues that to know shadow prices it is necessary to know the future development of the economy. Moreover, the assumptions made to arrive at the simple rule, such as absence of direct climate damages in social welfare, constant growth rates of GDP, and specific assumptions on damages, are discussed.

This paper also argues against using simple rules for policy making purposes. Economists’ capacity to solve large dynamic optimization problems is strong enough to present to policy makers solution paths over time, rather than rules. This recommendation is not novel at all and is implemented in practice already (see e.g. U.S. Interagency Working Group (2010)). Also e.g., Newell and Pizer (2003) present scenarios.

The sequel is organized as follows. Section 2 outlines the basic framework and derives the simple rules proposed by the five groups of prominent researchers mentioned above. Moreover, it is shown that the assumptions made to obtain the simple rule are incompatible with each other in some of the cases. Subsequently, Section 3 critically assesses these simple rules as well as the Ramsey Rule employed in the literature. Assumptions underlying the rules that will be discussed, include the absence of natural capital in the instantaneous welfare functions, constancy of the elasticity of marginal utility of consumption, constant growth rates, and particular assumptions regarding marginal damages. Moreover, the rules seem to be particularly attractive because they depend on current GDP in a simple way. But, what counts is optimal rather than actual current GDP, as already indicated above. Section 4 concludes.

[4] Analytical contributions that emphasize uncertainty such as Traeger (2015) are not included. For the case of certainty Traeger (2015) is close to Golosov et al.

2. Some simple rules

2,1. A prototype IAM model

In the present section a slightly modified version of the model developed by Rezai et al. (2014) is employed, that can be seen as a prototype of the (analytical) Integrated Assessment Models used to evaluate climate policy. It serves to illustrate all five 'simple rules' that will be considered. RP, GHKT, BGL, DV and CZL stand for Rezai and van der Ploeg (2016), Golosov et al. (2014), Van den Bijgaart et al. (2016), Dietz and Venmans (2019) and Chuan-Zhong Li (2019), respectively. The model is as simple as possible, while still capturing the essence of IAMs. Hence, it includes a climate module and describes the accumulation of capital, in a closed economy. The labor force L equals population and grows at an exogenous and constant growth rate π. Instantaneous welfare W depends positively on per capita consumption $c = C / L$ and negatively on temperature T. Instantaneous utility is discounted at a constant rate of pure time preference ρ. The time argument is omitted when there is no danger of confusion. Total welfare

$$\int_0^\infty e^{-\rho t} L W(c,T) dt,$$

is to be maximized. The economy's GDP is written as

(1) $Y(K, L, F, R, S, T, t) = \dot{K} + \mu K + C$,

where

$$Y(K, L, F, R, S, T, t) = Q(K, L, F + R, T, t) - G(S)F - bR.$$

Aggregate production is given by $Q(K, L, F + R, T, t)$, where K denotes man-made capital, F is the input from a non-renewable resource (fossil fuel), and R is the use of a renewable resource that is perfectly substitutable with the non-renewable resource. Temperature may have a negative impact on production. The time argument allows for exogenous technical progress. The unit extraction cost of the non-renewables stock S is represented by $G(S)F$ where G is a decreasing function. Renewables are produced with a linear technology, requiring an amount b of output per unit of production of the renewable, with b possibly exogenously declining over time.

Emissions of CO2 are proportional to fossil fuel use. Part φ_L of emissions stays in the atmosphere forever. The accumulated stock from these emissions thus follows from

$$(2)\ \dot{\hat{E}}_1 = \varphi_L F.$$

The transient stock of CO2 follows from

$$(3)\ \dot{\hat{E}}_2 = (1-\varphi_L)\varphi_0 F - \varphi \hat{E}_2,$$

with φ_0 a scale parameter and Total accumulated emissions are $\hat{E} = \hat{E}_1 + \hat{E}_2$. Temperature is governed by the effective CO2 stock E so that

$$(4)\ T = H(E),$$

with E following from

$$(5)\ \dot{E} = \frac{1}{\varphi_T}(\hat{E} - E) = \frac{1}{\varphi_T}(\hat{E}_1 + \hat{E}_2 - E).$$ [5]

Exhaustibility of fossil fuels gives

$$(6)\ \dot{S} = -F.$$

All initial stocks are given: $K_0, L_0, S_0, E_0, \hat{E}_{10}, \hat{E}_{20}, \hat{E}_0, T_0$. The optimal path of the economy is derived from maximizing social welfare subject to the conditions described above, including the obvious non-negativity conditions. Upon the use of $T = H(E)$, normalizing $L_0 = 1$ and omitting the arguments of Y, the Hamiltonian of the problem reads

$$\Lambda = e^{(\pi-\rho)t} W(e^{-\pi t}C, H(E)) + \lambda[-F] + \nu_1[\varphi_L F] + \nu_2[\varphi_0(1-\varphi_L)F - \varphi \hat{E}_2]$$
$$+\kappa[Y - C - \mu K] + \nu[\frac{1}{\varphi_T}(\hat{E}_1 + \hat{E}_2 - E)].$$

[5] This equation captures the discrete time version used by RP where $E_t - E_{t-1} = \frac{1}{\varphi_T}(\hat{E}_t - E_{t-1})$ to allow for delay between temperature and accumulated CO2. With $\varphi_T = 1$ we have $\hat{E}_t = E_t$, and adjustment is immediate. This corresponds with $\varphi_T = 0$ in this continuous time version.

Here $\lambda, \nu_1, \nu_2, \kappa$ and ν are the co-states or shadow prices corresponding with the resource stock, the permanent CO2 stock, the transient CO2 stock, capital and the effective CO2 stock, respectively. According to the Pontryagin maximum principle the necessary conditions for the optimization entail the maximization of the Hamiltonian with respect to the instruments C, F and $R.$ Moreover, the shadow price p of a state variable X satisfies $-\dot{p} = \partial H / \partial X$. This yields as necessary conditions:

(7) $e^{-\rho t} W_c(e^{-\pi t} C, H(E)) = \kappa, \quad (C)$,

(8) $Y_F = \dfrac{\lambda}{\kappa} + \dfrac{-\nu_1 \varphi_L - \nu_2 \varphi_0 (1-\varphi_L)}{\kappa}$ if $F > 0, \quad (F)$,

(9) $Y_R = 0$ if $R > 0, \quad (R)$,

(10) $-\dot{\lambda} = \kappa Y_S, \quad (S)$,

(11) $-\dot{\nu} = e^{(\pi-\rho)t} W_2(c, H(E)) H'(E) + \kappa Y_E - \nu \dfrac{1}{\varphi_T}, \quad (E)$,

(12) $-\dfrac{\dot{\kappa}}{\kappa} = Y_K - \mu, \quad (K)$,

(13) $-\dot{\nu}_1 = \nu \dfrac{1}{\varphi_T}, \quad (\hat{E}_1)$,

(14) $-\dot{\nu}_2 = \nu \dfrac{1}{\varphi_T} - \nu_2 \varphi, \quad (\hat{E}_2)$.

The interpretation of these conditions is straightforward. Equation (7), for example, says that in an optimum where fossil fuel is used, the marginal product of fossil fuel, net of extraction cost, should equal the sum of the Hotelling rent, λ, and the cost of the externality caused by the use of fossil fuel, where the social cost of carbon is the loss in welfare due to a marginal increase emissions. Hence

(15) $SCC = -\dfrac{\varphi_L \nu_1 + \varphi_0 (1-\varphi_L) \nu_2}{\kappa}$.

This is also the tax rate τ to be imposed in order to induce competitive firms to use the optimal fossil fuel input.

2.2. Solving the model

Generally, the expression for SCC is difficult to write in terms of primitives and time. But under special conditions, listed as assumptions below, a simple rule results.

Assumption 1. No damages in social welfare functions: $W(c,T)=U(c)$ for all T.

Assumption 2. The elasticity of marginal utility $\eta = -\frac{U''(c)c}{U'(c)}$ is constant.

Assumption 3. There exists a constant g such that $\dot{C}/C = \dot{K}/K = \dot{Y}/Y = g$.

Assumption 4. There exist constants Π and β such that $Y_E = \Pi e^{\beta t}$.

Hence, climate change does not cause direct damages to welfare, instantaneous utility of per capita consumption is isoelastic, GDP and consumption have constant growth rates g and marginal damages to GDP display a constant growth rate β .

With $z = \nu / \kappa$ it is straightforward to see that under Assumptions 1-4:

$$\dot{z} = -Y_E + (\frac{1}{\varphi_T} + \rho + \eta(g-\pi))z = -\Pi e^{\beta t} + (\frac{1}{\varphi_T} + \rho + \eta(g-\pi))z \,.$$

Hence

$$z(t) = \frac{\Pi e^{\beta t}}{\frac{1}{\varphi_T} + \rho + \eta(g-\pi) - \beta} \,.$$

Then

$$y(t) \equiv \frac{\nu_1(t)}{\kappa(t)} = \frac{\Pi e^{\beta t}}{(\rho + \eta(g-\pi) - \beta)(1 + \varphi_T(\rho + \eta(g-\pi) - \beta)} \,,$$

and

$$x(t) \equiv \frac{\nu_2(t)}{\kappa(t)} = \frac{\Pi e^{\beta t}}{(\varphi + \rho + \eta(g-\pi) - \beta)(1 + \varphi_T(\rho + \eta(g-\pi) - \beta)} \,.$$

It readily follows that

$$(16)\quad SCC = [\frac{\varphi_L}{(\rho+\eta(g-\pi)-\beta)} + \frac{\varphi_0(1-\varphi_L)}{(\varphi+\rho+\eta(g-\pi)-\beta)}]\frac{\Pi e^{\beta t}}{(1+\varphi_T(\rho+\eta(g-\pi)-\beta))}.$$

Alternatively, if a SCC results that has a constant growth rate β over time, then it must be the case that Assumptions 1-3 imply that Assumption 4 holds. This is easily seen as follows. A constant growth rate of SCC implies from (14) that ν_1/κ and ν_2/κ display the same constant growth rate. Since $\dot{\kappa}/\kappa$ is constant due to the fact that consumption has a constant growth rate and the elasticity of marginal utility is constant., it follows that $z = \nu/\kappa$ has a constant growth rate, so that Y_E has a constant growth rate equal to the growth rate of SCC.

2.3 Simple rules in the literature

Expression (16) boils down to the expression for the SCC obtained by RP (their equation (2) o.c. page 497) if $\Pi e^{\beta t} = \chi Y^{\varepsilon} Y(0)^{1-\varepsilon}$. Hence, if $\beta = \varepsilon g$ and $\Pi = \chi Y(0)$. This yields

$$(17)\quad SCC = [\frac{\varphi_L}{(\rho+\eta(g-\pi)-\varepsilon g)} + \frac{\varphi_0(1-\varphi_L)}{(\varphi+\rho+\eta(g-\pi)-\varepsilon g)}]\frac{\chi Y^{\varepsilon} Y(0)^{1-\varepsilon}}{(1+\varphi_T(\rho+\eta(g-\pi)-\varepsilon g))}.$$

Alternatively, $Y_E = \Pi e^{\beta t} = \chi Y^{\varepsilon} Y(0)^{1-\varepsilon}$. is not only sufficient to find their expression for the SCC, but also necessary. But, using $T = H(E),$ it holds in their model that (see their equation (6), o.c. page 501),

$$\begin{aligned} &Y(K,L,F,R,S,E,t) = Q(K,L,F+R,E,t) - G(S)F - bR \\ &= Z(K,L,F+R) - D(H(E))Z(K,L,F+R)^{\varepsilon} Z_0^{1-\varepsilon} - G(S)F - bR \end{aligned}$$

It is hard to see under which conditions $Y_E = \Pi e^{\beta t} = \chi Y^{\varepsilon} Y(0)^{1-\varepsilon}$. If $1 - D(E) = e^{-\delta(E-\bar{E})}$, as in GHKT, which seems the most favorable thing to do, then

$$Y_E = -\delta e^{-\delta(E-\bar{E})} Q(K,L,F+R,t) = -\delta[Y + G(S)F + bR].$$

It is not clear at all whether this is compatible with marginal damages written as $Y_E = \Pi e^{\beta t} = \chi Y^{\varepsilon} Y(0)^{1-\varepsilon}$. Also in case where the economy is carbon-free it is not straightforward that the assumption holds. Concluding, it would be worthwhile to find out what additional assumptions on the constituent parts of GDP need to be made to obtain expression (17).

In the model studied by BGL $E=\hat{E}=\hat{E}_2$. In particular it holds that $\dot{E}=F-\varphi E$. The motion of temperature over time reads $\dot{T}=\gamma(V(E)-T)$. Moreover,

$Y(K,L,F,T,t)=Z(K,F,t)-\omega T^{\psi}Z(K,F,t)^{\varepsilon}(L\bar{y})^{1-\varepsilon}$,

with $\bar{y}$ a constant, defined as "the reference per capita income at which one-degree temperature rise leads to relative damages ω." BGL assume that for all 'reasonable' ψ (i.e., ψ close to 2), $\frac{\partial(V(E))^{\psi}}{\partial E}=1.3c^{\psi}/m,$ with c and m constants. Moreover, the climate system is considered close to a stationary state.[6] Marginal damages are therefore constant, which is crucial in their derivation. They arrive at the following expression for the social cost of carbon, in our notation.

$$(18)\quad SCC=\frac{1.3\omega c^{\psi}}{m}\left(\frac{1}{\varphi+\rho+(\eta-\varepsilon)(g-\pi)-\pi}\right)\left(\frac{\gamma}{\gamma+\rho+(\eta-\varepsilon)(g-\pi)-\pi}\right)Z^{\varepsilon}(t)(L(t)\bar{y})^{\varepsilon-1}$$

As before, this equation can be derived under assumptions 1-4, and assumption 4 poses also a necessary condition. The issue raised with regard to compatibility of the assumptions in RP applies also now, as can be seen from their specification of national income. Indeed, if $GDP=Y$ is growing at a constant rate then, with constant temperature, Z and $Z^{\varepsilon}L^{1-\varepsilon}$ must grow at the same rate, which requires $\varepsilon=1$.

A second set of simple rules does not make the constant growth assumption. We first consider CZL. CZL makes assumption 1, does not include renewables and takes $E=\hat{E}=\hat{E}_1, T=\hat{E}_1$ so that $\dot{T}=\varphi_L F.$ Moreover, $Y(K,L,F,R,S,T,t)=e^{-\delta T}Q(K,L,F)$. With $g(t)$ GDP's growth rate at instant of time t It is shown that

$$(18)\quad SCC(t)=\varphi_L\delta Y(t)\int_t^{\infty}e^{-\int_t^s(r(v)-g(v))dv}ds,$$

where r is the interest rate. Hence, the social cost of carbon is proportional to to GDP.

[6] For the moment these assumptions are taken for granted although one may wonder whether the assumptions taken together do not uniquely determine temperature, the CO2 stock and fossil fuel use, through $d(V(E))^{\psi}/dE=1.3c^{\psi}/m,\ \Phi(\hat{E}_2)-T=0$ and $\dot{\hat{E}}_2=F-\varphi\hat{E}_2=0.$

The model by Rezai et al. presented in subsection 2.1, can be seen as a special case of the GHTK model. Expression (17) for the SCC closely resembles the one derived by GHKT (their equation (12) o.c. page 54) if $\eta = 1, \pi = 0, \varepsilon = 1, \beta = g$ and $\varphi_T = 0,$ because then

$$(19)\ \ SCC = [\frac{\varphi_L}{\rho} + \frac{\varphi_0(1-\varphi_L)}{(\varphi+\rho)}]\chi Y(t).$$

This is not precisely what GHKT do. They assume GDP can be written as $Y = (1 - D(E))Q(K, L, F, t)$, with $1 - D(E) = e^{-\delta(E-\bar{E})}$.

Hence $Y_E = -\delta e^{-\delta(E-\bar{E})}Q(K, L, F, t) = -\delta Y$ and assumption A4 holds if $GDP = Y$ has a constant growth rate. However, GHKT do not make the latter assumption. Instead, in addition to the assumptions described above, they assume a constant savings rate and a Cobb-Douglas production function with constant returns to scale in capital, labor and aggregate energy use, and they pick the discrete time interval in which they assume that all capital depreciates within one period of time. This is also sufficient to arrive at their SCC. Consequently, in view of the fundamentally different assumptions RP's claim that they generalize GHKT (o.c. page 495) is not warranted.

Finally, there is the study by Dietz and Venmans (2019). Their aim is not only to derive a closed-form solution for the SCC but also to show that many IAMs greatly overestimate the delay between carbon emissions and temperature increase. For the purpose of the present paper the focus is on the former part of their work. They make assumptions 1, 2 and 3. Labor is growing at a constant rate, as postulated before. Moreover there is Harrod technical change. GDP is be written as $Y(K, L, T, F, t) = Q(K, e^{\omega t}L)e^{-T^2/2}e^{\xi F - \xi F^2/2}$ with Q linearly homogeneous. They also postulate $\dot{E} = F$ and $\dot{T} = \gamma(\zeta E - T)$. $\dot{E} = F$ and $\dot{T} = \gamma(\zeta E - T)$. They assume a constant savings rate as well as a constant growth rate of GDP. By virtue of the Ramsey Rule, these assumptions imply constant growth rates of consumption, capital and GDP and a constant marginal product of capital, Y_K. Hence, $e^{-T^2/2}e^{\xi F - \xi F^2/2}$ is constant then. As is shown in Appendix B. this is incompatible with their postulates regarding the climate module. It is also shown in the appendix that, disregarding this, their SCC reads

$$(20)\ SCC = \frac{\varepsilon\zeta\gamma T_0 Y(0)e^{(\vartheta+g)t}}{(\rho+\eta(g-\pi)-\vartheta-g)(\varepsilon+\rho+\eta(g-\pi)-\vartheta-g)}.$$

2.4 Preliminary conclusions

Solow (1956, p. 65) states: ‘All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive’. Hence, from this perspective there is no reason to challenge the assumptions made to derive the five rules discussed in the previous subsection. Solow continues (o.c. page 65) ‘A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect’. These crucial assumptions will be discussed in the next section. The worrying point for now is that in the derivation of the simple rules assumptions are made that are or, to put it mildly, seem to be incompatible with each other. In particular, assumptions on constant growth rates are difficult to reconcile with the specifications of the technology that are used.

3. Further assessment of the simple rules

In this section we point at weaknesses of simple rules, apart from the inconsistencies in the assumptions made. This critique is related to the assumptions made and on the claim by the authors that the approximation is ‘good’. This section will start with the latter assertion and then go into the assumptions made. Finally, attention will be paid to the role of GDP in general in the expression for SCC.

3.1 “The simple rule is a good approximation”.

Obviously, in order to justify a rule being a ‘good approximation’ it is necessary to know the counterfactual ‘true’ SCC. For this true SCC several options present themselves. RP consider an extended calibrated version of their model. And they argue that the first-best SCC following from this model closely resembles the SCC given by their simple rule. BGL take another reference point. They “evaluate .quantitatively how well the formula ,….predicts the SCC of DICE (Nordhaus, 2008)” (o.c. page 81). Also CZL compares the outcomes with DICE. GHKT perform several robustness checks. However, one would expect the optimal solution of the model under consideration to be the reference point. Given the skills of the authors it should pose no insurmountable problem to actually calculate the optimum numerically.

Moreover, if one can derive the true optimum, the question arises why we as economists would offer rules to policy makers rather than the entire time path of the optimal SCC, once it is known. One could argue that policy makers prefer rules over numbers, but what counts in the end is a carbon tax to be implemented. Moreover, at least in many European countries policies are being developed to meet the Paris agreement, which for EU countries poses targets also for the immediate future. In this perspective the sometimes used argument that a rule is preferable because the future is uncertain, does not seem to be convincing either.

Another problem related to the claim of being a good approximation, is how to measure the fit. The SCC is extending over the entire future, or in any case until fossil fuel will be banned. Hence, the metric should involve dynamics. How should one then evaluate deviations in the short run from optimality in the short run to deviations from optimality in the long run? This is a relevant question as can be illustrated by looking at the SCCs discussed above. In the comparison of the growth rates of the SCC in the models differences can be observed. With a constant growth rate g of GDP the growth rate of the SCC in the GHKT and DV is g, in RP it is εg and in BGL it is $\varepsilon(g+\pi)-\pi$. For $\varepsilon=1$ the growth rates coincide, but for other elasticities they differ and diverge exponentially. At least in some topologies the resulting SCCs are not 'close'. The question which is then the 'good' rule, seems justified.

3.2 No direct welfare impact of climate change.

With a more general instantaneous welfare function, $LW(C/L,T)$, hence including direct climate change damages, the Ramsey Rule reads

$$r \equiv F_K - \mu = -\dot{\kappa}/\kappa = \rho + \eta(c,X)\frac{\dot{c}}{c} + \vartheta(c,T)\frac{\dot{T}}{T}, \text{ with } c = C/L,$$

$\eta(c,T) = -\dfrac{W_{11}(c,T)c}{W_1(c,T)}\dfrac{\dot{c}}{c}$ and $\vartheta(c,T) = -\dfrac{W_{12}(c,T)T}{W_1(c,T)}\dfrac{\dot{T}}{T}$. The elasticity of marginal utility of consumption η will be discussed in the next section. What matters for now is the final term on the right hand side. It is equal to zero under Assumption 1, $W_2 = 0$, i.e., there is no direct welfare impact of climate change. However, instantaneous welfare may directly depend on the stock of atmospheric CO2, for example due to increased health problems and the loss of biodiversity (see a.o., the latest report by the Intergovernmental Science-Policy Platform on Biodiversity and Ecosystem Services (2019)). Hence the elasticity of marginal welfare of consumption with respect to temperature does play a role. GHKT go briefly into the issue and argue that if the welfare function is additively separable and damages appear in a linear way, a simple rule can still be derived. But they also say that in other specifications their rule *fundamentally* changes. In the remaining literature on simple rules no attention is paid to justifying the usually made assumption of absence of CO2 or temperature in the instantaneous welfare function. This neglects the existence of a large literature on how to model damages (or ecosystem benefits) directly in the instantaneous welfare function. Van der Ploeg and Withagen (1990) derive the Ramsey Rule in case biodiversity, climate change or ecosystems appear directly in the social welfare function, albeit in an additively separable way (so that the cross-elasticity is zero). Michel and Rotillon (1995) emphasize the importance of non-separability. More recently Hoel and Sterner (2007) and Zhu et al. (2019) include climate change and ecosystem services, respectively, in a CES welfare function and study the effects in detail. However, most of the literature does not pay attention to the direct welfare aspects. The least one would expect is a sensitivity analysis with respect to alternative preference specifications. Not only is climate change relevant for the discount rate, it is also relevant for the SCC, representing all damages from a marginal increase of emissions. For example when atmospheric CO2 does not decay the SCC should include a term like

$$SCC(t) = \int_t^\infty e^{(\pi-\rho)s} \frac{W_2(c(s),T(s))}{W_1(c(t),T(t))} ds.$$

This might considerably increase the SCC compared to when there are no direct damages. Finally, the elasticity of marginal utility of consumption, $\eta(c,T)$, may be a function of the CO2 stock.

3.3 Constant elasticity of marginal utility

The assumption of a constant elasticity of marginal utility (Assumption 2) is crucial in deriving the simple rules discussed here. The constancy limits attention to specific utility functions. Groom and Maddison (2019) look into the elasticity of marginal utility for the UK and find a central estimate of 1.5. But they also mention other studies that find other estimates, ranging from 0.5 to 10. The differences can be attributed to the methodology used, but may also have to do with a.o. the dependence of the elasticity with respect to the level of consumption, in which case Assumption 2 is not supported in reality. See also the literature on superconcavity of utility functions (e.g. Mrazova and Neary (2014)). Disagreement among experts on the value of the elasticity is also apparent from the work of Drupp et al. (2018), where the values range from 0.5 to 5.

3.4 Constant growth rate of GDP.

This point has several aspects. First, what should the constant growth rate be, second, from what point in time on does the constant growth rate apply and, third, why would the growth rate be constant? These point are interrelated and will be discussed below, not in a particular order.

In the original Ramsey (1928) model of optimal growth in a one sector economy without population growth, technical change, emissions and a zero rate of pure time preference, the economy converges to a unique constant steady state of man-made capital and consumption. Convergence is monotonic. If the actual initial capital stock is smaller than the long run optimal one, capital and consumption will initially grow, otherwise they decline. Hence zero growth only occurs asymptotically unless the economy finds itself initially in the steady state. Extending this model with labor in the production function, growing at a constant rate, allowing for Harrod-neutral technical progress at a constant rate, for constant returns to scale with respect to labor and capital in production and for a positive rate of pure time preference, yields an asymptotic optimal long run per capita consumption growth rate equal to the rate of technical progress. The conclusion is that the growth rate of per capita consumption is constant forever only if the economy coincidentally happens to start in the state corresponding with steady state growth, even if the elasticity of marginal utility is constant.

Fossil fuels play a crucial role in climate change economics. However, in IAMs their exhaustibility is seldom taken into account. This may give rise to insufficient attention for its consequences for consumption growth, and therefore the Ramsey Rule. Stiglitz (1974) considered an optimal growth model in the vein of Ramsey, including, say, energy as a production factor, where energy comes from a non-renewable resource. He used a Cobb-Douglas production function and a utility function with elasticity of marginal utility equal to unity. Withagen (1990) extended the model so as to include non-unitary, but constant, elasticities of marginal utility. He showed that the long run growth rate of consumption converges to a constant. The constant includes not only the technological progress parameter but the elasticity of marginal utility as well, implying that the consumption growth rate and the elasticity of marginal utility are difficult to disentangle: In the Stiglitz model and the extension by Withagen postulating a given elasticity has an impact on the long run growth rate. The fact that this occurs in a very simple model with fossil fuel, does not make this issue an academic curiosity. Exhaustibility of fossil fuel is pertinent in the real world and should be included in any IAM. Another important finding of resource economics in the present context is due to Dasgupta and Heal (1974) who show that, even in the absence of technological progress, optimal consumption may be increasing for an initial interval of time and decreasing eventually. Hence, in that case, even with a constant elasticity of marginal utility, the social rate of discount is non-monotonic as well, and therefore not constant. Moreover, whether or not there is a phase with increasing consumption depends on the level of the pure rate of time preference (see Benchekroun and Withagen (2011)). Hence, in this version of the Ramsey Rule, the growth rate g is dependent on the rate of pure time preference ρ. Disentangling the growth rate and the rate of pure time preference is therefore not warranted.

There exist numerous analytical papers on capital accumulation and pollution. Early contributions include Forster (1973) and Keeler et al. (1971). A recent model that closely resembles the model of Section 2 was developed by Van der Ploeg and Withagen (2014). It focuses on the transition to a carbon-free economy. In their model, damages occur only at the direct welfare level, not in the technology. In the long run the economy will (asymptotically) converge to the carbon-free steady state, where, essentially, the standard Ramsey model applies that was discussed at the outset of this subsection. Without population growth and without technical progress, the rate of consumption will then be constant and the appropriate discount rate in that phase equals the rate of pure time preference. However, this should obviously not be the discount rate used on the transition path, because along the transition path optimal consumption can be increasing or decreasing. Actually, it is shown that if the economy is abundant in fossil fuel, consumption will initially rise, overshooting the steady state carbon-free consumption rate, and eventually decrease towards the steady state. Simulations show that in such an economy the carbon tax will typically monotonically increase, but it can also be inverted U-shaped, depending on the question whether the economy is still in the developing stage or mature, in terms of capital. In any case, a simple rule exists, obviously, but it may be far from optimal.

Hence, what is missing in the models of the previous section is a motivation of why the *GDP* growth should be constant.

3.5 GDP in the SCC rule

In all five rules that have been discussed thusfar, *GDP* appears in the expression for the SCC, sometimes in a linear way. The be more precise: SCC at time t is oftentimes a function of *GDP* at time t (and not of future GDP). GHKT claim that indeed, “no knowledge about future technology, productivity or labor supply is needed to calculate the marginal externality cost of emissions per *GDP* unit" (o.c. page 54). This is true, if only *GDP* were *optimal*. But precisely to determine the optimal *GDP*, and therefore the optimal SCC, knowledge of the entire future is indispensable. The same holds for the other models. In the case of RP *GDP* appearing in the expression for the SCC is actual *GDP* rather than optimal. This is clear from the fact that energy input in the first period is given, rather than optimally chosen (RP, o.c. p. 501). Also in BGL initial national income is exogenously given. DV and CZL do not mention that *GDP* in the SCC is optimal. The question arises why energy input is not chosen optimally. One reason could be that actual energy input is considered optimal. But that would imply that in the short run no action needs to be taken, which seems to contradict the politicians' wish to act now. One possible explanation could also be that Nordhaus and Sztorc (2013), which often serves as a reference, does not optimize with regard to energy inputs in the initial period under consideration. However, this cannot be a justification. Rather, working implicitly with actual rather than optimal GDP may be misleading. To illustrate this point, that is well known in the optimal control literature, reconsider equation (8) and assume that optimal initial fossil fuel input is positive. Then, for given initial stocks of capital, atmospheric CO2, and fossil fuel, optimal energy input follows from

$$Y_F(0) = \frac{\lambda(0)}{\kappa(0)} + \frac{-\nu_1(0)\varphi_L - \nu_2(0)\varphi_0(1-\varphi_L)}{\kappa(0)}.$$

Hence, optimal fossil fuel input depends on the initial shadow price of fossil fuel and the initial social cost of carbon. These initial values can typically only be determined from the boundary conditions on the entire solution of the differential equations for these co-state variables.

4. Conclusions

It has been demonstrated that one should be careful in proposing or believing in simple rules based on specific assumptions with regard to the constituent elements of the Ramsey Rule. The argument that policy makers want such rules, eventually giving numbers, and preferably constant numbers, is not convincing unless the scientific policy advisors can make clear that the model they have in mind, justifies this choice. The existing rules make heroic and sometimes contradictory assumptions, such as that actual current *GDP* or consumption is optimal *GDP* or consumption. One could argue that the difference is not large, but, without convincing evidence, this is hard to accept when it comes to climate change, which requires action in the short run. Moreover, even if the difference is small, say *only* 1% of world *GDP*, one may wonder why the *price* of a simple rule would be billions of euros.

Wouldn't it be better for the design of policy to confront policy makers with a set of scenarios, entailing the accumulation of atmospheric CO2, paths of capital accumulation, exhaustion of fossil fuel, consumption paths, consistent with each other, and let the policy maker make a choice? This procedure is followed in a.o., the US Interagency Working Group (2010) which by itself does not give preference of one scenario over another. Technically, the design and presentation of scenarios is relatively easy, given the enormous power we nowadays have in performing numerical calculations. One objection could be that preferences of policy makers should be revealed ex ante, not ex post. However, I would like to quote from Nobel Prize Winner Tjalling Koopmans: "Ignoring realities in adopting "principles" may lead one to search for a nonexistent optimum or to adopt an optimum that is open to unanticipated objections" (Koopmans, 1965). Indeed, if the policy maker does not understand what the ingredients of the model are, then it is better to confront her with entire time paths of all relevant variables, such as consumption, emissions, temperature, rather than with just optimal long run growth rates, discount rates or the social cost of carbon. Finally, in addition to the arguments given above for not relying on the simple Ramsey Rule, being more careful with this rule is also recommended by Drupp et al. (2018) who put forward: "….the prominence of the simple Ramsey Rule in public policy needs to be revised. When we impute the simple Ramsey Rule for all experts individually, we find wide discrepancies between these values and their recommended SDRs (*social discount rates)".*

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Appendix A

Model BGL

The time argument is omitted when there is no danger of confusion. The BGL model is expressed in the notation of the main text of the present paper.

Max $\int_0^\infty e^{-\rho t} L(U(C/L)dt$,

subject to

$\dot{L}/L = \pi,\ L(0) = 1$,

$GDP = Y(K, L, E, T, t) = C + \dot{K} + \mu K,\ K(0) = K_0,$,

$\dot{E} = F - \varphi E,\ E(0) = E_0$,

$\dot{T} = \gamma(V(E) - T), T(0) = T_0$,

$$Y(K, L, F, T, t) = Z(K, L, F, t)[1 - \omega T^{\psi}(\frac{Z(K, L, F, t)}{L\bar{y}})^{\xi-1}].$$

The Hamiltonian reads:

$$\Lambda = e^{(\pi-\rho)t}U(C/L) + \kappa[Y - C - \mu K] + \nu[F - \varphi E] + \lambda[\gamma(V(E) - T)].$$

Necessary conditions are:

$$e^{-\rho t}U'(C/L)=\kappa\ ,$$

$$-\dot{\kappa}=\kappa[Y_K-\mu],$$

$$-\dot{\nu}=-\nu\varphi+\lambda\gamma V'(E)\ ,$$

$$-\dot{\lambda}=\kappa Y_T-\gamma\lambda\ .$$

With a constant elasticity of marginal utility η and a constant growth rate of consumption g it holds that $\dot{\kappa}/\kappa=-\rho-\eta(g-\pi)$. The social cost of carbon is $SCC=-\nu/\kappa$.

Define $x=\nu/\kappa,\ y=\lambda/\kappa$. Then

$$\dot{x}=(\varphi+\rho+\eta(g-\pi))x-\gamma V'(E)y\ ,$$

$$\dot{y}=-Y_T+(\gamma+\rho+\eta(g-\pi))y\ .$$

Define $z=V'(E)y$. Then

$$\dot{x}=(\varphi+\rho+\eta(g-\pi))x-\gamma z$$

and

$$\dot{y}V'(E)=-Y_TV'(E)+(\gamma+\rho+\eta(g-\pi))z$$

In order to have a tuple of linear differential equations necessary to arrive at BGL's SCC, additional assumptions are made, namely $\dot{T}=0$ and $\frac{d(V(E))^{\psi}}{dE}=1,3c^{\psi}/m$ for all ψ close enough to 2 (see BGL, p.79). Note that these assumptions imply a constant given stock of CO2 and therefore also a specific constant temperature and emission flow. This seems hard to justify. Accepting this anyway, one arrives at

$$\begin{aligned}Y_TV'(E)&=-\psi\omega T^{\psi-1}Z(K,L,F,t)^{\xi}(L\bar{y})^{1-\xi}V'(E)=-\psi\omega(V(E))^{\psi-1}V'(E)Z(K,L,F,t)^{\xi}(L\bar{y})^{1-\xi}\\&=-\frac{d(V(E))^{\psi}}{dE}\omega Z(K,L,F,t)^{\xi}(L\bar{y})^{1-\xi}=-1.3\frac{c^{\psi}}{m}\omega Z(K,L,F,t)^{\xi}(L\bar{y})^{1-\xi}\end{aligned}$$

And, therefore,

$$\dot{z}=1.3\frac{c^{\psi}}{m}\omega Z(K,L,F,t)^{\xi}(L\bar{y})^{1-\xi}+(\gamma+\rho+\eta(g-\pi))z\ .$$

If $1.3\frac{c^{\psi}}{m}\omega Z(K,L,F,t)^{\xi}(L\bar{y})^{1-\xi}$ has a constant growth rate $g\xi+(1-\xi)\pi$ then it is straightforward to see that

$$SCC=\frac{1.3\omega c^{\psi}}{m}\frac{1}{(\varphi+\rho+(\eta-\xi)(g-\pi)-\pi)}\frac{\gamma}{(\gamma+\rho+(\eta-\xi)(g-\pi)-\pi)}Z^{\xi}(L\bar{y})^{1-\xi},$$

assuming the denominators are positive.

Appendix B.

Model DV.

In the notation of Section 2 the economic model analyzed by DV reads as follows.

Max $\int_0^{\infty} e^{-\rho t}L(U(C/L)dt$,

subject to

$\dot{L}/L=\pi,\ L(0)=L_0=1$,

$Y=C+\dot{K}+\mu K, K(0)=K_0$,

$\dot{E}=F,\ E(0)=E_0$,

$\dot{T}=\varepsilon(\zeta E-T), T(0)=T_0$,

$Y(K,L,F,T,t)=Q(K,Le^{\omega t})e^{-\gamma T^2/2}e^{\xi F-\xi F^2/2}$.

Hence, exhaustibility is not taken into account and renewables do not play a role. It is assumed that Q is linearly homogeneous. The Hamiltonian of the problem reads

$$\Lambda=e^{(\pi-\rho)t}U(C/L)+\nu F+\kappa[Y-C-\mu K]+\lambda[\varepsilon(\zeta E-T)].$$

Necessary conditions for an optimum are

$e^{-\rho t}(C/L)^{-\eta}=\kappa,$

$\kappa Y_F=-\nu$,

$-\dot{\lambda}=-\lambda\varepsilon+\kappa Y_T$,

$-\dot{\kappa} = \kappa[Y_K - \mu]$, ,

$-\dot{\nu} = \lambda\varepsilon\zeta$.

With $z = \nu / \kappa$ and $y = \lambda / \kappa$ we have

$$\dot{z} = -\varepsilon\zeta y - \frac{\dot{\kappa}}{\kappa} z$$
$$\dot{y} = (\varepsilon - \frac{\dot{\kappa}}{\kappa}) y - Y_T$$

Moreover, $Y_T = -\gamma TY$. Their Assumption 2 (page 115) reads $\dot{E} / E = \dot{T} / T = \vartheta,$ a constant. Then $Y_T = -\gamma T(0)Y(0)e^{(\vartheta+g)t}$ with $g = \dot{Y} / Y = \dot{C} / C = \dot{K} / K$ (following from the assumption of constant GDP growth rate and a constant savings rate, o.c. page 113). Hence, assumption 4 is satisfied. Then is it easily seen that

$$SCC = \frac{\varepsilon\zeta\gamma T(0)Y(0)e^{(\vartheta+g)t}}{(\rho + \eta(g-\pi) - \vartheta - g)(\varepsilon + \rho + \eta(g-\pi) - \vartheta - g)}.$$

The problem with these assumptions is that in the long run the marginal product of energy in product gets negative if $\vartheta > 0$. If $\vartheta = 0$ then $F = 0$ and one may wonder about the dynamic path of the SCC.

Actually, with growth and a constant savings rate, as is assumed in DV (o.c. p. 113), consumption has a constant growth rate so that κ has a constant growth rate, implying that Y_K is constant, and, due to constant returns to scale, $Q(K, Le^{\omega t})$ has the same constant growth rate as GDP. But then also $-\gamma T^2 / 2 + (\xi F - \xi F^2 / 2)$ is constant. If F would be bounded away from zero, then E is unbounded and $T \to \infty$ contradicting the constancy of $T^2 / 2 = \xi F - \xi F^2 / 2$. Hence $F \to 0$ and consequently $T \to 0$ but this contradicts $\dot{T} = \varepsilon(\zeta E - T), T(0) = T_0$ $\dot{T} = \varepsilon(\zeta E - T), T(0) = T_0$ with E bounded away from zero.