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A dynamic model of recycling with endogenous technological breakthrough[*]

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Abstract

We present a general equilibrium growth model in which the use of a non-renewable resource yields waste. Recycling waste produces materials of poor quality. These materials can be reused for production only once a dedicated R&D activity has made their quality reach a certain minimum threshold. The economy then switches to a fully recycling regime. We refer to this switch as the technological breakthrough.

We analyze the optimal trajectories of the economy and interpret the Ramsey-Keynes and Hotelling conditions in this specific context. We characterize the determinants of the date of the breakthrough, which is endogenous, as well as the discontinuity in the variables' paths that is induced by this breakthrough. We show, in particular, that the availability of a recycling technology leads to an over-exploitation of the resource and possibly to lower levels of consumption before the breakthrough. We also find that the breakthrough can have a negative impact on utility over a finite period.

Keywords: Recycling; Non-renewable resource; Technical change; Growth

JEL classifications: C61, O44, Q32, Q53

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1 Introduction

"Recycling is defined as any reprocessing of waste material in a production process that diverts it from the waste stream, except reuse as fuel. Both reprocessing as the same type of product, and for different purposes should be included. Recycling within industrial plants *i.e.* at the place of generation should be excluded." (United Nations[1]). By using waste as an input in the production process, recycling alleviates the scarcity of other resources. However, even if current levels of recycling greatly vary from sector to sector, recycling activity in the world is still low today. For instance, a UNEP report states that "many metal recycling rates are discouragingly low, and a "recycling society" appears no more than a distant hope" (UNEP, 2011). Geyer et al. (2017) underline that "between 1950 and 2015, cumulative waste generation of primary and secondary (recycled) plastic waste amounted to 6300 Mt. Of this, approximately (...) 600 Mt (9%) have been recycled".

The main reasons behind this low-level activity are first that it remains comparatively expensive – *i.e.* non-recycled materials remain relatively cheap. Regarding municipal solid waste, for example, "in some cases the value of recyclables are less than the extra costs associated with collecting the disturbed waste" (World Bank, 2012). Second, the recycled materials are not always perfect substitutes for the virgin materials, which entails reduced marketing possibilities. This substitution capacity is mainly driven by the degree of maturity of the recycling process itself and by the induced quality range of recycled goods. If recycling glass or pulp allows producing good-quality bottles and paper that are (almost) perfect substitutes for primary goods (see e.g. Alani et al., 2012 or ADEME, 2009) , the recycling capacity of more sophisticated products can be constrained by a deterioration of the physical characteristics of the virgin product during the recycling process. This concerns certain types of plastics (thermosets), for instance. "Thermosets (...) are characterized by their high resistance to mechanical force, chemicals, wear and heat. The robust properties of thermosets make them more difficult to recycle and they cannot be re-melted down and reformed like thermoplastics" (OECD, 2018). Similarly, carbon fiber reinforced polymer (CFRP) waste yields materials that cannot yet have the same industrial use as the virgin materials, particularly in advanced technology sectors such as aeronautics (Oliveux, 2015). This means that the quality of the recycled material is fundamental in some industries, which justifies investments to improve the physical properties of the recycled material, and not only the efficiency of the process.

[1] United Nations – Environmental indicators: http://unstats.un.org/unsd/environment/wastetreatment.htm

The aim of the present paper is to understand how an economy will invest in research so that recycled waste can be used at a large scale and to study how the economy switches to a fully recycling regime. To do so, we consider a model in which a recycling technology is available, but the current quality of recycled materials makes them non-usable by the production process. The only way to trigger the recycling activity is thus to improve the quality of these materials. This can be done by investing in a specific type of research and development (hereafter R&D). After a certain threshold quality level has been reached, a technological breakthrough occurs in the sense that the production of consumption goods starts using as inputs both the virgin (primary) resource and the recycled (secondary) waste. We characterize the optimal trajectories of the economy and their properties; in particular, we study the discontinuity occurring at the date of the technological breakthrough.

Natural resources and waste recycling is an economic issue that has been addressed in dynamic contexts by many authors.[2] In some models, recycling is motivated as an option to mitigate the pollution generated by waste disposal. Smith (1972) follows this approach but focuses on the two stationary corner solutions where it is optimal to recycle either 100% of the waste flow or nothing, depending on the comparison of the private cost of recycling with the public disutility of waste. Hoel (1978) analyzes the long-term path of an economy that consumes a non-renewable resource and a recycled resource. He shows how the environmental impact of the use of these resources affects the optimal trajectories of the economy. In an endogenous growth context, Di Vita (2001) studies an economy that uses both a non-renewable resource and recycled materials, the use of which harms the environment by producing waste. The model endogenizes the degree of recyclability of the accumulated waste: investing in a dedicated R&D sector allows improving recyclability. In an "AK" growth model without natural resource, Boucekkine and El Ouardighi (2016) introduce recycling to lighten the flow of waste generated both by capital accumulation and consumption. Waste storing is however not considered.

In other studies, as in the present one, recycling is justified only as a way to sustain production in the long run when the economy is constrained by the scarcity of non-renewable natural resources. André and Cerda (2006) study an economy that uses two types of natu-

[2]A related literature, closed to our dynamic approach, deals with the management of durable goods (see for instance Levhari and Pindyck, 1981, or Stewart, 1980). Also note that a large part of the studies on recycling can be found in the industrial organization literature (see for instance Ba and Mahenc (2018) for a survey on strategic behaviors of recycling firms). We have chosen to focus here on (general-equilibrium) dynamic contexts.

ral resources, only one of which being recyclable. They show that if recycling may alleviate resource scarcity in the short term, its ability to prevent negative long-run growth depends on how much the economy depends on non-renewable and renewable resources. Di Vita (2007) focuses on the degree of substitutability between the non-renewable resource and recycled waste in the production process. He then analyses its impact on the economy's growth path and the time profile of resource extraction. Pittel et al. (2010) also use an endogenous growth model with non-renewable and recycled resources; they consider that the waste flow resulting from the use of these resources depends on the level of economic activity. They carefully take into account the material balance equation (see also Ayres, 1999) and they show how a market for waste, and subsidies to resource extraction and recycling allow restoring the social optimum.

In all these studies, the recycling technology is immediately available and used by the economy. In the present paper however, we assume that the recycling technology initially produces materials of poor quality that cannot be used in the production process (*i.e.* at a large scale). Therefore, as in Di Vita (2001), we consider a sector of R&D devoted to improving the technical properties of the recycled resource. Here, technological improvements are needed so that the recycled resource reaches a minimum quality threshold. When this quality level is attained, the secondary material can be used and the recycling activity starts.

The growth model we develop can be sketched as follows. The production of a consumption good requires (general-purpose) knowledge, labor, and raw material. Raw material corresponds to flows of a non-renewable resource, in its virgin or recycled form. Its use yields waste flows that add to a pre-existing stock. The economy can invest in a dedicated research sector to improve the quality of recycled materials. Recycled waste starts being used as an input as soon as its quality meets a minimum threshold. This threshold is assumed to be exogenous for simplicity, but the date at which the recycling activity starts – referred as the technological breakthrough – is endogenous. The main trade-offs faced by the economy are the following: the intertemporal management of the stock of non-renewable resource, the intertemporal management of the stock of waste, the use of the virgin or the recycled resource, and the allocation of efforts between output production and R&D.

The general optimal conditions derived from the social planner's program feature a Ramsey-Keynes condition and a Hotelling rule, of which we provide a full interpretation.

The possibility of recycling makes these conditions more complex than in standard dynamic resource models. The social planner indeed manages both the remaining stock of virgin resource and the accumulated stock of waste, which is partially renewable as it is fed by flows entailed by the use of the virgin resource. The technological breakthrough entails a discontinuity in the trajectory followed by the economy. When it occurs, resource use jumps down and then declines at a slower pace. Besides, consumption can jump upwards or downwards depending on the values of parameters such as the maximal recycling rate of the virgin resource. In the case of a downward jump, the technological breakthrough has a negative impact on utility over a finite period.

We also study how the exogenous parameters affect the socially optimal trajectories and, in particular, the date of the technological breakthrough. Higher values of the growth rate of the total factor productivity, of the output elasticity of the material input, of the quality of the virgin resource or of the efficiency of the research sector make the breakthrough occur earlier. A larger initial stock of waste, higher waste content rates of the two resources or a higher maximal recycling rate also bring forward the date of the breakthrough. Conversely, this date is postponed with a higher social discount rate or a larger initial stock of virgin resource.

We finally consider the impact of the recycling activity and its timing on the economy. To do so, we compare the trajectories of the studied economy to those of a) an economy in which recycling is never possible and b) an economy in which recycling is immediately possible (the breakthrough occurs at date 0). We show that, as compared to both a) and b), the economy first over-exploits the virgin resource and then under-exploits it after the breakthrough has occurred. The remaining stock of virgin resource is, therefore, lower at each date. The use of the virgin resource makes the waste stock higher than in the never-recycling economy, but it becomes lower after a finite interval of time following the breakthrough, to remain so forever after. The availability of a recycling technology has more complex effects on consumption. We show, in particular, how it may reduce consumption before the breakthrough.

The general model is exposed in Section 2. Section 3 presents the optimal program of the economy. Then, we characterize the socially optimal trajectories and we study their properties in Section 4. In Section 5, we analyze how the availability of a recycling activity affects the economy. Section 6 concludes.

2 The model

We consider an economy where a final consumption good Y is produced from a raw material M and from labor L_Y according to the technology f. Denoting by A_Y the total factor productivity ("TFP" thereafter), the quantity produced at any time t is then given by the following expression:

$$Y(t) = f(A_Y(t), M(t), L_Y(t)), \tag{1}$$

where the production function $f(\cdot)$ is increasing and concave in each argument. We also assume that labor and physical materials are essential in production: $f(A_Y, 0, L_Y) = f(A_Y, M, 0) = 0$.

For simplicity, we take the growth of the TFP as exogenous. Denoting by g_{A_Y} the growth rate (positive and constant) of A_Y and by $A_{Y0} \equiv A_Y(0)$ the initial TFP index, we have $A_Y(t) = A_{Y0}e^{g_{A_Y}t}$.

The physical input M is made up of two types of materials: a non-renewable resource X – which we will hereafter refer to as the virgin resource – and a recycled secondary material Z – which we will refer to as the recycled resource. A quality index is associated to each of these materials. We denote by A_X the quality of the virgin resource, and by A_Z that of the recycled resource. $A_X(t)X(t)$ and $A_Z(t)Z(t)$ must then be viewed as the augmented material inputs that enter the production process at time t. In order to focus on the recycling-related activities, we assume that the quality index of the virgin resource is fixed and exogenous: $\bar{A} \equiv A_X(t) > 0$, $\forall t$. The quality index of the recycled resource is subject to improvements resulting from specific R&D activities. We assume that, as long as this quality is lower than a given fraction of the quality of the virgin resource, the recycled material cannot be introduced into the production process. Once its quality index has reached the minimum threshold $h\bar{A}$, with $h \in [0, 1]$, then it can be used in combination with the virgin material. In this case, we assume that both types of resources are perfect substitutes (as in Di Vita, 2001, or Pittel *et al.*, 2010). Consequently, the material input M can be expressed as follows:

$$M(t) = \begin{cases} \bar{A}X(t), & A_Z(t) < h\bar{A} \\ \bar{A}X(t) + A_Z(t)Z(t), & A_Z(t) \geq h\bar{A} \end{cases} . \tag{2}$$

Starting from a given initial level $A_{Z0} \equiv A_Z(0)$, the quality index $A_Z(t)$ of the recycled resource can be improved through the following endogenous R&D process:

$$\dot{A}_Z(t) = \delta L_A(t) A_Z(t), \tag{3}$$

where $\delta > 0$ is a parameter of productivity and $L_A(t)$ is the quantity of labor invested in this R&D activity at time t. For the problem to be meaningful, we clearly must assume that $0 < A_{Z0} < h\bar{A}$.

The economy is endowed with a fixed labor amount L, which can be devoted either to production or to R&D:

$$L_A(t) + L_Y(t) = L. \tag{4}$$

The virgin resource is extracted from a non-renewable stock according to a one-to-one technology: one unit of extracted resource yields one unit of virgin material. We assume that the extraction cost is negligible and then, the virgin resource cost is only captured by its scarcity rent. Denoting by $S(t)$ the stock of resource at time t, and by $S_0 \equiv S(0)$ the initial reserves, we have the following standard depletion process:

$$\dot{S}(t) = -X(t). \tag{5}$$

The consumption of $C(t)$ units of final good generates an instantaneous utility $u\left(C(t)\right)$ to consumers. The utility function $u(\cdot)$ satisfies the standard properties (increasing, concave, Inada conditions). Moreover, the utility flows are discounted by consumers at the social discount rate ρ, supposed to be positive and constant.

The final output production process generates waste that can be saved and reused. We assume that recycling is instantaneous, meaning that waste production and dismantling occur instantaneously and at the same time.[3] Within the production process, only the primary physical inputs – virgin and recycled resources – yield waste. For simplicity, the waste content rates of the virgin and recycled materials, α and β respectively, with $\alpha, \beta \in (0, 1)$, are taken as exogenous and constant. Moreover, we also assume that there is no natural degradation process. At any time, the incoming flow of waste is then $\alpha X(t) + \beta Z(t)$. Let $W(t)$ be the cumulative amount of waste at time t, and $W_0 \equiv W(0)$ the initial stock

[3]Note that, in this model, we do not consider specific costs of resource extraction or recycling. One could have considered that storing waste so that it can be used in the future is costly. Such costs could be expressed in terms of consumption good, labor, or we could assume that storing additional waste partially degrades the existing stock of waste. Such a feature of the model would complexify the intertemporal management of the stock of the non-renewable resource. Indeed, beyond reducing the remaining stock available for future use, the use of the resource at each date would have an instantaneous cost. We assume away such a cost in order to maintain the model tractable.

inherited from the past. As a flow of waste $Z(t)$ is eventually used by the recycling sector and thus removed from the accumulated stock $W(t)$, we can write:

$$\dot{W}(t) = \alpha X(t) - (1-\beta)Z(t). \tag{6}$$

In the remainder of the paper, we adopt the following conventional notations. We denote by φ_x the partial derivative of any function $\varphi(\cdot)$ with respect to variable x when this function contains more than one argument: $\varphi_x \equiv \partial\varphi(\cdot)/\partial x$. As usual, g_x characterizes the growth rate of variable x: $g_x(t) \equiv \dot{x}(t)/x(t)$. Last, for simplicity, we drop the time index when this causes no confusion.

3 The optimal program

The social planner program consists in determining the trajectories of resource extraction, waste recycling and efforts in R&D and production, that maximize the discounted sum of utility flows subject to the set of technical constraints. However, the problem turns out to be discontinuous since the final output has two different expressions depending on whether the quality index of the recycled material is smaller or larger than the threshold $h\bar{A}$. In this section, we consider separately these two successive phases.

Let T be the (endogenous) time at which A_Z is equal to $h\bar{A}$, *i.e.* the date at which recycling becomes operational. As $g_{A_Z} = \delta L_A \geq 0$, the trajectory of A_Z is always non-decreasing. Henceforth, if such a finite time T exists, then it is unique. We define respectively by $\mathcal{P}_1$ and $\mathcal{P}_2$ the social planner programs before and after time T, and we solve them backwards.

3.1 Recycling phase

Once the recycling option becomes available, *i.e.* after time T, the optimal program is:

$$(\mathcal{P}_2): \max_{\{X,Z,L_A,L_Y\}} \int_T^\infty u(C)e^{-\rho(t-T)}dt,$$

subject to the technological condition $C = f(A_Y, \bar{A}X + A_Z Z, L_Y)$, to the labor use condition (4), to the dynamic constraints (3), (5) and (6), and to the initial condition $A_Z(T) = h\bar{A}$. The following constraints on the control variables must also be satisfied:

$$X(t) \geq 0 \tag{7}$$

$$Z(t) \geq 0 \tag{8}$$

$$L_A(t), L_Y(t) \in [0, L]. \tag{9}$$

For the moment, we omit these non-negativity conditions which will be verified ex-post.

Denoting by λ_A, λ_S and λ_W the co-state variables associated with A_Z, S and W respectively, the optimal interior solution must satisfy the following first-order conditions:

$$u'(C)f_M\bar{A} = \lambda_S - \alpha\lambda_W \tag{10}$$

$$u'(C)f_M A_Z = (1-\beta)\lambda_W \tag{11}$$

$$u'(C)f_{L_Y} = \delta A_Z \lambda_A \tag{12}$$

$$\dot{\lambda}_S = \rho\lambda_S \tag{13}$$

$$\dot{\lambda}_W = \rho\lambda_W \tag{14}$$

$$\dot{\lambda}_A = (\rho - \delta L_A)\lambda_A - u'(C)f_M Z\,. \tag{15}$$

The transversality conditions are:

$$\lim_{t\to\infty} e^{-\rho(t-T)}\lambda_\kappa(t)\kappa(t) = 0, \quad \kappa = \{A_Z, S, W\}\,. \tag{16}$$

Conditions (10)-(12) state that the marginal social gain (in terms of utility) of one unit of input must be equal to its corresponding social marginal cost. More precisely, in (10), the marginal social gain of one unit of virgin resource equals the scarcity rent λ_S of the non-renewable resource stock, reduced by $\alpha\lambda_W$ to take into account that this unit generates waste up to α%, which accumulates into the stock W whose shadow value is given by λ_W. Note that, as long as no negative externality is associated with the stock of waste, λ_W works as a scarcity rent and is unambiguously positive. The same interpretation applies in (11) for the recycled resource, except that it does not involve the stock of natural resource but directly the stock of waste. Last, equations (12) is a standard static arbitrage condition relative to the labor allocation between either production or R&D. The left-hand side reads as the marginal social gain (in terms of utility) of increasing by one unit labor in production while the right-hand side represents the marginal social cost (in terms of knowledge value) of these labor reallocation resulting from a diminution of the effort devoted to R&D.

Conditions (13) and (14) imply that $\lambda_S(t) = \lambda_S(T)e^{\rho(t-T)}$ and $\lambda_W(t) = \lambda_W(T)e^{\rho(t-T)}$. Consequently, and conditionally on the fact that both resource stocks have a positive value at time T (*i.e.* $\lambda_S(T) > 0$ and $\lambda_W(T) > 0$), the transversality conditions (16) associated with S and W reduce to $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} W(t) = 0$. The stock of natural resource and the stock of waste must be asymptotically exhausted:

$$S(T) = \int_T^\infty X(t)dt \quad \text{and} \quad W(T) = \int_T^\infty [(1-\beta)Z(t) - \alpha X(t)]dt. \tag{17}$$

By replacing λ_W into (10) by its expression in (11), we obtain the following equation:

$$\frac{\lambda_S}{u'(C)} = \bar{A} f_M + \left(\frac{\alpha}{1-\beta}\right) A_Z f_M \,. \tag{18}$$

This equation states that the marginal social gain of virgin resource use expressed in units of good, that is the left-hand side of (18), and the total marginal productivity of the virgin resource, expressed by the right-hand side of (18), must be equal. Note that the total marginal productivity of the resource embodies the recycling possibility. Any unit of virgin resource extracted is indeed used a first time, which increases the output by $\bar{A} f_M$. This unit then generates α% of waste from which $(1-\beta)$% can be valued through recycling. The ratio $\alpha/(1-\beta)$ can be interpreted as the recyclability factor of the virgin resource. Multiplying this rate by the marginal productivity $A_Z f_M$ of the recycled material yields the second increase in production induced by the virgin resource through recycling. To simplify the forthcoming equations, let us denote by Ψ the total marginal productivity of the resource: $\Psi \equiv \bar{A} f_M + \left(\frac{\alpha}{1-\beta}\right) A_Z f_M$.

We now use Equation (18) to derive the two main conditions that characterize the socially-optimal intertemporal use of the virgin resource. Denoting by $\sigma(C)$ the inverse of the elasticity of intertemporal substitution, *i.e.* $\sigma(C) \equiv -Cu''(C)/u'(C)$, the growth rate of the marginal utility can be simply expressed as $-\sigma(C) g_C$. Log-differentiating (18) with respect to time and using (13), we obtain the first following intertemporal arbitrage condition:

$$\rho + \sigma(C) g_C = \frac{\dot{\Psi}}{\Psi}. \tag{19}$$

This is the Ramsey-Keynes condition in the specific context of our economy. The standard Ramsey-Keynes condition characterizes the socially optimal arbitrage made between consumption and capital accumulation. Here, the arbitrage is made between consumption and the use of the virgin resource. Assume, at date t, a marginal reduction of the production of consumption good through the diminution of resource use. At date $t + dt$, the economy uses this amount of resource whose total marginal productivity (as expressed in (18)) has increased while it was kept *in situ*. The extra amount of consumption good accordingly produced, represented by the right-hand side of (19), must be the amount of consumption that compensates households for the loss of one unit of consumption at date t, represented by the left-hand side of (19). What is new here is that, as previously mentioned, the total marginal productivity Ψ of the virgin resource features the term $\left(\frac{\alpha}{1-\beta}\right) A_Z f_M$, which accounts for the fact that the waste induced by the use of the virgin resource is recycled and used as an input for consumption good production.

The second dynamic arbitrage condition is obtained as follows. By log-differentiating (12) and replacing $\dot{\lambda}_A$ by its expression in (15), one obtains a new expression of the term $\rho + \sigma(C)g_C$. Inserting it in Condition (19) yields:

$$\frac{\dot{\Psi}}{\Psi} = \frac{\dot{f}_{L_Y}}{f_{L_Y}} + \frac{\delta A_Z Z f_M}{f_{L_Y}}. \tag{20}$$

Equation (20) can be seen as a Hotelling condition in the context of a dynamic general equilibrium framework, though modified in two ways. First, there is no physical capital but knowledge (intellectual capital) accumulation with intertemporal spillovers. Second, the virgin resource use yields waste that can be reused in the production process. The economic reasoning behind this condition is the following.

We first consider a given set of time profiles of all variables of the economy: we will refer to this benchmark as situation 1. We then assume a modification of the time profile of certain variables that keep the level of consumption good production unchanged at each date. We refer to this new set of time profiles as situation 2. We thus have $Y^1(t) = Y^2(t)$ for all t, equivalently $C^1(t) = C^2(t)$. At date t, the economy reallocates one unit of labor from production to research: $L_Y^2(t) - L_Y^1(t) \equiv \Delta L_Y(t) = -\Delta L_{A_Z}(t) = -1$. In order to maintain the level of output, the economy increases its use of virgin resource by $\Delta X(t) = f_{L_Y}/\Psi$ that is, the marginal productivity of labor in output production expressed in terms of resource[4]. Then, $X^2(t) = X^1(t) + \Delta X(t)$.

At date $t+dt$ (with $dt \to 0$), the economy reallocates the unit of labor from research to production: $L_Y^2(t+dt) = L_Y^1(t+dt)$ and $L_{A_Z}^2(t+dt) = L_{A_Z}^1(t+dt)$. Between t and $t+dt$, the stock of knowledge A_Z has increased due to the additional unit of labor in research within this interval and due to the (accordingly) increased level of knowledge A_Z (through intertemporal knowledge spillovers). This allows the economy to maintain the level of output production ($Y^2(t+dt) = Y^1(t+dt)$) while saving a certain amount of virgin resource, $\Delta X(t+dt)$. To express $\Delta X(t+dt)$, one has to consider the extra amount of output produced through the increased effort in research: $d\left[\delta A_Z Z f_M\right]/dt + \delta A_Z Z f_M$, where $\delta A_Z Z f_M$ is the increase in output production induced by a marginal increase in labor devoted to research[5]. Expressed in terms of resource, this amount is: $\left[d\left(\delta A_Z Z f_M\right)/dt + \delta A_Z Z f_M\right]/\Psi$. Then,

[4] From $\Delta Y = f_{A_Y}\Delta A_Y + f_M \Delta M + f_{L_Y}\Delta L_Y$, with $\Delta L_Y = -1$ and $\Delta A_Y = 0$ (as the discrete changes in stocks at a given time t are nil), then ΔM must be equal to $\frac{f_{L_Y}}{f_M}$ to maintain the same level of output. The economy can thus compensate for the amount of labor devoted to production by increasing its use of virgin resource X and/or recycled resource Z (recall that $\Delta M = \bar{A}\Delta X + A_Z \Delta Z + Z \Delta A_Z$, with $\Delta A_Z = 0$). Moreover, as $\Delta W = 0$, we must have $\Delta Z = \left(\frac{\alpha}{1-\beta}\right)\Delta X$ from (6). Then, $\Delta X = \frac{\Delta M}{\bar{A} + \left(\frac{\alpha}{1-\beta}\right)A_Z}$.

[5] Indeed, δA_Z is the marginal productivity of labor in research (see (3)) and $Z f_M$ the marginal productivity of knowledge dedicated to the recycled resource.

by dividing this expression by the rate of growth of the total marginal productivity of the resource, we take into account that the productivity of the resource evolves over the interval $(t; t+dt)$. We thus have $\Delta X(t+dt) = \frac{\Psi}{d\Psi/dt} \times \frac{d[\delta A_Z Z f_M]/dt + \delta A_Z Z f_M}{\Psi}$. Since $\delta A_Z Z f_M = f_{L_Y}$, that is, the marginal productivity of labor is the same in production and research, we obtain $\Delta X(t+dt) = \frac{\dot{f}_{L_Y} + \delta A_R f_{A_R}}{d\Psi/dt}$. Condition (20) states that, at the optimum, $\Delta X(t)$ and $\Delta X(t+dt)$ must be equal; in other words, the labor transfer from output production to research does not allow to save virgin resource.

Finally, note that from (10)-(11) and (13)-(14), the marginal productivity of the natural resource and of the recycled material, respectively $\bar{A} f_M$ and $A_Z f_M$, must grow at the same rate. This result is driven by the assumption of perfect substitution between the two types of resources. The additional assumption of a constant quality index for the virgin resource then simplifies the analysis as it implies that the quality index of the recycled resource must also be constant. An immediate consequence is that no more effort in R&D is made once the quality of the secondary raw material has reached the required (minimum) threshold:

$$\forall t \geq T: \; A_Z(t) = h\bar{A}, \quad L_A(t) = 0 \quad \text{and} \quad L_Y(t) = L. \tag{21}$$

3.2 Pre-recycling phase

Before time T, as the secondary material cannot be used for production yet, we have $M = \bar{A}X$ from (2). Denoting by $V_2(A_Z(T), S(T), W(T))$ the value function of program $\mathcal{P}_2$ at time T, we can write the initial program $\mathcal{P}_1$ as follows:

$$(\mathcal{P}_1): \max_{\{X, L_A, L_Y\}} \int_0^T u(C) e^{-\rho t} dt + e^{-\rho T} V_2(A_Z(T), S(T), W(T)),$$

subject to the condition $C = f(A_Y, \bar{A}X, L_Y)$, to the labor use constraint (4), and to the dynamic constraints (3), (5), (6). Note that, as the cumulative waste equation (6) is now reduced to $\dot{W} = \alpha X$, the trajectories of the resource reserves and of the waste stock are linked through the following relation:

$$W(t) = W_0 + \alpha(S_0 - S(t)), \; \forall t \in [0, T).$$

The first-order conditions of $\mathcal{P}_1$ are very similar to those of $\mathcal{P}_2$. Conditions (10), (12), (13) and (14) are the same. Condition (11) is no longer valid, whereas (15) becomes:

$$\dot{\lambda}_A = (\rho - \delta L_A)\lambda_A. \tag{22}$$

Note that even if some conditions have the same expression, the anticipation of the recycling option availability at time T is captured by the shadow prices, which may follow different trajectories than under $\mathcal{P}_2$. The transversality conditions at time T are:

$$\lambda_A(T^-) = \frac{\partial}{\partial A_Z(T^+)} V_2(A_Z(T^+), S(T^+), W(T^+)) \tag{23}$$

$$\lambda_S(T^-) = \frac{\partial}{\partial S(T^+)} V_2(A_Z(T^+), S(T^+), W(T^+)) \tag{24}$$

$$\lambda_W(T^-) = \frac{\partial}{\partial W(T^+)} V_2(A_Z(T^+), S(T^+), W(T^+)). \tag{25}$$

Last, the intertemporal trade-off condition writes:

$$\rho + \sigma(C) g_C = \frac{\dot{f}_M}{f_M} = \frac{\dot{f}_{L_Y}}{f_{L_Y}}, \tag{26}$$

which means that, under $\mathcal{P}_1$, the productivity of the resource and of labor must grow at the same rate.

4 Optimal trajectories of the economy

To illustrate the recycling problem with endogenous technical breakthrough and provide an example of optimal trajectories, we consider the following standard functional forms. Utility is characterized by a CES function of parameter $\sigma > 0$ and output production is described by a Cobb-Douglas function of parameter $\epsilon \in (0,1)$: $u(C) = C^{1-\sigma}/(1-\sigma)$ and $f(A_Y, M, L_Y) = A_Y M^{\epsilon} L_Y^{1-\epsilon}$.

Using these specified analytical forms, we study in this section the main qualitative properties of the optimal paths we have obtained. In particular, we explain their behavior at the time the economy switches from the pre-recycling to the recycling phases. We also analyze the sensitivity of the optimal variables to some key parameters of the model.

4.1 Qualitative properties of the optimal trajectories

The computational details of the social planner's solution as well as the existence conditions of this solution are described in Appendix A.1.

Labor allocation

Formal expressions of L_A and L_Y are given by:

$$L_Y(t) = \begin{cases} L_{Y0} e^{-\tilde{k}t} & , t < T \\ L & , t \geq T \end{cases} \quad \text{and} \quad L_A(t) = \begin{cases} L - L_{Y0} e^{-\tilde{k}t} & , t < T \\ 0 & , t \geq T \end{cases}, \tag{27}$$

where the initial level of productive labor is $L_{Y0} \equiv L_Y(0) = \frac{\tilde{k}(1-\epsilon)(1-\beta)S_0}{\delta \epsilon h(W_0 + \alpha S_0)}$. The initial quality index of recycled material A_{Z0} is not high enough to allow using this input. Before time

T, the effort L_A devoted to the improvement of the recycled resource quality continuously rises and it stops once the required quality threshold is reached. The effort in R&D thus instantaneously falls to zero at time T, as depicted in Figure 1-a. Consequently, since the total labor supply L is constant (c.f. (4)), the effort in production L_Y declines throughout the first phase, then jumps to level L at date T, and remains at this level onwards.

The quality index of the recycling material is exponentially increasing until T and then forever equal to $h\bar{A}$ (see expression (A.27) in the appendix). The optimal switching time T is endogenously determined in such a way that A_Z is continuous.[6]

Virgin resource use

The optimal path of the virgin resource use is given by:

$$X(t) = \begin{cases} \tilde{k} S_0 e^{-\tilde{k}t} & , t < T \\ k S_0 e^{(k-\tilde{k})T - kt} & , t \geq T \end{cases} \quad \text{with } \tilde{k} \equiv \frac{\rho - (1-\sigma) g_{A_Y}}{\sigma}, \; k \equiv \frac{\rho - (1-\sigma) g_{A_Y}}{1 - \epsilon(1-\sigma)}. \tag{28}$$

Resource use is always exponentially decreasing through time – at rate $\tilde{k}$ during the pre-recycling phase and at rate k during the recycling phase – and it asymptotically tends towards zero. In this sense, it follows a standard Hotelling depletion process, but discontinuous here. Let $\Delta X(T) \equiv X(T^+) - X(T^-)$ denote the magnitude of the jump made by X at time T. From (28), we have $\Delta X(T) = (k - \tilde{k}) S_0 e^{-\tilde{k}T}$, which is negative as $\tilde{k} - k = (1-\sigma)(1-\epsilon)k/\sigma > 0$. This means that virgin resource use jumps down at time T and then follows a less sloping declining path, as illustrated in Figure 1-b. At that time indeed, the constraint on the virgin resource consumption is partially relaxed since i) recycling becomes operational, and ii) the whole labor flow is allocated to production (see below). Consequently, the resource stock, as given by expression (A.24) in Appendix A.1, is continuously declining until its full exhaustion, but its trajectory is less steep declining after T than before T.

Recycling activity and waste accumulation

The optimal trajectories of Z and W are given by:

$$Z(t) = \begin{cases} 0 & , t < T \\ k\Phi S_0 e^{-k(t-T)} & , t \geq T \end{cases} \quad \text{with } \Phi \equiv \frac{W_0 + \alpha S_0}{(1-\beta) S_0}, \tag{29}$$

$$W(t) = \begin{cases} W_0 + \alpha S_0 (1 - e^{-\tilde{k}t}) & , t < T \\ \left[W_0 + \alpha S_0 (1 - e^{-\tilde{k}T}) \right] e^{-k(t-T)} & , t \geq T \end{cases} . \tag{30}$$

[6] We show in Appendix A.1 that the optimal date T is determined as the solution of the following equation: $\delta L T = \ln\left(\frac{h\bar{A}}{A_{Z0}}\right) + \frac{(1-\epsilon)(1-\beta)S_0}{\epsilon h (W_0 + \alpha S_0)} \left(1 - e^{-\tilde{k}T}\right)$.

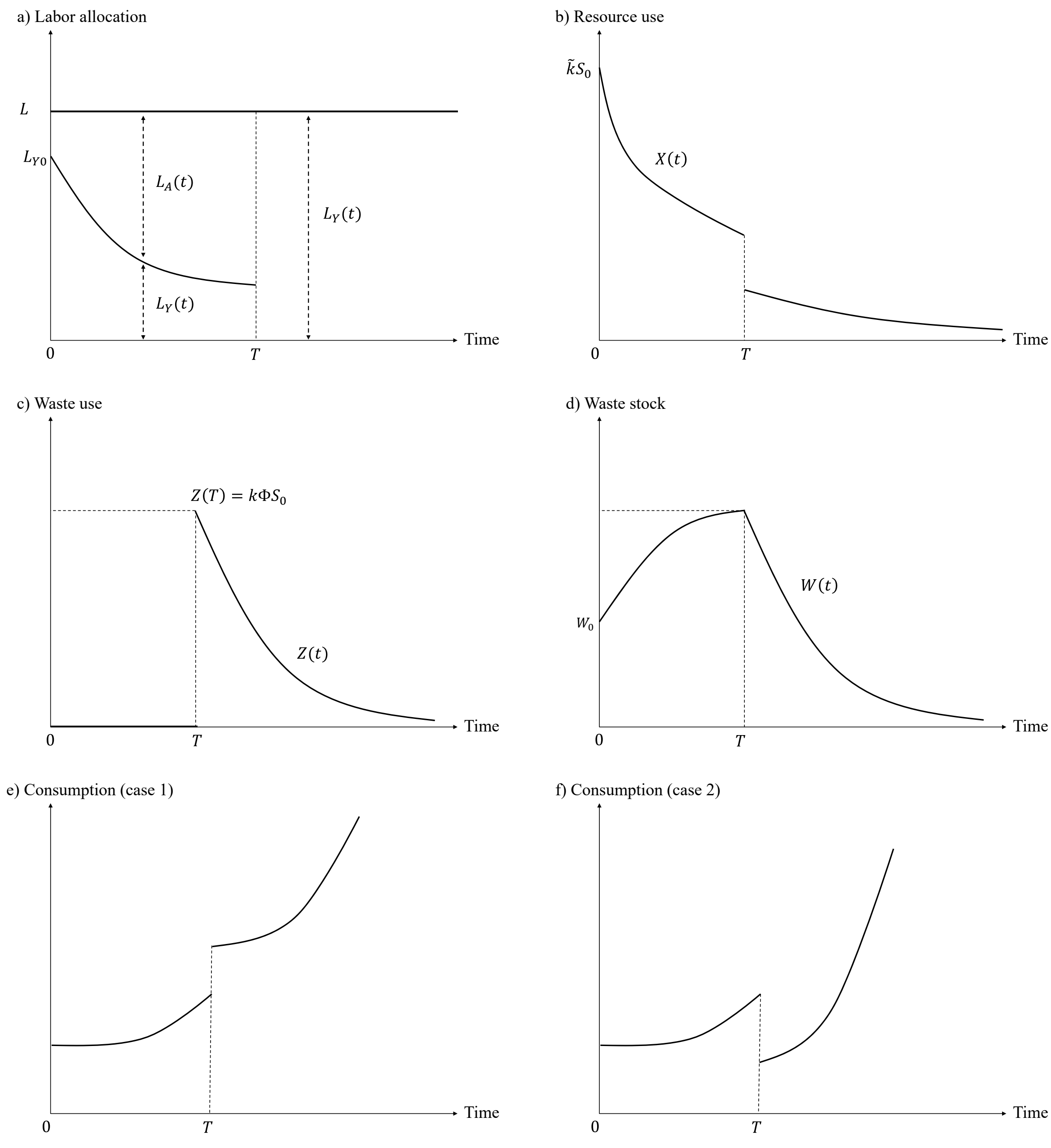

Figure 1: Optimal trajectories of the model

In (29), Φ can be interpreted as the maximal recycling rate of the virgin resource stock. Indeed, we can verify that the total use of the recycled resource $\int_T^\infty Z(t)dt$ amounts to $(W_0 + \alpha S_0)/(1 - \beta)$, which formally reads as the maximal quantity of waste that can be generated over the planning horizon divided by the net recycling rate of the secondary material. By dividing this expression by S_0, we obtain a ratio reflecting the maximal recycling potential of the virgin resource.

As shown in Figure 1-c, the flow of recycled material is first nil. At time T, it jumps upwards to its maximal value $Z(T) = k\Phi S_0$ and then behaves as $X(t)$ by following a trajectory that exponentially declines at rate k and that asymptotically converges towards zero. This partially explains the fact that, as previously mentioned (see Figure 1-b), virgin resource use is more intensive before T. During the pre-recycling phase indeed, resource use has two purposes: first, the immediate production of output in order to meet consumption needs; second, the accumulation of a stock of waste that will be used to produce recycled material during the second phase. Since the maximum of the recycling activity is reached at time T and then steadily declines, the waste stock needs to be high enough at this date.

The stock of waste, yielded by the use of the virgin and recycled resources, is first increasing before T and next declining until exhaustion (see Figure 1-d).

Consumption

The translation of the previous results in term of consumption is complex. On the one hand, as we show below, the optimal consumption trajectory can be either increasing or decreasing. On the other hand, the nature of the jump at time T depends on the relative magnitude of the jumps made by the different inputs in the production function.

The consumption path is given by:

$$C(t) = \begin{cases} C_0 e^{\tilde{g}_C t} & , \ t < T \\ C_T e^{g_C (t-T)} & , \ t \geq T \end{cases} , \tag{31}$$

where $C_0 \equiv C(0) = A_{Y0}(\bar{A}X_0)^\epsilon L_{Y0}^{1-\epsilon}$ and $C_T \equiv C(T) = A_Y(T)[\bar{A}X(T) + h\bar{A}Z(T)]^\epsilon L^{1-\epsilon}$. The growth rates of consumption during the pre-recycling and recycling phases, respectively denoted by $\tilde{g}_C$ and g_C, are:

$$\tilde{g}_C = \frac{g_{A_Y} - \rho}{\sigma} \quad \text{and} \quad g_C = \frac{g_{A_Y} - \epsilon\rho}{1 - \epsilon(1 - \sigma)} . \tag{32}$$

We show in Appendix A.1 that $g_C > \tilde{g}_C$. In other words, consumption grows faster – or decreases more slowly in case of negative rates – after date T.

We first consider the sign of these growth rates. We know that consumption is a combination of three factors (see (1)): TFP (A_Y), material input (M) and labor (L_Y). Its potential growth is only (exogenously) driven by A_Y, as L_Y is declining during the pre-recycling phase and constant afterwards, and M is always decreasing (as a linear combination of virgin and recycled resources, both being declining as previously shown). More precisely, we show that $\tilde{g}_C$ and g_C can be positive or negative depending on the level of the TFP growth rate g_{A_Y} relative to the social discount rate ρ and the input substitution parameter ϵ. The three following cases can occur:

$$\begin{cases} \rho < g_{A_Y} & \Rightarrow \quad g_{\tilde{C}} > 0 \text{ and } g_C > 0\,, \\ g_{A_Y} \leq \rho < g_{A_Y}/\epsilon & \Rightarrow \quad \tilde{g}_C \leq 0 \text{ and } g_C > 0\,, \\ g_{A_Y}/\epsilon \leq \rho & \Rightarrow \quad g_{\tilde{C}} < 0 \text{ and } g_C \leq 0\,. \end{cases}$$

As usual, time impatience favors immediate consumption to the detriment of future consumption. Consequently, the larger the social discount rate, the weaker the consumption growth rate, with negative values below a given threshold. For intermediate values of ρ, the optimal growth path may be U-shaped (with a discontinuity at the bottom of the U): decreasing over time during the non-recycling phase, and then increasing during the recycling phase. For simplicity, and to reduce the number of scenarios, we only illustrate in Figures 1-e and 1-f cases where both g_C and $\tilde{g}_C$ are positive, that is cases where the social discount rate is not too high.

We now turn to the discontinuity of C at time T. The combination of a downward jump in virgin resource and upward jumps in both recycled material and productive labor (see previous findings) results, *a priori*, in an undetermined overall jump in consumption. Conclusions depend on the relative magnitude of jumps in inputs and on the substitutability properties of the production function. We show in Appendix A.2 that large values of Φ (the maximal recycling rate of the virgin resource stock) are a sufficient condition for consumption to jump upwards at time T. This case is illustrated in Figure 1-e while the other case – the case of a downward jump in consumption – is represented in Figure 1-f. However, we also show in Appendix A.2 that the jump is always positive in the particular case where $\sigma = 1$ (i.e. with a logarithm utility function).

Recall that utility only depends on consumption here. For this reason, the preceding analysis of the jump in consumption shows that the technological breakthrough may have a negative impact on utility at the time T of its occurrence and over a certain period of adjustment (see Figure 1-f), before the pre-recycling utility level is reached again.

4.2 Comparative dynamic analysis

We now perform some comparative dynamics so as to analyze the sensitivity of the most relevant variables, including the date T of occurrence of the technological breakthrough, to the exogenous parameters of the model. We do not intend to be exhaustive here, we rather focus on the main insights – a complete presentation of the comparative dynamics is provided in Appendix A.3, Table 1.

Parameters characterizing the preferences

An increase in the social discount rate ρ means that the representative household obtains more utility from current consumption relative to future consumption. In order to increase consumption today, the social planner increases early extraction, which means that future extraction gets lower. The extraction of the virgin resource before T is thus accelerated: g_X decreases (recall that $g_X = \tilde{k}$). Simultaneously, a higher ρ entails stronger initial efforts in production: L_{Y0} is increased. As a consequence, the initial effort in research gets lower, that is L_{A0} decreases, and g_{A_Z}, the growth of the quality of the recycled resource, is slower. Hence, for a given initial quality A_{Z0}, the date at which the recycled resource reaches the minimum quality threshold $h\bar{A}$ – and starts to get used in the production process – is postponed.

An increase in the elasticity of marginal utility σ means that the representative household derives more utility from a uniform consumption path, *ceteris paribus*. Here, in order to achieve a flatter consumption path, the social planner invests less in R&D (investing would imply a higher consumption tomorrow): L_{A0} and g_{A_Z} both decrease. This too entails a later occurrence of the technological breakthrough: T increases.

Parameters characterizing the output production technology

In our framework, the total factor productivity (TFP) A_Y is taken as exogenous and growing at constant rate g_{A_Y}. A higher growth of TFP allows the planner to slow down resource extraction between 0 and T: less resource is used during the early periods and more in the future periods. In other words, g_X (that is, $\tilde{k}$) gets higher for all $t < T$. For a given level of A_{Y0}, the increase in g_{A_Y} entails higher A_Y for all t. Therefore, the planner devotes less labor to the production of output: L_{Y0} and g_{L_Y} both get lower. Consequently, the effort in research L_A is higher at each date $t < T$. Thus R&D is more intensive, the quality of the recycled resource grows faster, and the technological breakthrough is reached earlier.

An increase in the output elasticity of raw material ϵ means that the material input M gets more productive (relative to labor). In this case, the economy relies more heavily on the flows of resource and hence the need for an additional source of material is more pressing. Therefore the economy transfers labor from production to research and by this makes the technological breakthrough happen sooner: T gets lower.

Parameters characterizing the research sector

Parameter $\bar{A}$ characterizes the quality level of the virgin resource. With a higher $\bar{A}$, the economy requires that the recycled resource reaches a higher quality threshold before it starts to get used in the production process. *Ceteris paribus* and, in particular, for a given investment in R&D, this postpones the date T of the breakthrough. The impact of A_{Z0}, the initial quality level of the recycled resource, is obviously opposite. A higher A_{Z0} means that the distance to (a given) level $h\bar{A}$ is shorter; the quality threshold is thus reached faster, and the technological breakthrough occurs at an earlier date.

A higher δ means that the R&D sector is more efficient. In such a case, L_{Y0} is lower while g_{L_Y} is unchanged. This means that, at each date $t < T$, the effort in research L_A is higher and the quality of the recycled resource grows faster. The breakthrough occurs earlier here too.

Parameters characterizing the recycling sector

With a higher S_0, the initial stock of virgin resource is larger and the need for a complementary resource is thus less urgent. Hence the social planner can use more virgin resource at each date between 0 and T, that is X gets higher, and devotes more labor to production and thus less to research: L_A is lower. Thus the quality of the recycled resource grows less fast and the technological breakthrough occurs later.

The initial stock of waste W_0 and the waste content rates of the two resources, α and β have similar effects on the economy's trajectories and henceforth on the date of the technological breakthrough. A larger initial stock of waste W_0 means that, for a given path of resource use between 0 and T, the stock of recycled resource to be used from date T on is larger. This makes the technological breakthrough more significant. Hence the economy increases its effort in R&D, that is $L_A(t)$ gets higher for all $t < T$, and the recycled resource starts to get used sooner in the production process. Similarly, if the waste content of the virgin resource, α, and the waste content of the recycled resource, β, are higher, the planner gets an incentive to invest more in research, and the technological breakthrough occurs earlier. We have introduced $\Phi \equiv \frac{W_0+\alpha S_0}{(1-\beta)S_0}$ as the maximal recycling rate; a higher Φ thus also brings forward the date of the breakthrough.

5 Effects of the availability of a recycling technology

We analyze here how the availability of a recycling technology affects the trajectory followed by the economy. To do so, we consider three cases. First, an economy in which no recycling technology is available at any date t. Hereafter, we refer to this case as the "never-recycling economy". This is obviously a particular case of the economy studied in the last sections, in which date T tends towards infinity (equivalently, $h \to \infty$, see (2)). Second, we consider an economy in which the recycling option is available at date 0. In other words, the quality threshold at which the recycled resource starts to be used is instantaneously met: $T = 0$, and thus $A_{Z0} = h\bar{A}$. We will refer to this case as the "always-recycling economy". The third case is the general case studied in Sections 2-4 in which recycling is possible after date T. We will refer to this case as the "T-recycling economy".

We first characterize the trajectories of the never and always-recycling economies. In both cases, we can easily show that the marginal social value of (A_Z-type) R&D is nil. Consequently, at any point in time, the total amount of available labor is allocated to production and the set of control variables reduces to the uses of raw material only. However, in this "cake-eating" problem, the nature and the availability of the stocks of resources differ in each case. In the never-recycling economy, only the virgin resource can be used and the final consumption good is produced according to the following technological form: $C = A_Y(\bar{A}X)^\epsilon L^{1-\epsilon}$. In the always-recycling economy, thanks to recycling activities, the initial stock of waste W_0 can be exploited at rate Z as a complementary resource: $C = A_Y(\bar{A}X + h\bar{A}Z)^\epsilon L^{1-\epsilon}$.

The optimal trajectories of such a program are the following: [7]

$$
\begin{aligned}
X_n(t) &= X_a(t) = kS_0e^{-kt} \quad \text{and} \quad S_n(t) = S_a(t) = S_0e^{-kt}\,, && (33)\\
Z_n(t) &= 0 \quad \text{and} \quad Z_a(t) = k\Phi S_0e^{-kt}\,, && (34)\\
W_n(t) &= W_0 + \alpha S_0\left(1 - e^{-kt}\right) \quad \text{and} \quad W_a(t) = W_0e^{-kt}\,, && (35)\\
C_n(t) &= C_{n0}e^{g_C t}, \text{ with } \quad C_{n0} = A_{Y0}(\bar{A}kS_0)^{\epsilon}L^{1-\epsilon}\,, && (36)\\
C_a(t) &= C_{a0}e^{g_C t}, \text{ with } \quad C_{a0} = A_{Y0}(\bar{A}kS_0)^{\epsilon}(1+h\Phi)^{\epsilon}L^{1-\epsilon}\,, && (37)
\end{aligned}
$$

From (33), virgin-resource use in the never-recycling economy is identical to virgin-resource use in the always-recycling economy. As shown in Section 4.1, the emergence of a technological breakthrough at date $T > 0$, which characterizes the T-recycling economy, yields a discontinuity in the trajectory of resource use at date T. These three extraction paths are depicted in Figure 2-a. Before the breakthrough, we can see that the T-recycling economy over-exploits the resource – as compared to the never and always-recycling economy. Indeed, part of the labor flow L_Y directed to production in the never and always-recycling economies is affected to R&D in the T-recycling economy. The T-recycling economy thus compensates for this lower input use by using higher flows of resource X. Moreover, this yields additional flows of waste: the stock of waste that can be recycled from date T on is thus higher. After the breakthrough, the whole amount of labor available L is devoted to production and the recycled resource Z starts to get used. This allows the T-recycling economy to use lower levels of virgin resource X at each date $t > T$. Figure 2-b illustrates the over-exploitation of the virgin resource stock S by the T-recycling economy at each date $t > 0$.

Figure 2-c depicts the trajectories of the waste stock. From (35), the stock of waste W_n in the never-recycling economy grows over time and asymptotically tends to its upper limit level $W_0 + \alpha S_0$ (see the dashed curve in Figue 2-c). In other words, as the never-recycling economy uses the non-renewable resource, the associated waste adds to the existing stock until the whole resource stock is exhausted. The stock of waste W_a in the always-recycling economy exponentially declines from W_0 down to 0. This means that the amount of waste produced is instantaneously re-used by the production process and that the stock of waste follows a continuous depletion process. The impact of the immediate availability of a

[7]The subscript n denotes variables characterizing the never-recycling economy and the subscript a, variables characterizing the always-recycling economy.

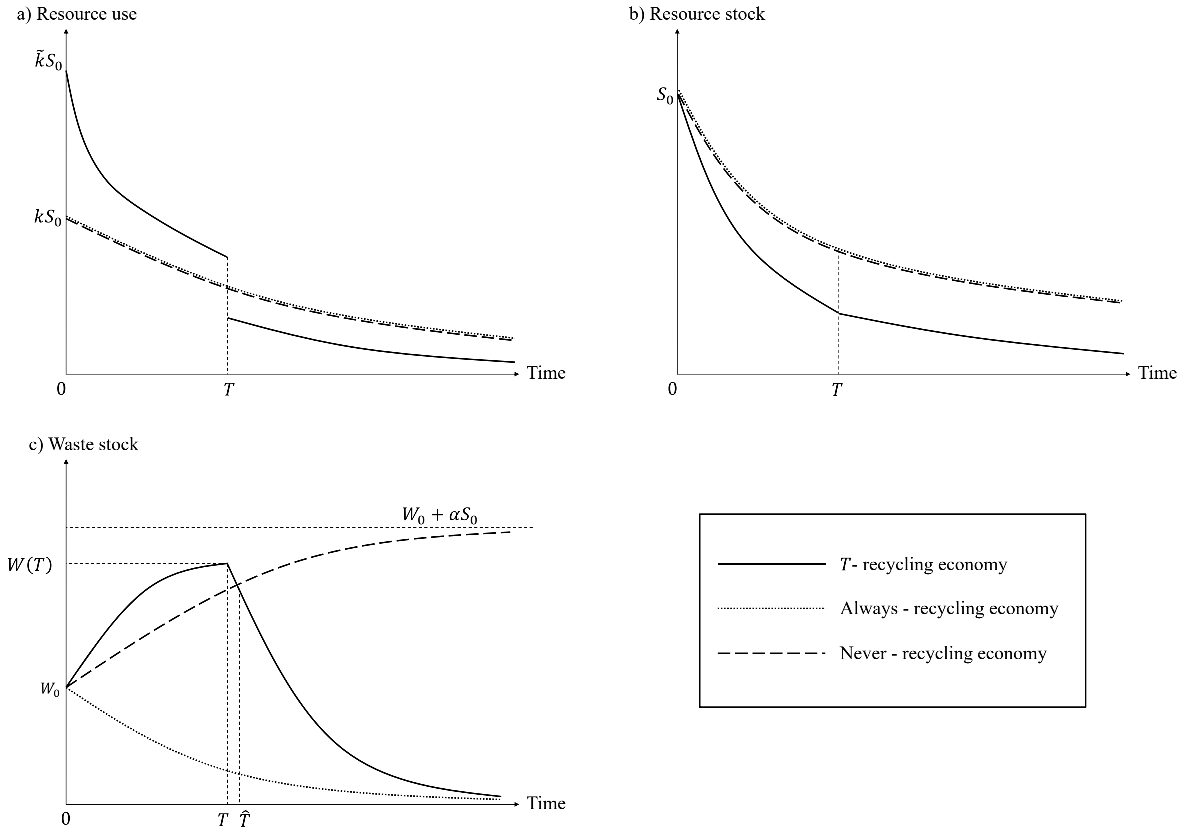

Figure 2: Effect of the availability of the recycling technology

recycling technology is therefore unambiguous: the stock of waste is lower at each date $t > 0$.

Equation (30) presents the trajectory of the stock of waste in the T-recycling economy. Before date T, W increases and is higher than W_n at each date. At date T, the stock reaches its maximum level $W_0 + \alpha S_0(1 - e^{-\tilde{k}T})$ and then starts to steadily decline, to asymptotically converge towards 0 since, in the long-run, the T-recycling economy uses waste until its stock gets exhausted (see Section 4.1). Conversely, the never-recycling economy keeps accumulating waste after date T and W_n gets higher than W at any date $\hat{T} > T$. In other words, the recycling option makes the stock of waste larger until date $\hat{T}$. After this date, the stock of waste is lower in the T-recycling economy.[8]

From the expressions (36) and (37), one can see that the growth rate of consumption is identical in the never-recycling and the always-recycling economies. Moreover, it is straightforward that $C_{a0} > C_{n0}$. One can thus conclude that the immediate availability of a recycling technology has a positive impact on consumption: $C_a(t) > C_n(t)$ for all t.

Comparing consumption in the T-recycling economy with consumption in the always and never-recycling cases involves numerous complex scenarios that depend on various parameters of the model, notably including Φ. Deriving clear conclusions is not possible at this stage. However, in the specific case of a logarithm utility, that is when $\sigma = 1$ (a case in which the jump in consumption in the T-recycling economy is always positive as already mentioned), we show in Appendix A.2 that the consumption paths verify the following ranking: $C_{a0} > C_{n0} > C_0$ and $C(T^+) > C_a(T) > C_n(T)$, as illustrated in Figure 3. For any $t < T$, the T-recycling economy is characterized by consumption levels lower than those in the always and never-recycling economies. Conversely, once the technological breakthrough has occurred, consumption in the T-recycling economy is higher than in the other two economies.

6 Concluding remarks

The aim of this paper is first to study the socially optimal path of an economy that needs to invest in research so that recycling becomes operational. We also want to understand how

[8]Regardless of its date of occurrence, the maximal level reached by a stock of pollutant is a serious concern for many – since irreversible damages may occur after certain thresholds. Here, this maximal level $W(T)$ is reached earlier by the T-recycling economy; however, it is lower than the maximal level reached by the never-recycling economy. Indeed, the difference between the two is equal to $\left[W_0 + \alpha S_0(1 - e^{-\tilde{k}T})\right] - (W_0 + \alpha S_0)$, which is negative since $e^{-\tilde{k}T} > 0$.

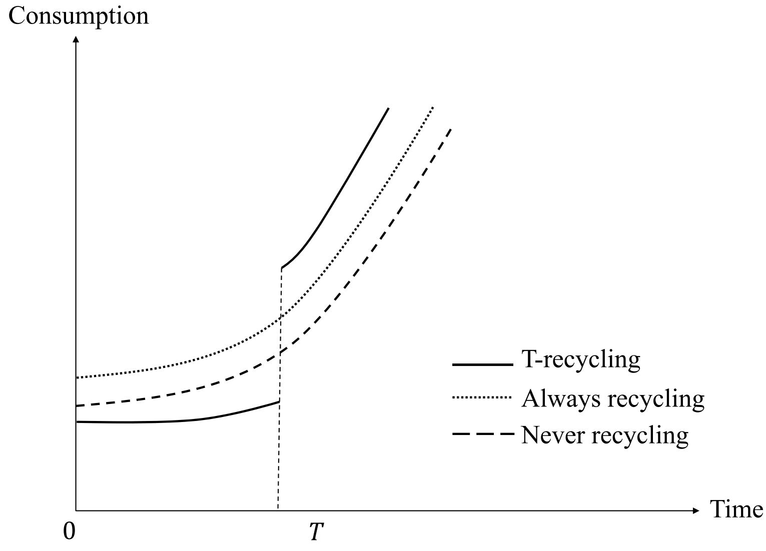

Figure 3: Optimal consumption – Case $\sigma = 1$

the emergence of the recycling technology – what we call the technological breakthrough – affects the economy. To do so, we utilize a dynamic model with recycling and R&D, in which the use of recycled materials produced from waste requires a prior investment in research so that the quality of the recycled resource meets a certain standard.

We first present the general optimality conditions derived from the social planner's program. We characterize a Ramsey-Keynes condition and a Hotelling condition that take new forms in the presence of a recycling activity; we also provide a full interpretation of these conditions. In particular, we show how these conditions tackle the joint dynamics of the remaining stock of virgin resource and of the accumulated stock of waste, the latter stock being partially renewable as it is fed by flows entailed by the virgin resource use.

We then study the socially optimal trajectories of the economy, and especially the discontinuity induced by the technological breakthrough. At the time of the breakthrough, resource use jumps down and then follows a less sloping declining path. The jump in consumption is ambiguous. If the social discount rate is not too high, consumption grows over time; however, depending on the values of parameters such as the maximal recycling rate of the virgin resource, consumption can jump upwards or downwards at the time of the breakthrough. In the latter case, the technological breakthrough has a negative impact on utility during a finite period. We also consider the sensitivity of the optimal trajectories to the exogenous parameters. In particular, we study how these parameters affect the date of the technological breakthrough.

Next, we compare the general trajectories of this economy to two particular cases: an economy in which recycling is never possible and an economy where recycling is immediately operational. This allow us analyzing the impact of the recycling activity and its timing. We show that, as compared with these two particular cases, the economy over-exploits the virgin resource before the breakthrough, and then uses lower flows after this date. Besides, the waste stock is initially higher than in the never-recycling economy, but eventually becomes lower after a finite interval of time following the breakthrough and remains so forever after. The impact of the availability of a recycling technology on consumption is more complex. It results from combined effects on the different inputs. As previously mentioned, the virgin resource use is accelerated; besides, part of the whole labor force is directed from production to research until the breakthrough occurs. According to the relative strengths of these effects, which depend on the exogenous parameters of the model (and notably the maximal recycling rate), the availability of the recycling technology may reduce consumption before the breakthrough; this is what we illustrate in the simple case of a logarithm utility function.

To get a tractable framework, we have made simplifying hypotheses. One major simplification is that we assume away environmental externalities. The use of many resources like, say plastics, yields flows of waste and their accumulated stocks are a major concern for public health and ecosystems - see for instance the "Great Pacific Garbage Patch" (UNEP, 2016). One characteristic of most recycling activities is that, besides from producing additional inputs for production, they allow reducing such pollution stocks and thus induce additional welfare gains. In other respects, it is well known that the use of many non-renewable resources in the production process yields greenhouse gas (GHG) emissions. However, using virgin materials to produce output does not yield the same amount of GHG than using recycled materials. "Producing new products with secondary materials can save significant energy. For example, producing aluminum from recycled aluminum requires 95% less energy than producing it from virgin materials." (World Bank, 2012). As mentioned before, the availability of a recycling technology accelerates resource use before the breakthrough. This means that, in the absence of environmental externality, the recycling option accelerates GHG emissions. This result may be nuanced in a framework in which, for instance, utility is negatively affected by carbon accumulation.

Finally, the introduction of environmental externalities in this framework paves the way for the study of pro-recycling policy that may be necessary for a decentralized economy to achieve the type of first-best outcomes analyzed in the present paper. We leave this for future research.

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Appendix A.1 Computation of the optimal trajectories

A.1.1 Analytical solution of program $\mathcal{P}_2$

We know that, for $t \geq T$, $A_Z(t) = h\bar{A}$, $L_A(t) = 0$ and $L_Y(t) = L$, which implies $g_{A_Z} = g_{L_Y} = 0$. Given that $C = A_Y M^\epsilon L_Y^{1-\epsilon}$, equations (19) and (20) can be rewritten as:

$$\rho + \sigma g_C = \frac{\dot{f}_M}{f_M} \quad \Rightarrow \quad \rho - (1-\sigma) g_C = -g_M \tag{A.1}$$

$$\rho + \sigma g_C = g_C - g_{L_Y} + \frac{\delta h \bar{A} Z f_M}{f_{L_Y}} \quad \Rightarrow \quad \rho - (1-\sigma) g_C = \frac{\delta \epsilon h \bar{A} L Z}{(1-\epsilon) M}\,. \tag{A.2}$$

As $g_C = g_{A_Y} + \epsilon g_M + (1-\epsilon) g_{L_Y} = g_{A_Y} + \epsilon g_M$, equation (A.1) implies:

$$g_C = \frac{g_{A_Y} - \epsilon \rho}{1 - \epsilon(1-\sigma)}\,. \tag{A.3}$$

Combining (A.1) with (A.2), we obtain $g_M M = -(\delta \epsilon h \bar{A} L Z)/(1-\epsilon)$, which imposes g_M to be negative. Replacing M by $\bar{A}(X + hZ)$ in this last expression, we have:

$$\frac{X}{hZ} = -\frac{\epsilon \delta L}{(1-\epsilon) g_M} - 1\,. \tag{A.4}$$

As g_C is constant (from (A.3)), g_M is constant too (from (A.1)). Log-differentiating (A.4) with respect to time, we obtain $g_X = g_Z$. Next, differentiating $M = \bar{A}(X + hZ)$ with respect to time yields $\dot{M} = \bar{A}(g_X X + g_Z hZ) = g_X M$, and then $g_M = g_X$. Finally, we obtain:

$$g_X = g_Z = g_M = -\left[\frac{\rho - (1-\sigma) g_{A_Y}}{1 - \epsilon(1-\sigma)}\right]\,. \tag{A.5}$$

Last, as X and Z grow at the same constant (negative) rate given by (A.5), we can easily solve the linear differential equation system (5)-(6) to get the optimal trajectories of resource extraction and of waste recycling. To conclude, the entire solution of $\mathcal{P}_2$ is characterized in the following lemma.

Lemma 1 *For $t \in [T, \infty)$, the optimal trajectories of the model are:*

$$\begin{aligned}
X(t) &= X(T) e^{-k(t-T)}; \quad X(T) = kS(T) \\
Z(t) &= Z(T) e^{-k(t-T)}; \quad Z(T) = \frac{k[W(T) + \alpha S(T)]}{(1-\beta)} \\
S(t) &= S(T) e^{-k(t-T)}; \quad W(t) = W(T) e^{-k(t-T)} \\
C(t) &= C(T) e^{g_C (t-T)}; \quad C(T) = A_Y(T) \left[\bar{A} X(T) + h \bar{A} Z(T)\right]^\epsilon L^{1-\epsilon}
\end{aligned}$$

where $k \equiv [\rho - (1-\sigma) g_{A_Y}]/[1 - \epsilon(1-\sigma)]$ and where g_C is given by (A.3).

From the expression of C as given in Lemma 1 and noting that $g_C = (\rho - k)/(1-\sigma)$, the optimal value of program $\mathcal{P}_2$ can be simply expressed as:

$$\begin{aligned} V_2(A_Z(T), S(T), W(T)) &= \int_T^{\infty} u(C) e^{-\rho(t-T)} dt \\ &= \frac{C(T)^{1-\sigma}}{(1-\sigma)} \int_T^{\infty} e^{-k(t-T)} dt = \frac{C(T)^{1-\sigma}}{(1-\sigma)k}, \end{aligned} \tag{A.6}$$

where $C(T) = A_Y(T)\left[\bar{A}X(T) + A_Z(T)Z(T)\right]^{\epsilon} L^{1-\epsilon}$ and $A_Y(T) = A_{Y0}e^{g_{A_Y}T}$. We can then compute the following derivatives:

$$\frac{\partial V_2}{\partial A_Z(T)} = \frac{\epsilon Z(T)}{kM(T)} C(T)^{1-\sigma} \tag{A.7}$$

$$\frac{\partial V_2}{\partial S(T)} - \alpha \frac{\partial V_2}{\partial W(T)} = \frac{\epsilon \bar{A}}{M(T)} C(T)^{1-\sigma}. \tag{A.8}$$

A.1.2 Analytical solution of program $\mathcal{P}_1$

In what follows, growth rates with an upper tilde refer to the optimal trajectories under program $\mathcal{P}_1$. As long as $t < T$, consumption amounts to $C = A_Y(\bar{A}X)^{\epsilon} L_Y^{1-\epsilon}$. The intertemporal trade-off condition (26) directly implies that $\tilde{g}_X = \tilde{g}_{L_Y} = (1-\sigma)\tilde{g}_C - \rho$. As $\tilde{g}_C = g_{A_Y} + \epsilon \tilde{g}_X + (1-\epsilon)\tilde{g}_{L_Y}$, we can deduce the optimal growth rate of consumption for $t \in [0, T)$:

$$\tilde{g}_C = \frac{g_{A_Y} - \rho}{\sigma}. \tag{A.9}$$

As we know the growth rates of all the control variables (note also that $L_A = L - L_Y$), we can then solve the state equations (3)-(6) and characterize the solution of program $\mathcal{P}_1$ as follows.

Lemma 2 *For $t \in [0, T)$, the optimal trajectories of the model are:*

$$\begin{aligned} X(t) &= \tilde{k} S_0 e^{-\tilde{k}t}; \quad S(t) = S_0 e^{-\tilde{k}t}; \quad W(t) = W_0 + \alpha S_0 (1 - e^{-\tilde{k}t}) \\ L_Y(t) &= L_Y(0) e^{-\tilde{k}t}; \quad L_A(t) = L - L_Y(t) \\ A_Z(t) &= A_{Z0} \exp\left[\delta L t - \frac{\delta L_Y(0)}{\tilde{k}} (1 - e^{-\tilde{k}t})\right] \\ C(t) &= C(0) e^{\tilde{g}_C t}; \quad C(0) = A_{Y0} \left(\bar{A}\tilde{k}S_0\right)^{\epsilon} L_Y(0)^{1-\epsilon}, \end{aligned}$$

where $\tilde{k} \equiv [\rho - (1-\sigma)g_{A_Y}]/\sigma$, where $\tilde{g}_C$ is given by (A.9), and where $L_Y(0)$ and T are endogenous variables that must be determined from the set of continuity and transversality conditions at time T.

A.1.3 Transversality conditions at time T

Given the expression of the state variables provided by Lemma 1 and 2, we can deduce the following continuity conditions at time T:

$$S(T^-) = S(T^+) \;\Leftrightarrow\; S(T) = S_0 e^{-\tilde{k}T} \tag{A.10}$$

$$W(T^-) = W(T^+) \;\Leftrightarrow\; W(T) = W_0 + \alpha S_0 (1 - e^{-\tilde{k}T}) \tag{A.11}$$

$$A_Z(T^-) = h\bar{A} \;\Leftrightarrow\; \delta L T = \frac{\delta L_Y(0)}{\tilde{k}}\left(1 - e^{-\tilde{k}T}\right) + \ln\left(\frac{h\bar{A}}{A_{Z0}}\right). \tag{A.12}$$

Next, we analyze the transversality conditions as given by (23)-(25). We need first to characterize the trajectories of the co-state variables. Solving the differential equations (13), (14) and (22) for $t \in [0, T)$ results in:

$$\lambda_S(t) = \lambda_S(0) e^{\rho t} \tag{A.13}$$

$$\lambda_W(t) = \lambda_W(0) e^{\rho t} \tag{A.14}$$

$$\lambda_A(t) = \frac{\lambda_A(0) A_{Z0}}{A_Z(t)} e^{\rho t}, \tag{A.15}$$

where $\lambda_S(0)$, $\lambda_W(0)$ and $\lambda_A(0)$ are endogenous variables that must satisfy the first-order conditions (10) and (12) at time $t = 0$:

$$\lambda_S(0) - \alpha\lambda_W(0) = \frac{\epsilon}{\tilde{k}S_0} C(0)^{1-\sigma} \tag{A.16}$$

$$\lambda_A(0) = \frac{(1-\epsilon)}{\delta A_{Z0} L_Y(0)} C(0)^{1-\sigma}. \tag{A.17}$$

Combining (A.13)-(A.15) with (A.16)-(A.17), we have:

$$\lambda_S(T^-) - \alpha\lambda_W(T^-) = \frac{\epsilon}{\tilde{k}S_0} C(0)^{1-\sigma} e^{\rho T^-} \tag{A.18}$$

$$\lambda_A(T^-) = \frac{(1-\epsilon)}{\delta h\bar{A} L_Y(0)} C(0)^{1-\sigma} e^{\rho T^-}. \tag{A.19}$$

Next, using the transversality conditions (23)-(25) together with (A.18)-(A.19) and rearranging the outcome, expressions (A.7)-(A.8) can be rewritten as follows:

$$M(T)\left[\frac{C(0)}{C(T)}\right]^{1-\sigma} e^{\rho T} = \frac{\epsilon\delta h\bar{A} L_Y(0) Z(T)}{k(1-\epsilon)} \tag{A.20}$$

$$M(T)\left[\frac{C(0)}{C(T)}\right]^{1-\sigma} e^{\rho T} = \tilde{k} S_0 \bar{A} \tag{A.21}$$

These two last equations allow for determining the optimal initial level of effort in production. Last, given this optimal value of $L_Y(0)$, the optimal switching time T is obtained as the solution of the continuity equation (A.12).

A.1.4 Optimal trajectories: Summary

The optimal solution is characterized by the following trajectories:

$$X(t) = \begin{cases} \tilde{k} S_0 e^{-\tilde{k}t} & , \quad t < T \\ k S_0 e^{(k-\tilde{k})T - kt} & , \quad t \geq T \end{cases} \tag{A.22}$$

$$Z(t) = \begin{cases} 0 & , \quad t < T \\ \frac{k(W_0 + \alpha S_0)}{(1-\beta)} e^{-k(t-T)} & , \quad t \geq T \end{cases} \tag{A.23}$$

$$S(t) = \begin{cases} S_0 e^{-\tilde{k}t} & , \quad t < T \\ S_0 e^{(k-\tilde{k})T - kt} & , \quad t \geq T \end{cases} \tag{A.24}$$

$$W(t) = \begin{cases} W_0 + \alpha S_0 (1 - e^{-\tilde{k}t}) & , \quad t < T \\ \left[W_0 + \alpha S_0 (1 - e^{-\tilde{k}T}) \right] e^{-k(t-T)} & , \quad t \geq T \end{cases} \tag{A.25}$$

$$L_Y(t) = L - L_A(t) = \begin{cases} L_{Y0} e^{-\tilde{k}t} & , \quad t < T \\ L & , \quad t \geq T \end{cases} \tag{A.26}$$

$$A_Z(t) = \begin{cases} A_{Z0} \exp \left[\delta L t - \frac{\delta L_{Y0}}{\tilde{k}} (1 - e^{-\tilde{k}t}) \right] & , \quad t < T \\ h\bar{A} & , \quad t \geq T \end{cases} \tag{A.27}$$

where the initial level of productive labor is given by:

$$L_{Y0} \equiv L_Y(0) = \frac{\tilde{k}(1-\epsilon)(1-\beta) S_0}{\delta \epsilon h (W_0 + \alpha S_0)}; \tag{A.28}$$

and where the optimal switching time T is solution of the following equation:

$$\delta L T = \ln \left(\frac{h\bar{A}}{A_{Z0}} \right) + \frac{S_0 (1-\epsilon)(1-\beta)}{\epsilon h (W_0 + \alpha S_0)} \left(1 - e^{-\tilde{k}T} \right). \tag{A.29}$$

We can easily verify that that such an interior solution exists, *i.e.* that the non-negativity constraints (7)-(9) hold, if and only if $L_{Y0} \in (0, L)$, $k > 0$ and $\tilde{k} > 0$. This corresponds to a set of parameters that must satisfy the following conditions:

$$(1-\sigma) g_{A_Y} \leq \rho \, , \tag{A.30}$$

$$L_{Y0} = \frac{\tilde{k}(1-\epsilon)(1-\beta) S_0}{\delta \epsilon h (W_0 + \alpha S_0)} \leq L \, . \tag{A.31}$$

Condition (A.30) states that, to justify resource extraction and recycling, the social discount rate must be large enough as compared with the exogenous trend parameter of technical progress (this condition guarantees that both k and $\tilde{k}$ are positive). Condition (A.31) says that the total amount of effort (*i.e.* labor) must be large enough.

Appendix A.2 Properties of the consumption path

A.2.1 General case, for any σ

We first focus on the discrete jump in consumption at time t. The size of this jump is given by $\Delta C(T) \equiv C(T^+) - C(T^-)$, where $C(T^+) = A_Y(T) M(T^+)^\epsilon L^{1-\epsilon}$ and $C(T^-) =$

$A_Y(T)M(T^-)^{\epsilon}L_Y(T^-)^{1-\epsilon}$. Then, we can write:

$$\Delta C(T) \geq 0 \;\Leftrightarrow\; \left[\frac{M(T^+)}{M(T^-)}\right]^{\epsilon} \geq \left[\frac{L_Y(T^-)}{L}\right]^{1-\epsilon}.$$

Denoting by $\Delta M\%$ and $\Delta L_Y\%$ the percentage of instantaneous variation at time T of M and L_Y respectively, the previous expression becomes:

$$\Delta C(T) \geq 0 \quad \Leftrightarrow \quad (\Delta M\% + 1)^{\epsilon} \times (\Delta L_Y\% + 1)^{1-\epsilon} \geq 1\,,$$

which states that if the multiplier of M (first term in bracket) is larger than one, then, given that $\Delta L_Y\% > 0$, consumption jumps up. For the other cases, that is if this multiplier is smaller than one, the multiplier of L (second term in brackets, always larger than one) must be large enough and/or the substitution parameter ϵ must be small enough to compensate for the instantaneous decrease in material inputs and then get a positive jump in C.

A positive jump in the material input would thus be a sufficient condition for consumption to jump upwards. From (2), this jump in M is given by $\Delta M(T) = \bar{A}[\Delta X(T) + h\Delta Z(T)]$. Developing this expression, we get:

$$\begin{aligned}
\Delta M(T) &= \bar{A}k\left[h\left(\frac{W_0+\alpha S_0}{1-\beta}\right) - \frac{(1-\sigma)(1-\epsilon)S_0}{\sigma}e^{-\tilde{k}T}\right] \\
\Rightarrow \Delta M(T) &\geq 0 \quad \Leftrightarrow \quad \Phi e^{\tilde{k}T} \geq \frac{(1-\sigma)(1-\epsilon)}{h\sigma}
\end{aligned} \tag{A.32}$$

where $\Phi \equiv (W_0+\alpha S_0)/(1-\beta)S_0$ and where, from (A.29), T can be expressed as a function of Φ:

$$T(\Phi) \text{ is s.t. } \delta LT = \ln\left(\frac{h\bar{A}}{A_{Z0}}\right) + \frac{(1-\epsilon)}{\epsilon h\Phi}\left(1 - e^{-\tilde{k}T}\right). \tag{A.33}$$

Let consider the function $g(\Phi) = \Phi e^{\tilde{k}T}$, with $g'(\Phi) = \left(1 + \Phi\tilde{k}\frac{dT}{d\Phi}\right)e^{\tilde{k}T}$. Differentiating (A.33) with respect to Φ, we obtain:

$$\frac{dT}{d\Phi} = \frac{\frac{-(1-\epsilon)}{\epsilon h\Phi^2}(1-e^{-\tilde{k}T})}{\delta L - \frac{\tilde{k}(1-\epsilon)}{\epsilon h\Phi}e^{-\tilde{k}T}},$$

which is proved to be negative (see Appendix A.3, Table 1). As the numerator of this ratio is negative, the denominator must be positive. After computation, we get:

$$g'(\Phi) = \frac{\left[\delta L - \frac{(1-\epsilon)\tilde{k}}{\epsilon h\Phi}\right]e^{\tilde{k}T}}{\delta L - \frac{\tilde{k}(1-\epsilon)}{\epsilon h\Phi}e^{-\tilde{k}T}}.$$

As just shown, the denominator of this expression is positive. Moreover, the numerator is also positive given the existence condition (A.31). Then, function $g(.)$ is increasing in Φ. Coming back to (A.32), we can conclude that $\Delta M(T)$ is positive if and only if Φ is larger than $g^{-1}\left(\frac{(1-\sigma)(1-\epsilon)}{h\sigma}\right)$.

A.2.2 Particular case $\sigma = 1$

When $\sigma = 1$, *i.e.* when $u(C) = \ln C$, the optimal trajectories of the T-recycling economy are given by:

$$\begin{array}{rclcrcl} X(t) & = & \rho S_0 e^{-\rho t} & \text{and} & S(t) & = & S_0 e^{-\rho t} \\ Z(t) & = & \left\{ \begin{array}{l} 0 \\ \rho \Phi S_0 e^{-\rho(t-T)} \end{array} \right. & \text{and} & W(t) & = & \left\{ \begin{array}{l} W_0 + \alpha S_0 (1 - e^{-\rho t}) \\ \left[W_0 + \alpha S_0 (1 - e^{-\rho T}) \right] e^{-\rho(t-T)} \end{array} \right. \\ L_Y(t) & = & \left\{ \begin{array}{l} L_{Y0} e^{-\rho t} \\ L \end{array} \right. & \text{and} & C(t) & = & \left\{ \begin{array}{l} C_0 e^{(g_{A_Y} - \rho)t} \\ C_T e^{(g_{A_Y} - \epsilon\rho)(t-T)} \end{array} \right. \end{array}$$

where $L_{Y0} = \frac{\rho(1-\epsilon)}{\epsilon \delta h \Phi}$, $C_0 = A_{Y0}(\bar{A}\rho S_0)^\epsilon L_{Y0}^{1-\epsilon}$, $C_T = A_Y(T)(\bar{A}\rho S_0)^\epsilon (e^{-\rho T} + h\Phi)^\epsilon L^{1-\epsilon}$ and where T is solution of:

$$\delta L T = \ln\left(\frac{h\bar{A}}{A_{Z0}}\right) + \frac{L_{Y0}}{\rho}\left(1 - e^{-\rho t}\right).$$

In this case, the resource extraction trajectory is no longer discontinuous. The other trajectories evolve as in the general CES utility case. In particular, the optimal consumption grows faster after T. However, we are now able to precisely characterize the jump in consumption at time T. As labor L_Y jumps up, we know that a positive jump in the material input M would be a sufficient condition for consumption to jump upwards. As resource extraction is now continuous at time T, the size of this jump is simply given by the jump in waste recycling: $\Delta M(T) = \bar{A}[\Delta X(T) + h\Delta Z(T)] = h\bar{A}\rho\Phi S_0 > 0$. Then, without any ambiguity, consumption now jumps upwards.

With $\sigma = 1$, consumption in the never-recycling and in the always-recycling economies are, respectively:

$$C_n(t) = A_{Y0}(\bar{A}\rho S_0)^\epsilon L^{1-\epsilon} e^{(g_{A_Y} - \epsilon\rho)t} \quad \text{(A.34)}$$

$$C_a(t) = A_{Y0}(\bar{A}\rho S_0)^\epsilon (1 + h\Phi)^\epsilon L^{1-\epsilon} e^{(g_{A_Y} - \epsilon\rho)t}. \quad \text{(A.35)}$$

Obviously, we have $C_a(t) > C_n(t)$ for any t. As $C_n(0) - C(0) = A_{Y0}(\bar{A}\rho S_0)^\epsilon (L^{1-\epsilon} - L_{Y0}^{1-\epsilon}) > 0$, we can write:

$$C_a(0) > C_n(0) > C(0).$$

Moreover, as $C_a(T) - C(T) = A_Y(T)(\bar{A}\rho S_0)^\epsilon L^{1-\epsilon}\left[(1 + h\Phi)^\epsilon - (1 + h\Phi e^{\rho T})^\epsilon\right] e^{-\epsilon\rho T} < 0$, then:

$$C(T) > C_a(T) > C_n(T).$$

The resulting consumption trajectories are illustrated in Figure 3.

Appendix A.3 Comparative dynamic analysis

We conduct a sensitivity analysis of the key variables of the model, with respect to the set of parameters. The results are described in Table 1. Each box indicates the sign of the partial derivative of the variable mentioned in line with respect to the parameter given in column. This sign can be positive ("+") or negative ("−"). An empty box means that there is no relation between the variable and the parameter whereas "?" indicates that the sign is ambiguous.

Partial differentiation of the growth rates k, $\tilde{k}$, g_C and $\tilde{g}_C$, and of the initial values $X(0)$, $Z(T)$, $W(T)$, L_{Y0} and $C(0)$ are trivial so that their computations are not detailed here. However, the sensitivity analysis of the switching date T is less obvious as we cannot get a closed-form expression. We simply know that T is characterized by the implicit function (A.29). Let us define the following functions i and j:

$$\begin{aligned} i(t) &= \delta L t - \ln\left(\frac{h\bar{A}}{A_{Z0}}\right) \\ j(t) &= \frac{S_0(1-\epsilon)(1-\beta)}{\epsilon h(W_0+\alpha S_0)}\left(1-e^{-\tilde{k}t}\right) = \frac{(1-\epsilon)}{\epsilon h \Phi}\left(1-e^{-\tilde{k}t}\right). \end{aligned}$$

These functions are depicted in Figure 4. Given their analytical properties, we can observe graphically that the solution T to the equation $i(T) = j(T)$ is unique. Moreover, for this solution to exist, we must have $i'(T) > j'(T)$.

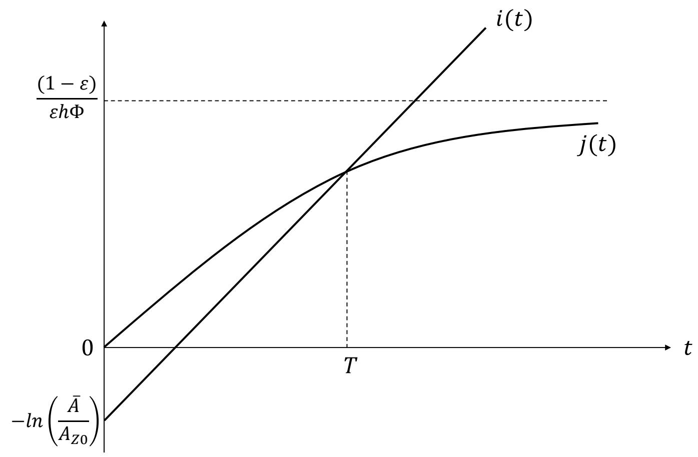

Figure 4: Graphical identification of T

We apply now the implicit function theorem to i and j. For any parameter x, we obtain:

$$\frac{dT}{dx} = \frac{\partial j/\partial x - \partial i/\partial x}{i'(T) - j'(T)} \quad \Rightarrow \quad \text{sign}\left(\frac{dT}{dx}\right) = \text{sign}\left(\frac{\partial j}{\partial x} - \frac{\partial i}{\partial x}\right). \tag{A.36}$$

This equation, together with the computations of $\partial i/\partial x$ and $\partial j/\partial x$, thus allows identifying the sign of the derivatives of T with respect to any parameter x.

Table 1: Comparative dynamic analysis

	ρ	σ	g_{A_Y}	ϵ	$\bar{A}$	δ	S_0	W_0	α	β	Φ
k	+	+	−	+							
$\tilde{k}$	+	+	−								
g_C	−	−	+	−							
$\tilde{g}_C$	−	−	+								
T	+	+	−	−	+	−	+	−	−	−	−
$X(0)$	+	+	−				+				
$Z(T)$	+	+	−	+			+	+	+	+	+
$W(T)$	+	+	−	−	+	−	+	?	?	−	+
L_{Y0}	+	+	−	−		−	+	−	−	−	−
$C(0)$	+	+	−	?	+	−	+	−	−	−	−