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Natural cycles and pollution*

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Abstract

In this paper, we study a competitive Ramsey model where a pollution externality, coming from production, impairs a renewable resource which affects the consumption demand. A proportional tax, levied on the production level, is introduced to finance public depollution expenditures.

In the long run, two steady states may coexist, the one with a low resource level, the other with a high level. Interestingly, a higher green tax rate lowers the resource level of the low steady state, giving rise to a Green Paradox (Sinn, 2008). Moreover, the green tax may be welfare-improving at the high steady state but never at the low one. Therefore, at the latter, it is optimal to reduce the green tax rate as much as possible. Conversely, the optimal tax rate is positive when the economy experiences the high steady state. This rate is unique.

In the short run, the two steady states may collide and disappear through a saddle-node bifurcation. Since consumption and natural resources are substitutable goods, a limit cycle may arise around the high stationary state. To the contrary, this kind of cycles never occur around the low steady state whatever the resource effect on consumption demand. Finally, focusing on the class of bifurcations of codimension two, we find a Bogdanov-Takens bifurcation.

Keywords: nature, logistic dynamics, Ramsey model, depollution, saddle-node bifurcation, Hopf bifurcation, Bogdanov-Takens bifurcation.

JEL Classification: E32, O44.

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1 Introduction

Paleontologists define a mass extinction as a situation in which Earth looses more than three-quarters of its species in a geologically short interval (Barnosky et al. 2011). In the past 540 million years, five mass extinctions occurred and biologists conjecture that a sixth mass extinction (also known as Holocene extinction) is under way (Barnosky et al. 2011). This new extinction comes principally from human activities (deforestation, global warming and climate change) (Ceballos et al. 2015). The strong loss of biodiversity has a large impact on human wellbeing. For instance, as pointed out by Ceballos et al. (2015), this alters crop pollination or water purification. That is, production activities generate pollution promoting global warming and climate change that impair biodiversity and human wellbeing in turn.

The interplay between renewable resource (species, forest...) and economic activities has already been studied in the theoretical literature. To the best of our knowledge, the first attempt to consider a renewable resource dynamics in a Ramsey framework dates back to the seminal paper by Beltratti et al. (1994). Those authors have considered a renewable resource which serves as an input of production and affects household's utility. They assume also that a pollution externality, coming from consumption activities, impairs the renewable resource. Considering that nature (i.e. the natural resource) has a small impact on production, they show the existence of a unique stable steady state (saddle-point). A very similar result was obtained by Ayong Le Kama (2001): he shows that, when pollution comes from production instead of consumption, then the assumption that nature as a small impact on production is no longer necessary to ensure the existence of a unique stable steady state (saddle-point).

Reconsidering the framework studied by Beltratti et al. (1994) and Ayong Le Kama (2001), Wirl (2004) proves that, when pollution only affects household's utility, then the economy exhibits two steady states. More precisely, by considering a Pearl-Verhulst logistic function for the reproduction of the natural resource, Wirl (2004) shows that each branch of the reproduction function possesses a steady state. Interestingly, Wirl (2004) points out the possibility of the emergence of a limit cycle through a Hopf bifurcation around the low steady state (located on the upward-sloping branch of the reproduction function) and prove the impossibility of such a complex dynamics around the high steady state (located on the downward-sloping branch). This existence of endogenous cycles matters from an environmental point of view because it entails the potential emergence of intergenerational inequalities in environmental terms: some generations face a high level of natural resource while others face a low level.

All these contributions rest on the assumption of a separable utility function between consumption and natural resource. Nevertheless, intuition suggests that the stock of natural resource affects the marginal utility of consumption and, then, the consumption demand. Indeed, if nature increases the consumption demand, then nature and consumption are complement: it is the case when households like to consume in a pleasant environment, in presence, for instance, of large biodiversity. Conversely, if nature lowers consumption demand, then nature and consumption are substitutable: in this case, the household compensates the utility loss due to a lower natural resource (for instance, a lower biodiversity) by increasing her consumption demand. Both these situations are impossible in Beltratti et al. (1994), Ayong Le Kama (2001) or Wirl (2004) because they consider only separable preferences. One may expect that both these potential effects of nature on consumption demand matter and change substantially Wirl's conclusions (2004) on the occurrence of endogenous cycles. In addition, Beltratti et al. (1994), Ayong Le Kama (2001) and Wirl (2004) focus only on the central planner's solution. It is important to understand the short and long-run consequences of the interplay between natural and capital accumulation with a pollution externality. The market representation is pertinent when households face prices without choosing the external effects. We aim at addressing all these important issues by considering non-separable preferences. In a context of a market economy, the government is allowed to levy a proportional tax on production activities to finance depollution expenditures according to a balanced budget rule.

In the long run, we find that the economy experiences multiple steady states depending upon the environmental impact of production. In particular, the economy has no steady state under an excessive impact while a low impact ensures the existence of two steady states located on each branch of the reproduction function of renewable resource. The first one is characterized by a low natural resource level while the other by a high level. We observe that the effect of a higher green tax rate depends on the steady state. In particular, it lowers the natural resource level of the low steady state. Such counter-intuitive negative relation suggests that a greener policy may exacerbate the environmental damage. This case is very close to the Green Paradox pointed out by Sinn (2008) in a resource extraction context.

In the environmental literature, a Green Paradox is a situation in which a green tax exacerbates the environmental damages instead of mitigating them. To understand the mechanism, consider a owner of fossil fuel maximizing the profit and a government announcing that a green tax will be introduced tomorrow. The owner rationally expects higher extraction costs tomorrow and, then, speeds up the extraction today thus exacerbating the environmental damages. According to a recent survey by Jensen et al. (2015), this paradox is observed in various contexts and, in particular: when agents behave strategically in the resource markets (Gerlagh and Liski, 2011), when resource and capital markets interact (Van der Meijden et al., 2015), when future policies are uncertain (Hoel, 2010). Even if the Green Paradox historically refers to a dynamic effect (transitional dynamics), broadly speaking, it states that a higher green tax rate impairs the environmental quality. In this sense, the negative relation between the green tax rate and the natural resource level observed in our paper along the increasing branch of the reproduction function at the steady state, gives rise to a new category of Green Paradox differing from the traditional one in two respects: (1) it is outside the resource extraction context and (2) it is a static relation (in terms of comparative statics) rather than a dynamic effect. Recently, Bosi and Desmarchelier (2016a and 2016b) have pointed out the possibility of such Green Paradox at the steady state in a Ramsey model with pollution but without natural resource dynamics. In this sense, the static Green Paradox seems to be a robust property of Ramsey models. In addition, we prove that the green tax may be welfare-improving at the high steady state but never at the low one. Finally, we prove that an optimal green tax rate exists and is unique for the high steady state while the optimal policy consists in lowering as much as possible the green tax rate at the low steady state.

In the short run, we find that the low stationary state is always unstable while a limit cycle may emerge near the high steady state because nature and consumption are substitutable goods. This result is surprising since Wirl (2004) shows that limit cycles only occurs near the low steady state in the case of central planner. In addition, we show that the two steady states may collide and disappear through a saddle-node bifurcation under a sufficiently high environmental impact of production. Finally, we prove that there exists a parameter region for which, at the saddle-node bifurcation point, the low steady state coalesces with the limit cycle surrounding the high steady state. This unusual situation in econmics is known as a Bogdanov-Takens bifurcation. Recently, Barnett and Ghosh (2013) have pointed out that such a Bogdanov-Takens bifurcation is possible in an endogenous growth model. Our contribution adds a value to the existing literature by proving the existence of a Bogdanov-Takens bifurcation in an (environmental) exogenous growth model à la Ramsey.

The rest of the paper is organized as follows: sections 2 to 4 present the model, section 5 analyzes the local dynamics, section 6 presents an isoelastic example and section 7 provides numerical simulations. Section 8 concludes. All the mathematical proofs are gathered in the appendix.

2 Model

We consider an economy in the spirit of Wirl (2004), but with three main differences: (1) a market economy instead of a social planner, (2) a non-separable utility function between consumption and the natural resource, (3) a proportional tax on production.

2.1 Firms

The firm chooses the amount of capital and labor to maximize the profit taking as given the real interest rate r. In addition, the government levies a proportional tax $\tau \in (0,1)$ on polluting production $F(k_j, l_j)$ of firm j to finance the maintenance of natural resource.

Assumption 1 *The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^2, homogeneous of degree one, strictly increasing and concave. Inada conditions hold.*

The profit maximization $\max_{K_j,N_j} [F(K_j, L_j) - rK_j - wL_j - \tau F(K_j, L_j)]$ entails the following first-order conditions:

$$r = (1-\tau) f'(k_j) \text{ and } w = (1-\tau)[f(k_j) - k_j f'(k_j)]$$

where $k_j \equiv K_j/L_j$ is the capital intensity and $f(k_j) \equiv F(k_j, 1)$ the average productivity of the firm j.

All the firms share the same technology and address the same demand for capital.

Corollary 1 *Let $k \equiv K/L$ with $K \equiv \sum_{j=1}^{J} K_j$ and $L \equiv \sum_{j=1}^{J} L_j$. In aggregate terms, $Y = F(K, L)$ and profit maximization yields*

$$r = (1-\tau)\rho(k) \text{ and } w = (1-\tau)\omega(k)$$

with $\rho(k) \equiv f'(k)$ and $\omega(k) \equiv f(k) - kf'(k)$.

We introduce the capital share in total disposable income and the elasticity of capital-labor substitution:

$$\alpha(k) \equiv \frac{rk}{(1-\tau) f(k)} = \frac{kf'(k)}{f(k)} \text{ and } \sigma(k) = \alpha(k) \frac{\omega(k)}{k\omega'(k)}$$

In addition, we determine the elasticities of factor prices:

$$\frac{k\rho'(k)}{\rho(k)} = -\frac{1-\alpha(k)}{\sigma(k)} \text{ and } \frac{k\omega'(k)}{\omega(k)} = \frac{\alpha(k)}{\sigma(k)}$$

Example The Cobb-Douglas production function

$$f(k) = Ak^{\alpha} \tag{1}$$

gives $\rho(k) = \alpha Ak^{\alpha-1}$ and $\omega(k) = (1-\alpha) Ak^{\alpha}$. Notice that the zero-profit condition holds for this constant returns to scale specification.

2.2 Households

The representative household earns a capital income rh where h denotes the individual wealth at time t and a labor income wl where $l = 1$ (inelastic labor supply). Thus, the household consumes and saves her income according to the budget constraint:

$$c + \dot{h} \leq (r-\delta) h + w \tag{2}$$

where $\dot{h}$ denotes the time-derivative of wealth. The gross investment includes the capital depreciation at the rate δ.

As in Beltratti et al. (1994), Ayong Le Kama (2001) or Wirl (2004), we assume that the aggregate natural ressource enters the household's utility function $u(c, N)$ with $u_N > 0$. Differently from them, we suppose that the nature affects the marginal utility of consumption ($u_{cN} \neq 0$). Indeed, intuition suggests that nature plays a role in consumption demand. As discussed ealier, if nature increases the consumption demand, then nature and consumption are complement ($u_{cN} > 0$): this happens when households like to consume in a pleasant environment, in presence, for instance, of large biodiversity. Conversely, if nature lowers consumption demand, then nature and consumption are substitutable

($u_{cN} < 0$): in this case, the household compensates the utility loss due to a lower natural resource (for instance, a lower biodiversity) by increasing her consumption demand.

Assumption 2 *Preferences are rationalized by a non-separable utility function* $u(c, N)$*. First and second-order restrictions hold on the sign of derivatives:* $u_c > 0$*,* $u_N > 0$ *and* $u_{cc} < 0$*, jointly with the limit conditions:* $\lim_{c \to 0} u_c = \infty$ *and* $\lim_{c \to \infty} u_c = 0$*.*

We introduce the second-order elasticities:

$$\begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cN} \\ \varepsilon_{Nc} & \varepsilon_{NN} \end{bmatrix} \equiv \begin{bmatrix} \frac{cu_{cc}}{u_c} & \frac{Nu_{cN}}{u_c} \\ \frac{cu_{Nc}}{u_N} & \frac{Nu_{NN}}{u_N} \end{bmatrix} \tag{3}$$

$-1/\varepsilon_{cc}$ represents the intertemporal elasticity of substitution in consumption while ε_{cN} captures the effect of the natural resource on the marginal utility of consumption. Typically, if $\varepsilon_{cN} > 0$ (< 0), then the natural resource and consumption are complement (substitute) for households.

In a Ramsey model, the representative household maximizes an intertemporal utility functional:

$$\int_0^\infty e^{-\theta t} u(c, N) \, dt$$

under the budget constraint (2) where $\theta > 0$ denotes the rate of time preference.

Proposition 2 *The first-order conditions of the consumer's program are given by a static relation* $\mu = u_c$*, a dynamic Euler equation* $\dot{\mu} = \mu(\theta + \delta - r)$ *and the budget constraint (2), now binding,* $\dot{h} = (r - \delta) h + w - c$ *jointly with the transversality condition* $\lim_{t \to \infty} e^{-\theta t} \mu(t) h(t) = 0$*.* μ *denotes the multiplier associated to the budget constraint.*

Applying the Implicit Function Theorem to the static relation $\mu = u_c(c, N)$, we obtain the consumption function $c \equiv c(\mu, N)$ with elasticities

$$\frac{\mu}{c}\frac{dc}{d\mu} = \frac{1}{\varepsilon_{cc}} < 0 \text{ and } \frac{N}{c}\frac{dc}{dN} = -\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \tag{4}$$

Example The isoelastic utility function

$$u(c, N) = \frac{(cN^\eta)^{1-\varepsilon}}{1-\varepsilon} \tag{5}$$

with $\varepsilon, \eta \geq 0$, yields

$$\begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cN} \\ \varepsilon_{Nc} & \varepsilon_{NN} \end{bmatrix} = \begin{bmatrix} -\varepsilon & \eta(1-\varepsilon) \\ 1-\varepsilon & \eta(1-\varepsilon) - 1 \end{bmatrix} \tag{6}$$

2.3 Government

The government uses all the tax revenues to finance depollution expenditures (G) according to a balanced budget rule:

$$G = \tau F(K, L) \tag{7}$$

2.4 Nature

In the spirit of Ayong Le Kama (2001) and Wirl (2004) the dynamics of natural resource is given by

$$\dot{N} = g(N) - P \tag{8}$$

where $g(N)$ and P represent the reproduction function and the net pollution level respectively.

Following Wirl (2004) and Bella (2010), we specify $g(N)$ as a Pearl-Verhulst logistic function: $g(N) \equiv N(1-N)$ with $0 < N < 1$.

Interestingly, since $g'(N) = 1 - 2N$, the maximal sustainable yield occurs at $N = 1/2$. By considering the central planner solution, Wirl (2004) has pointed out that limit cycles can occur if and only if $N < 1/2$ (the maximal sustainable yield) at the steady state. In a competitive context, we will show that, in the case of non-separable preferences, limit cycles may occur even for $N > 1/2$.

Pollution is assumed to be a flow coming from production activity:

$$P = aY - bG \tag{9}$$

where a and b capture respectively the environmental impact of production and the depollution efficacity.

Considering (7), (8) and (9), we find the natural resource accumulation law:

$$\dot{N} = N(1-N) - aF(K,L) + b\tau F(K,L) \tag{10}$$

Let us introduce an additional assumption.

Assumption 3 *$a > b\tau$.*

3 Equilibrium

In the capital market, the aggregate demand for capital by firms is equal to the aggregate supply by households: $K = Lh$. Therefore, we obtain the equality between individual wealth and capital intensity: $h = k$.

For simplicity, we normalize the population to one: $L = 1$. Hence, $f(k) \equiv F(k,1) = F(K,L)$ and (10) is written as

$$\dot{N} = N(1-N) + (b\tau - a) f(k)$$

with $a - b\tau > 0$ according to Assumption 3. Gathering the first-order conditions and the accumulation of natural resource, we obtain the dynamic system.

Proposition 3 *Equilibrium dynamics are driven by a three-dimensional dynamic system:*

$$\begin{aligned} \dot{\mu} &= g_1(\mu,k,N) \equiv \mu\left[\theta + \delta - (1-\tau)\rho(k)\right] && (11) \\ \dot{k} &= g_2(\mu,k,N) \equiv \left[(1-\tau)\rho(k) - \delta\right]k + (1-\tau)\omega(k) - c(\mu,N) && (12) \\ \dot{N} &= g_3(\mu,k,N) \equiv N(1-N) + (b\tau - a) f(k) && (13) \end{aligned}$$

jointly with the transversality condition.

We observe that there are two predetermined variables (k and N) and one jump variable (μ) which inherits this status from consumption demand.

4 Steady states

We introduce the critical environmental impact of production:

$$a^* \equiv b\tau + \frac{1}{4}\frac{1-\tau}{\theta+\delta}\frac{\alpha(k)}{k}$$

(see (9)). a^* is well defined. Indeed, the RHS does not depend on a.

Proposition 4 *The Modified Golden Rule holds with*

$$k = \rho^{-1}\left(\frac{\theta+\delta}{1-\tau}\right) \tag{14}$$

$$c = k\left[\theta + (\theta+\delta)\frac{1-\alpha(k)}{\alpha(k)}\right] \tag{15}$$

Under Assumption 1, the capital intensity and the consumption demand are unique.

Under Assumption 3, the steady states of natural resource are given by

$$N_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - (\theta+\delta)\frac{a-b\tau}{1-\tau}\frac{k}{\alpha(k)}} \tag{16}$$

$$N_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - (\theta+\delta)\frac{a-b\tau}{1-\tau}\frac{k}{\alpha(k)}} \tag{17}$$

If $a < a^*$, there are two steady states with $0 < N_1 < 1/2 < N_2 < 1$.
If $a = a^*$, the steady state becomes unique with $N_1 = N_2 = 1/2$.
If $a > a^*$, there are no steady states.

Proposition 5 *Assume that $\sigma = 1$ (Cobb-Douglas production function). The impacts of the tax rate τ on c, k and N_i are given by:*

$$\frac{\tau k'(\tau)}{k(\tau)} = \frac{\tau c'(\tau)}{c(\tau)} = -\frac{\tau}{1-\tau}\frac{1}{1-\alpha} < 0 \tag{18}$$

$$\frac{\tau N_i'(\tau)}{N_i(\tau)} = \frac{\tau}{1-\tau}\frac{1-N_i}{2N_i-1}\frac{\alpha(a-b)+(1-\tau)b}{(1-\alpha)(a-b\tau)} \tag{19}$$

These elasticities deserve an economic interpretation. Since the green tax is levied on production activities, a higher τ lowers the income and the capital level in the long run. A decrease in income means also a lower consumption demand. Therefore, the effects of τ on k and c are straightforward. Conversely and surprisingly, according to (19), the effects of τ on N_1 and on N_2 are opposite.

Corollary 6 *If $a > b$,*

$$\frac{\tau N_1'(\tau)}{N_1(\tau)} < 0 \text{ (Green Paradox) while } \frac{\tau N_2'(\tau)}{N_2(\tau)} > 0$$

Let us provide an intuition. Keeping in mind that the reproduction function $g(N)$ is logistic, at the steady state, we get

$$g(N) = (a - b\tau) f(k(\tau)) = \phi(\tau)$$

and, therefore,

$$\frac{\tau\phi'(\tau)}{\phi(\tau)} = -\left(\frac{b\tau}{a - b\tau} + \frac{\alpha}{1-\alpha}\frac{\tau}{1-\tau}\right) < 0$$

Thus, a higher τ implies a lower value for the reproduction function g. According to Proposition 4, N_1 lies on the upward-sloping branch of the reproduction function while N_2 on the decreasing branch. Since a higher τ means a lower g, this entails a lower N_1 and a higher N_2. In other words, the opposite effect of τ on N_1 and N_2 comes precisely from the shape of the reproduction function. The effect of τ on N_2 is intuitive: a higher green tax implies a lower pollution level (because of the drop in production and higher depollution expenditures) and permits a better regeneration of nature. Conversely, the effect of τ on N_1 implies that a greener tax rate promotes a deterioration of nature. Such a counter-intuitive effect is very close to the so-called Green Paradox in the recent literature (see Sinn 2008 among others).

The Green Paradox appears historically in a resource extraction context (see Jensen et al., 2015). In order to understand the paradox, consider a owner of fossil fuel maximizing her profit. If the government announces that a green tax will be levied tomorrow, the rational owner expects a higher extraction cost tomorrow and, then, she speeds up the extraction today raising in fine the pollution emissions. In a broader sense, the Green Paradox states that a higher green tax rate impairs the environmental quality. The negative relation between τ and N_1 (see Corollary 6) captures a similar phenomenon but with two main differences: (1) there is no reference to any extraction activity and (2) the paradox is static (at the steady state) instead of dynamic (along the transition path). The paradox is usually considered as a dynamic effect in the environmental literature. To the best of our knowledge, the static Green Paradox was introduced by Bosi and Desmarchelier (2016a, 2016b). They have found the same result of the current paper but within a different economic model.

Indeed, in a Ramsey model with pollution accumulation, they have shown that a positive pollution effect on consumption demand (so-called compensation effect by Michel and Rotillon (1995)) implies that a higher green tax rate stimulates the pollution stock at the steady state either when pollution comes from production (Bosi and Desmarchelier 2016a) or when pollution comes from consumption (Bosi and Desmarchelier 2016b). The present paper proves that a static Green Paradox also arises in a simple model with natural dynamics. This means that the Green Paradox is a robust feature of environmental Ramsey models with non-separable preferences.

Bosi and Desarchelier (2016b) have also discussed the interplay between the Laffer Curve, the (static) Green Paradox and the possibility of endogenous fluctuations. They have found in particular that the (static) Green Paradox rules out endogenous cycles while the Laffer Curve promotes endogenous cycles. This question will be readdressed later in our new framework.

We can also compute the impact of the tax rate on welfare, that is the elasticity of welfare function with respect to τ. Because of the representative agent, the welfare function coincides with her utility function:

$$W_i(\tau) = \int_0^{\infty} e^{-\theta t} u(c(\tau), N_i(\tau))\, dt = \frac{u(c(\tau), N_i(\tau))}{\theta} \tag{20}$$

with $i = 1, 2$. We introduce the first-order elasticities

$$\varepsilon_c \equiv \frac{cu_c}{u} \text{ and } \varepsilon_N \equiv \frac{Nu_N}{u} \tag{21}$$

Proposition 7 *Under Assumption 2, the impact of taxation on welfare is positive if and only if:*

$$(0 <) \frac{\varepsilon_c}{\varepsilon_N} < -\frac{c(\tau)}{\tau c'(\tau)} \frac{\tau N'(\tau)}{N(\tau)} = (1-\alpha) \frac{1-\tau}{\tau} \frac{\tau N'(\tau)}{N(\tau)} = \frac{1-N}{2N-1} \frac{\alpha(a-b) + (1-\tau) b}{a - b\tau} \tag{22}$$

Corollary 8 *In the lower steady state N_1, the impact of taxation on welfare is negative.*

Unsurprisingly, in the lower steady state, a higher green tax rate always implies a drop in utility because of a lower consumption (Proposition 5) and a lower environmental quality N_1 (Corollary 6).

In higher steady state, the higher green tax lowers the consumption demand but raises the environmental quality N_2. The positive effect will dominate if only if inequality (22) holds. In this case, the RHS of inequality (22) is positive. The inequality is satisfied when the slope of the indifference curve of u becomes flatter

$$\frac{\varepsilon_c}{\varepsilon_N} = -\frac{c}{N} \frac{dN}{dc} \to 0$$

that is when the households overweight nature with respect to consumption ($\varepsilon_c \to 0$ and $\varepsilon_N \to +\infty$).

Interestingly, if the government plays with the tax rate in order to maximize the welfare at the steady state, it follows that, if the economy is in N_1, the government has to lower the tax rate as much as possible. Since a lower τ means a higher N_1 and a lower N_2, we expect that τ^* is the lowest possible tax rate, such that $N_1 = N_2$ (if $\tau < \tau^*$, according to Proposition 4, the economy has no longer a steady state). The existence of an optimal τ^* such that $N_1 = N_2$ will be discussed in section 6.1.

5 Local dynamics

We linearize the dynamic system (11)-(13) around each steady state.

Lemma 9 *The trace, the sum of minors of order two and the determinant of the three-dimensional Jacobian matrix evaluated at the steady state are given by*

$$T = \theta + 1 - 2N \tag{23}$$

$$S = \theta(1-2N) + \alpha\gamma(1-N)\frac{\varepsilon_{cN}}{\varepsilon_{cc}} + (\theta+\delta)\frac{1-\alpha}{\sigma}\frac{\gamma}{\varepsilon_{cc}}$$

$$D = (1-2N)(\theta+\delta)\frac{1-\alpha}{\sigma}\frac{\gamma}{\varepsilon_{cc}} \tag{24}$$

where $\alpha = \alpha(k)$, $\sigma = \sigma(k)$ *and* $\gamma \equiv \theta + (\theta+\delta)(1-\alpha)/\alpha = c/k$.

Clearly, the values taken by T, S and D depend on the steady state N we focus on.

We know that, in terms of information, the vector (T, S, D) is equivalent to the vector of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. More precisely, we have $T = \lambda_1 + \lambda_2 + \lambda_3$, $S = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $D = \lambda_1\lambda_2\lambda_3$. The characteristic polynomial becomes

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - T\lambda^2 + S\lambda - D$$

5.1 Local determinacy

System (11)-(13) is three-dimensional with a jump variable (μ) and two predetermined variables (k and N). Thus, multiple equilibria (local indeterminacy) arise when the three eigenvalues of the Jacobian matrix evaluated at the steady state have negative real parts: either $\lambda_1, \lambda_2, \lambda_3 < 0$ or $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 < 0$ and $\lambda_3 < 0$.

Proposition 10 (local determinacy) *The equilibrium is locally unique if* $N^* > 1/2$.

5.2 Local bifurcations

In continuous time, a local bifurcation generically arises when the real part of an eigenvalue $\lambda(p)$ of the Jacobian matrix crosses zero in response to a change in a parameter p. Denoting by p^* the critical parameter value of bifurcation, we get generically two cases: (1) when a real eigenvalue crosses zero: $\lambda(p^*) = 0$, the system undergoes a saddle-node bifurcation (either an elementary saddle-node or a transcritical or a pitchfork bifurcation depending on the number of steady states), (2) when the real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero, the system undergoes a Hopf bifurcation. More precisely, in the second case, we require $a(p^*) = 0$ and $b(p) \neq 0$ in a neighborhood of p^* (see Bosi and Ragot, 2011, p. 76).

Proposition 11 (saddle-node bifurcation) *A saddle-node bifurcation occurs at* $a = a^*$.

A Hopf bifurcation generates limit cycles either attractive (supercritical) or repulsive (subcritical).

Lemma 12 (Hopf bifurcation) *In the case of a three-dimensional system, a Hopf bifurcation generically arises if and only if* $D = ST$ *and* $S > 0$.

5.2.1 Stability properties of the lower steady state

Focus now on the lower steady state N_1.

Proposition 13 *If* $a < a^*$*, then the steady state* N_1 *is a saddle point with a one-dimensional stable manifold.*

Since the dynamic system (11)-(13) has two predetermined variables (k and N), any equilibrium trajectory starting in a small neighborhood of this steady state, generically, move away from the stable manifold and, so, from the steady state. In other terms, the saddle-path stability fails because agents are unable to play with the non-predetermined variable (μ) to jump on the stable manifold.

Corollary 14 *A limit cycle through a Hopf bifurcation never occurs around* N_1.

Let us bridge the stability properties and the steady state to understand the implications of this important corollary. We have found that a Green Paradox arises at N_1 but never at N_2 (Corollary 6). We do not know whether endogenous cycles can occur at N_2, but we know from Corollary 14 that endogenous cycles never occur at N_1. These results imply an incompatibility between the (static) Green Paradox and the existence of endogenous cycles. Such an incompatibility was also previously pointed out by Bosi and Desmarchelier (2016b) in a very different context: a Ramsey model with a pollution externality, viewed as a stock variable, coming from consumption and affecting its marginal utility. In their model, limit cycles may appear along a Laffer Curve but are ruled out under a (static) Green Paradox. In the present framework, a Green Paradox always occurs at N_1 but never at N_2 while cycles never arise around N_1. Endogenous cycles are possible around N_2 as we will see in the next section. In the previous section, we have observed that the (static) Green Paradox seems to be a robust feature of environmental Ramsey models with non-separable utility function. An incompatibility between the (static) Green Paradox and the occurrence of endogenous cycles seems to be a robust feature of such models.

5.2.2 Stability properties of the higher steady state

Focus now on $N_2 > 1/2$.

Proposition 15 *A Hopf bifurcation generically arises at*

$$\varepsilon_{cN}^{*} = \theta \frac{T\left(2N-1\right)\varepsilon_{cc} - \gamma\left(\theta+\delta\right)\frac{1-\alpha(k)}{\sigma(k)}}{\alpha\gamma T\left(1-N\right)} \tag{25}$$

where N and $T = \theta + 1 - 2N$ are evaluated at the steady state, provided that

$$\frac{1}{2} < N < \frac{1+\theta}{2} \tag{26}$$

The double inequality (26) is equivalent to $0 < T < \theta$.

In order to provide an intuition of Proposition 15, we need to avoid the potential interdependence between the elasticities leading to a misleading interpretation. In the isoelastic case, the elasticities are constant and a clear-cut interpretation of a limit cycle is possible as we will see in section 6.

5.2.3 Codimension two

For now, we have considered only one bifurcation parameter (η). The codimension of a bifurcation is the number of parameters to vary for the bifurcation to occur (see Kuznetsov (1998) among others). We introduce the definition of a codimension-two bifurcation which is pertinent in our model.

We know that a Hopf bifurcation may occur around N_2 (Proposition 15). A saddle-node bifurcation may also occur (Proposition 11). Thus, we address the issue of the simultaneous occurrence of these two bifurcations. More precisely, when a saddle-node bifurcation takes place, it is possible to observe a limit cycle around N_2 coalescing with the saddle point N_1 instead of a saddle-node bifurcation involving two saddle-points. Such a bifurcation is known as Bogdanov-Takens.

Definition 16 (Bogdanov-Takens bifurcation) *Consider the curve in a parametric (p_1, p_2)-plane along which a real eigenvalue λ_1 remains equal to zero. Assume that, when the pair (p_1, p_2) moves along this curve, an additional real eigenvalues λ_2 becomes zero at $(p_1, p_2)^*$. In this case, the central manifold becomes two-dimensional and a Bogdanov-Takens (or double-zero) bifurcation arises at $(p_1, p_2)^*$.*

Proposition 17 *A Bogdanov-Takens bifurcation generically occurs if and only if $D = S = 0$.*

6 Isoelastic case

As seen above, to provide clear interpretations, we need to introduce isoelastic fundamentals and compute explicit bifurcation values. In this case, the elasticities appearing in the matrix (3) are constant and independent of each other. To this purpose, we adopt the technology and preferences (1) and (5). The second-order elasticities of preferences are given by (6).

6.1 Optimal taxation

We maximize the welfare at the steady state and, therefore, we suppose the tax rate constant over time. The government computes the optimal tax rate to maximize the utility of the representative agent.

Assumption 4 *$a > b$.*

This assumption is more restrictive than Assumption 3.

Lemma 18 *There is a unique tax rate τ_S solution to $N = 1/2$.*

We observe that τ_S is also the saddle-node bifurcation value. It is equivalent to a^*.

We are now interested in the tax rate which maximizes the welfare at N_1 and N_2.

Proposition 19 *Let W_i be the welfare evaluated at the steady state N_i with $i = 1, 2$. Let $\tau_i^* = \arg\max_\tau W_i(\tau)$. Then,*

$$\tau_1^* = \tau_S < \tau_2^* < 1$$

Let us provide an interpretation. After Proposition 7, we have seen that a higher green tax lowers the welfare at N_1. Intuitively, at this steady state, the optimal solution consists in lowering the green tax rate as far as possible. According to Corollary 6, a lower τ raises N_1 and lowers N_2. Thus, τ reaches a value such that $N_1 = N_2 = 1/2$. According to Lemma 18, such a value τ_S exists and is unique. τ_S is the lowest possible τ for which a steady state exists, and is precisely the optimal green tax rate at N_1. Hence, the optimal fiscal policy when the economy is at N_1 is lowering sufficiently τ until N_1 collides with N_2.

Conversely, at N_2, the green tax turns out to be welfare-improving (Proposition 7). This is possible because a higher green tax rate increases N_2 (Corollary 6) and raises the utility level (Assumption 2). However, at the same time, a higher green tax rate lowers the consumption level (Proposition 5) and, so, the utility level (Assumption 2). This trade-off suggests the existence of an interior solution as optimal tax rate. Proposition 19 shows that this optimal tax rate (τ_2^*) exists and is unique.

We know that $\tau_2^* \in (\tau_S, 1)$ but we do not know its explicit expression. However, we can know the qualitative impact of η on τ_2^*, that is how the relative preference for nature with respect to consumption affects the green tax rate. Intuition suggests that the higher the relative preference for environmental quality, the higher the green tax.

Proposition 20 *Under Assumption 4, $d\tau_2^*/d\eta > 0$.*

We observe that

$$\frac{dN}{dc} = -\frac{u_c}{u_N} = -\frac{\varepsilon_c}{\varepsilon_N}\frac{N}{c} = -\frac{1}{\eta}\frac{N}{c}$$

Then,

$$\frac{1}{\eta} = -\frac{c}{N}\frac{dN}{dc}$$

is the consumption elasticity of nature in the (c, N)-plane. When η is larger, the indifference curve becomes flatter. This means that a smaller variation in nature is required to compensate a change in consumption. In other terms, the household weights more nature in the utility function. Proposition 20 means that, when nature becomes more important (higher η), the green tax has to become larger to finance natural maintenance.

6.2 Local dynamics

Corollary 21 (saddle-node bifurcation) *In this case, the saddle-node critical value is explicitly given by*

$$a^* \equiv b\tau + \frac{1}{4}\left(\frac{1}{A}\left[\frac{\theta+\delta}{\alpha(1-\tau)}\right]^{\alpha}\right)^{\frac{1}{1-\alpha}}$$

The critical parameter to understand the cross effect of nature on consumption demand is η. The analysis of this effect is an added value of our contribution with respect to the existing literature.

Proposition 22 (Hopf bifurcation) *If*

$$\frac{1}{2} < N < \frac{1+\theta}{2} \tag{27}$$

then, a limit cycle arises near the steady state through a Hopf bifurcation at

$$\eta = \eta_H \equiv \frac{\varepsilon\theta(2N-1) + \gamma(1-\alpha)(\theta+\delta) + \varepsilon D/T}{\alpha\gamma(\varepsilon-1)(1-N)} \tag{28}$$

In the isoelastic case, we obtain an explicit bifurcation value and a straightforward interpretation. We know that limit cycles arise only near N_2 (Corollary 14 and Proposition 15). Then, $\eta_H > 0$ if and only if $\varepsilon > 1$.

According to (4) jointly with (6), we find that a necessary (but not sufficient) condition for the occurrence of a limit cycle near N_2 is that nature and consumption are substitutable goods ($\varepsilon > 1$): a higher natural quality implies a lower consumption demand. Now, let us interpret the occurrence of endogenous cycles. Let the economy be at N_2 and consider an exogenous rise in the pollution level today. This implies a lower environmental quality which increases the consumption demand (because consumption and nature are substitutable). The household reduces her savings. This implies a drop tomorrow in the capital as well as in the production level and, finally, in the pollution stock. Thus, a higher pollution level today leads to a lower pollution tomorrow giving rise to an endogenous fluctuation.

We observe that this critical value is well-defined because N, D, T and γ in the RHS do not depend on η.

A Bogdanov-Takens bifurcation involves two parameters (codimension two). In our model, a^* does not depend upon η. This suggests the possibility of a Bogdanov-Takens bifurcation, namely because the limit cycle around N_2 considered in Proposition 22 collides with the saddle-point N_1 (Proposition 13). Our strategy consists of fixing $a = a^*$ to obtain a saddle-node bifurcation (Proposition 11) and then to recompute η_H with $a = a^*$ to ensure the existence of a Hopf bifurcation in a neighborhood of the saddle-node bifurcation point ($a = a^*$).

Proposition 23 *A Bogdanov-Takens (BT) bifurcation occurs at* $(a, \eta) = (a, \eta)^*$ *where*

$$\eta^* \equiv 2\frac{1-\alpha}{\alpha}\frac{\theta+\delta}{\varepsilon-1}$$

is the critical BT bifurcation value.

7 Simulation

To study the stability of the limit cycle around N_2 and the possible degeneracy of both the saddle-node and the Hopf bifurcation, we perform a numerical simulation under the following yearly calibration:

Parameter	θ	δ	τ	α	A	ε	b
Value	0.05	0.1	0.1	0.33	1	2	0.1

(29)

Calibration (29) implies:

$$(a^*, \eta^*) = (0.18858, 0.60909)$$

We perform an equilibrium continuation using the MATCONT package for MATLAB with calibration (29) (see Fig.1). In this figure, LP, H and BT stand for Limit Point (saddle-node), Hopf and Bogdanov-Takens. This points are computed and represented by MATCONT when a saddle-node, a Hopf and a Bogdanov-Takens bifurcation occur near the steady state.

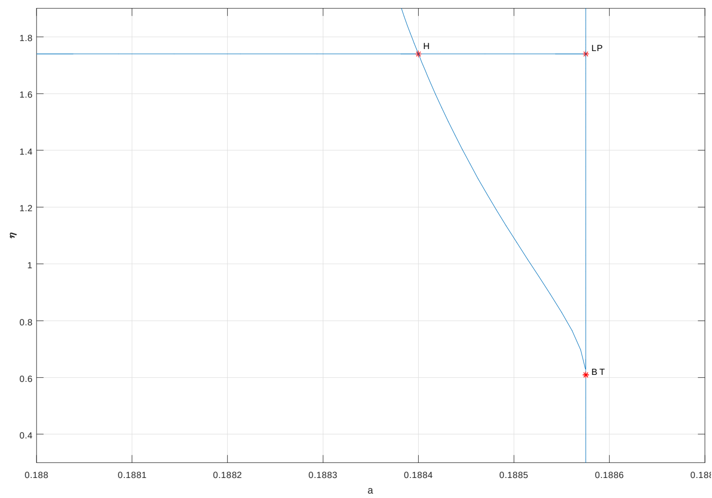

Fig. 1. Equilibrium continuation in the (a, η)-space.

Since a Hopf bifurcation occurs only around the higher steady state, the continuation exercise focuses only on N_2. The curve represents the locus of Hopf bifurcations: $\{(a, \eta_H(a))\}$, where

$$\eta_H(a) \equiv \frac{\varepsilon\theta\left[2N_2(a) - 1\right] + \gamma(1-\alpha)(\theta+\delta) + \varepsilon\frac{D(N_2(a))}{T(N_2(a))}}{\alpha\gamma(\varepsilon-1)\left[1 - N_2(a)\right]}$$

where $N_2(a)$, $T(N_2(a))$ and $D(N_2(a))$ are given by (17), (23) and (24) respectively.

We start by considering an arbitrary value $a_0 = 0.1884 < 0.18858 = a^*$ (Proposition 4). The corresponding Hopf critical value for η is $\eta_H = 1.7392$.

For any η the saddle-node bifurcation value for a is $a^* = 0.18858$ (the line $LP - BT$ is vertical because a^* does not depend on η). In particular, the Limit Point (LP) corresponding to $\eta = 1.7392$ is $LP = (0.18858, 1.7392)$.

Increasing a from $a_0 = 0.1884$ to $a^* = 0.18858$ we obtain all the Hopf bifurcation points $(a, \eta_H(a))$ along the curve $C \equiv \{(a, \eta_H(a))\}_{a\in[a_0,a^*]}$ from H to BT. In the range $[a_0, a^*) \ni a$, we have two distinct steady states. When a attains the maximal value a^* these two steady states coalesce and the Hopf bifurcation point $(a, \eta_H(a))$ reaches the ending point BT along the curve C while the economy experiences a Bogdanov-Takens bifurcation.

At the Hopf bifurcation point (H), the steady state is given by

$$(\mu, k, N) = (3.2759601, 2.7719, 0.515665)$$

with eigenvalues

$$(\lambda_1, \lambda_2, \lambda_3) = (-0.172903i, 0.172903i, 0.0186708)$$

The corresponding first Lyapunov coefficient is given by $l_1 = 2.441148 > 0$. Its positivity means that the Hopf bifurcation is subcritical, that is the limit cycle arising near N_2 is unstable (Fig. 2).

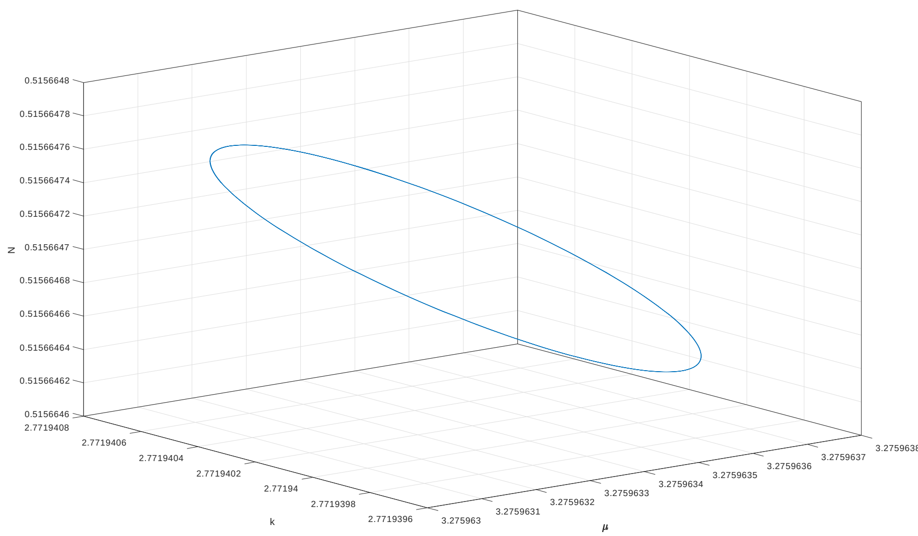

Fig. 2. Unstable limit cycle.

At the saddle-node bifurcation (LP), the steady state becomes:

$$(\mu, k, N) = (3.4565301, 2.7719, 0.5)$$

with eigenvalues[1]

$$(\lambda_1, \lambda_2, \lambda_3) = (0.0250006 - 0.180085i, 0.0250006 + 0.180085i, 0)$$

For now, we have considered codimension-one bifurcations (saddle-node and Hopf).

According to Proposition 23 we focus now on codimension two. The Bogdanov-Takens bifurcation arises at $(a, \eta) = (a^*, \eta^*) = (0.18858, 0.60909)$. The Bogdanov-Takens bifurcation (BT) occurs when conditions for the saddle-node bifurcation and for the Hopf bifurcation meet each other.[2]

[1] The bifurcation is non-degenerate because the quadratic coefficient associated with the normal form of the saddle-node bifurcation is $a\,(0) = 0.04467276 \neq 0$ (see Kuznetsov (1998), p.85, among others).

[2] The BT bifurcation is non-degenerate because the two quadratic coefficients associated with the normal form of the Bogdanov-Takens bifurcation are nonzero: $(a\,(0)\,, b\,(0)) = (-0.1643409, -7.824758) \neq (0, 0)$ (see Kuznetsov (1998), p320, among others).

At the Bogdanov-Takens point, the steady state becomes:

$$(\mu, k, N) = (1.5792, 2.7719, 0.5)$$

with real eigenvalues

$$(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0.0499996)$$

As in Kuznetsov et al. (2014), at the Bogdanov-Takens point, the orbit describes a parasitic loop near the saddle-point (Fig. 3). The parasitic loop typically arises when the limit cycle and the saddle-point collides.

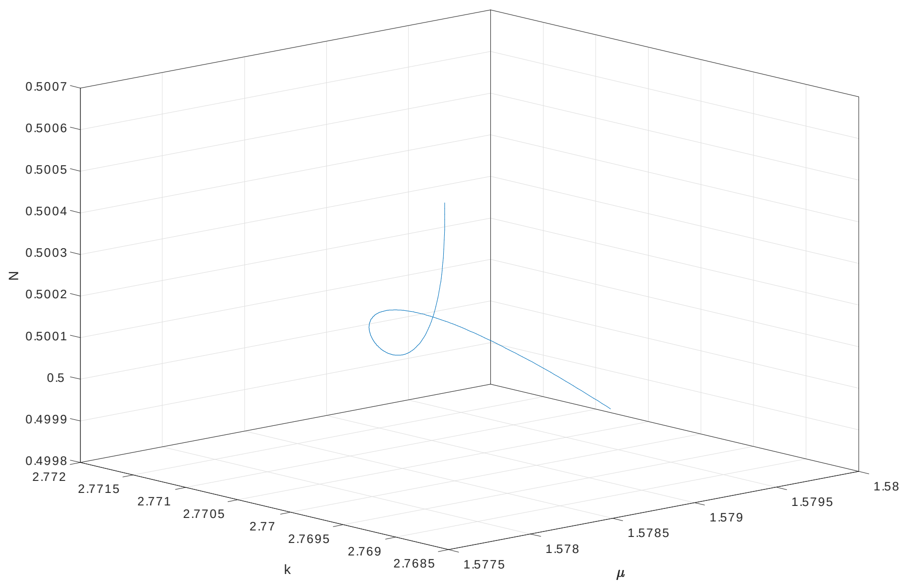

Fig. 3. Parasitic loop.

8 Conclusion

In this paper, we have studied a competitive Ramsey model where a pollution externality, coming from production, impairs a renewable resource which affects the consumption demand. A proportional tax, levied on the production level, was introduced to finance public depollution expenditures. Within this stylized framework, we have shown that the environmental impact of production determines the number of steady state: two steady states coexist (the one with low resource level, the other with high level) under a low impact while under an excessive impact, we have shown that the two steady states collide through a saddle-node bifurcation and disappear. In the long run, we have proven that a higher green tax rate always reduces the resource level in the low steady state. Such a result is counter-intuitive but very close to the Green Paradox (Sinn, 2008): a greener tax may exacerbate environmental damages and a higher green tax rate is not always the best way to clean the environment. In addition, we have found that the green tax may be welfare-improving at the high steady state but never at the low one. Then, the optimal policy will consist in lowering the tax rate as much as possible at the low stationary state. At the high steady state, an optimal green tax rate exists, which is positive and unique. In the short run, we have pointed out the possible emergence of a limit cycle near the high steady state through a Hopf bifurcation if and only if the resource and consumption are substitutable goods. Conversely, any endogenous cycle is ruled out around the low steady state whatever the resource effect on consumption demand. This result is surprising since, in a very similar framework, Wirl (2004) has observed that cycles occur only around the low steady state. Even if Wirl (2004) considers a separable utility function, the difference with our result rests on the fact that he focuses only on the central planner solution while we consider instead the competitive equilibrium. Finally, we have investigated the possibility of codimension-two bifurcations and we have proven analytically the existence of a Bogdanov-Takens bifurcation, so unusual in economic models.

9 Appendix

Proof of Proposition 2

According to the Pontryagin's approach, we derive the first-order conditions:

$$\begin{aligned} \partial H/\partial \lambda &= \dot{h} \\ \partial H/\partial h &= -\dot{\lambda} = \lambda (r-\delta) \quad (30) \\ \partial H/\partial c &= 0 = e^{-\theta t} u_c - \lambda \quad (31) \end{aligned}$$

jointly with the transversality condition: $\lim_{t\to\infty} \lambda(t) h(t) = 0$. We introduce a new multiplier: $\mu \equiv \lambda e^{\theta t}$, to obtain from (30) and (31) Proposition 2.

Proof of Proposition 4

By definition of steady state, $\dot{\mu} = \dot{k} = \dot{N} = 0$. (11) gives (14), the capital intensity of MGR, while (12) gives (15), the consumption demand of MGR. Under Assumption 1, ρ is invertible. (13) yields

$$N(1-N) = (a-b\tau) f(k) = (\theta+\delta) \frac{a-b\tau}{1-\tau} \frac{k}{\alpha(k)}$$

Solving for N, we get (16) and (17). Solutions are real iff $a \leq a^*$.

Proof of Proposition 5

We differentiate (14), (15), (16) and (17).

Proof of Corollary 6

Consider (19) jointly with (16) and (17), and remark that, since $a > b$, then:

$$\frac{\alpha(a-b) + (1-\tau) b}{(1-\alpha)(a-b\tau)} > 0$$

Proof of Proposition 7
We derive (20) using (21) to obtain

$$\frac{\tau W'(\tau)}{W(\tau)} = \varepsilon_c \frac{\tau c'(\tau)}{c(\tau)} + \varepsilon_N \frac{\tau N'(\tau)}{N(\tau)} \tag{32}$$

Proof of Proposition 8
Indeed, in this case,

$$\frac{\varepsilon_c}{\varepsilon_N} > 0 > \frac{1-N_1}{2N_1-1} \frac{\alpha(a-b)+(1-\tau)b}{a-b\tau}$$

Proof of Lemma 9
Noticing that $\omega(k)/[k\rho(k)] = (1-\alpha)/\alpha$, we obtain the following Jacobian matrix

$$J \equiv \begin{bmatrix} \frac{\partial g_1}{\partial \mu} & \frac{\partial g_1}{\partial k} & \frac{\partial g_1}{\partial N} \\ \frac{\partial g_2}{\partial \mu} & \frac{\partial g_2}{\partial k} & \frac{\partial g_2}{\partial N} \\ \frac{\partial g_3}{\partial \mu} & \frac{\partial g_3}{\partial k} & \frac{\partial g_3}{\partial N} \end{bmatrix} = \begin{bmatrix} 0 & (\gamma-\theta)\frac{\alpha}{\sigma}\frac{\mu}{k} & 0 \\ -\gamma\frac{k}{\mu}\frac{1}{\varepsilon_{cc}} & \theta & \gamma\frac{k}{N}\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \\ 0 & \alpha(N-1)\frac{N}{k} & 1-2N \end{bmatrix}$$

Usual computations give T, S and D.
Proof of Proposition 10
Indeed, in this case, $D > 0$ and there is at least one positive real eigenvalue.
Proof of Proposition 11
We observe that $N = 1/2$ implies $D = 0$ and that $N_1 = N_2 = 1/2$ when $a = a^*$.
Proof of Lemma 12
Necessity In a three-dimensional dynamic system, we require at the bifurcation value: $\lambda_1 = ib = -\lambda_2$ with no generic restriction on λ_3 (see Bosi and Ragot (2011) or Kuznetsov (1998) among others). The characteristic polynomial of J is given by: $P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - T\lambda^2 + S\lambda - D$. Using $\lambda_1 = ib = -\lambda_2$, we find $D = b^2\lambda_3$, $S = b^2$, $T = \lambda_3$. Thus, $D = ST$ and $S > 0$.
Sufficiency In the case of a three-dimensional system, one eigenvalue is always real, the others two are either real or nonreal and conjugated. Let us show that, if $D = ST$ and $S > 0$, these eigenvalues are nonreal with zero real part and, hence, a Hopf bifurcation generically occurs.
We observe that $D = ST$ implies

$$\lambda_1\lambda_2\lambda_3 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)$$

or, equivalently,

$$(\lambda_1+\lambda_2)\left[\lambda_3^2 + (\lambda_1+\lambda_2)\lambda_3 + \lambda_1\lambda_2\right] = 0 \tag{33}$$

This equation holds if and only if $\lambda_1 + \lambda_2 = 0$ or $\lambda_3^2 + (\lambda_1+\lambda_2)\lambda_3 + \lambda_1\lambda_2 = 0$. Solving this second-degree equation for λ_3, we find $\lambda_3 = -\lambda_1$ or $-\lambda_2$. Thus, (33) holds if and only if $\lambda_1 + \lambda_2 = 0$ or $\lambda_1 + \lambda_3 = 0$ or $\lambda_2 + \lambda_3 = 0$. Without loss of generality, let $\lambda_1 + \lambda_2 = 0$ with, generically, $\lambda_3 \neq 0$, a real eigenvalue. Since

$S > 0$, we have also $\lambda_1 = -\lambda_2 \neq 0$. We obtain $T = \lambda_3 \neq 0$ and $S = D/T = \lambda_1 \lambda_2 = -\lambda_1^2 > 0$. This is possible only if λ_1 is nonreal. If λ_1 is nonreal, λ_2 is conjugated, and, since $\lambda_1 = -\lambda_2$, they have a zero real part.

Proof of Proposition 13

$N_1 < 1/2$ implies $T > 0$ and $D < 0$. $D < 0$ implies that one eigenvalue is real and negative (say $\lambda_3 < 0$). $T > 0$ entails that (1) at least one eigenvalue is real and positive (say $\lambda_1 > 0$) or (2) there are two nonreal conjugate eigenvalues with a positive real part (say $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > 0$).

(1) In the first case, since $\lambda_1\lambda_2\lambda_3 < 0$, we have $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$.

(2) In the second case, $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > 0$ and $\lambda_3 < 0$.

In both the cases, the stable manifold is one-dimensional and the unstable manifold is two-dimensional. But, in case (1), the trajectories departing from the steady state along the unstable manifold are monotonic, while, in case (2), these trajectories are spiral-shaped.

Proof of Corollary 14

As seen above, a Hopf bifurcation occurs when the real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero. More precisely, we require $a(p^*) = 0$ and $b(p) \neq 0$ in a neighborhood of p^*.

The proof of Proposition 13 considers two cases: (1) $\lambda_1 > 0$ or (2) $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > 0$. Thus, the case of nonreal and conjugate eigenvalues with $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 < 0$ is excluded. A Hopf bifurcation occurs when the real part of two nonreal and conjugate eigenvalues ($a(p) = \operatorname{Re}\lambda_1(p) = \operatorname{Re}\lambda_2(p)$) crosses zero at p^*. In this case, $\operatorname{Re}\lambda_1(p) = \operatorname{Re}\lambda_2(p) < 0$ for some p in a neighborhood of p^*. But this is impossible.

Proof of Proposition 15

We apply the Lemma 12: the equality (25) is equivalent to $D = ST$, while the inequality $S > 0$ to

$$\varepsilon_{cN}^* < -\frac{\gamma(\theta+\delta)\frac{1-\alpha(k)}{\sigma(k)} + \theta(1-2N)\varepsilon_{cc}}{\alpha\gamma(1-N)} \tag{34}$$

Replacing expression (25) in the LHS of inequality (34) and solving for N, we get (26).

Proof of Proposition 17

A Bogdanov-Takens bifurcation arises if and only if two real eigenvalues cross zero (say, $\lambda_1 = \lambda_2 = 0$). Therefore, $D = \lambda_1\lambda_2\lambda_3 = 0$ and $S = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0$. Conversely, if $D = 0$, at least one eigenvalue is zero, say λ_1. $S = \lambda_2\lambda_3 = 0$ implies that another eigenvalue is zero, say λ_2.

Proof of Lemma 18

Equation $N = 1/2$ is equivalent to

$$(a - b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}} = \frac{1}{4A}\left(\frac{\theta+\delta}{\alpha A}\right)^{\frac{\alpha}{1-\alpha}} \tag{35}$$

The LHS of (35) is strictly decreasing in τ, while the RHS does not depend on τ. Thus, the solution is unique.

Proof of Proposition 19

We evaluate the welfare at the generic steady state N:

$$W(\tau)=\int_0^{\infty}e^{-\theta t}u(c(\tau),N(\tau))\,dt=\frac{u(c(\tau),N(\tau))}{\theta}$$

We maximize this value with respect to τ.

$$W'(\tau)=\frac{1}{\theta}\left[u_c c'(\tau)+u_N N'(\tau)\right]=\frac{u}{\theta\tau}\left[\frac{cu_c}{u}\frac{\tau c'(\tau)}{c(\tau)}+\frac{Nu_N}{u}\frac{\tau N'(\tau)}{N(\tau)}\right]$$

Since α is constant, we have

$$\begin{aligned}\frac{\tau k'(\tau)}{k(\tau)}&=\frac{\tau}{1-\tau}\frac{\rho(k)}{k\rho'(k)}=-\frac{\tau}{1-\tau}\frac{1}{1-\alpha}\\ \frac{\tau c'(\tau)}{c(\tau)}&=\frac{\tau k'(\tau)}{k(\tau)}=-\frac{\tau}{1-\tau}\frac{1}{1-\alpha}\\ \frac{\tau N'(\tau)}{N(\tau)}&=\frac{1-N}{1-2N}\left[\frac{\tau}{1-\tau}\frac{a-b}{a-b\tau}+\frac{\tau k'(\tau)}{k(\tau)}\right]=\frac{\tau}{1-\tau}\frac{1-N}{1-2N}\left(\frac{a-b}{a-b\tau}-\frac{1}{1-\alpha}\right)\end{aligned}$$

Therefore,

$$W'(\tau)=\frac{1}{1-\tau}\frac{u}{\theta}\left[\frac{Nu_N}{u}\frac{1-N}{1-2N}\left(\frac{a-b}{a-b\tau}-\frac{1}{1-\alpha}\right)-\frac{cu_c}{u}\frac{1}{1-\alpha}\right]$$

In the isoelastic case (function (5)), we obtain

$$\varepsilon_c\equiv\frac{cu_c}{u}=1-\varepsilon \text{ and } \varepsilon_N\equiv\frac{Nu_N}{u}=\eta(1-\varepsilon) \tag{36}$$

Thus,

$$W'(\tau)=\frac{1}{1-\tau}\frac{(cN^{\eta})^{1-\varepsilon}}{\theta}\left[\eta\frac{1-N}{1-2N}\left(\frac{a-b}{a-b\tau}-\frac{1}{1-\alpha}\right)-\frac{1}{1-\alpha}\right]$$

and

$$W'(\tau)>0\Leftrightarrow\eta\frac{1-N}{1-2N}\left(\frac{a-b}{a-b\tau}-\frac{1}{1-\alpha}\right)-\frac{1}{1-\alpha}>0$$

where N is lower or higher:

$$\begin{aligned}N_1&=\frac{1}{2}\left[1-\sqrt{1-4A(a-b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha A}{\theta+\delta}\right)^{\frac{\alpha}{1-\alpha}}}\right]>0\\ N_2&=\frac{1}{2}\left[1+\sqrt{1-4A(a-b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha A}{\theta+\delta}\right)^{\frac{\alpha}{1-\alpha}}}\right]>0\end{aligned}$$

We observe that $N_1'(\tau)<0<N_2'(\tau)$.

We now that τ_S is the unique solution of $N = 1/2$. N_1 and N_2 are real iff

$$(a - b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}} \leq \frac{1}{4A}\left(\frac{\theta+\delta}{\alpha A}\right)^{\frac{\alpha}{1-\alpha}}$$

The LHS decreases from a to 0 and crosses the RHS iff

$$a > \frac{1}{4A}\left(\frac{\theta+\delta}{\alpha A}\right)^{\frac{\alpha}{1-\alpha}}$$

Then, $N_1 = N_2 = 1/2$ when $\tau = \tau_S$ and N_1 and N_2 are real iff $\tau \geq \tau_S$.
Under Assumption 4,

$$(1-\alpha)\frac{a-b}{a-b\tau} - 1 < 0$$

and

$$\begin{aligned} W_1'(\tau) &> 0 \Leftrightarrow 1 + \frac{2}{\eta\left[1-(1-\alpha)\frac{a-b}{a-b\tau}\right]} < -\frac{1}{\sqrt{1-4A(a-b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha A}{\theta+\delta}\right)^{\frac{\alpha}{1-\alpha}}}} \\ W_2'(\tau) &> 0 \Leftrightarrow 1 + \frac{2}{\eta\left[1-(1-\alpha)\frac{a-b}{a-b\tau}\right]} < +\frac{1}{\sqrt{1-4A(a-b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha A}{\theta+\delta}\right)^{\frac{\alpha}{1-\alpha}}}} \end{aligned}$$

We notice that

$$1 + \frac{2}{\eta\left[1-(1-\alpha)\frac{a-b}{a-b\tau}\right]} > 0$$

increases from

$$1 + \frac{2}{\alpha\eta + (1-\alpha)\eta\frac{b}{a}} \text{ to } 1 + \frac{2}{\alpha\eta}$$

Then, at the steady state N_1, the optimal policy is to lower τ to τ_S.
Focus now on N_2. We observe that $W_2'(\tau) > 0$ iff

$$\varphi(\tau) \equiv \left(1 + \frac{2}{\eta\left[1-(1-\alpha)\frac{a-b}{a-b\tau}\right]}\right)^{-2} > 1 - 4A(a-b\tau)(1-\tau)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha A}{\theta+\delta}\right)^{\frac{\alpha}{1-\alpha}} \equiv \psi(\tau)$$

φ is decreasing while ψ is increasing in τ. Moreover,

$$\begin{aligned} \varphi(\tau_S) &\equiv \left(1 + \frac{2}{\eta\left[1-(1-\alpha)\frac{a-b}{a-b\tau_S}\right]}\right)^{-2} > 0 \equiv \psi(\tau_S) \\ \varphi(1) &\equiv \left(1 + \frac{2}{\alpha\eta}\right)^{-2} < 1 \equiv \psi(1) \end{aligned}$$

In other terms, $W_2'(\tau) > 0 \Leftrightarrow \tau < \tau_2^* \in (\tau_S, 1)$. Then, $\tau_2^* \in (\tau_S, 1)$ is the $\arg\max_\tau W_2(\tau)$.

Proof of Proposition 20

τ_2^* maximizes the welfare evaluated at N_2. Thus, $W_2'(\tau_2^*) = 0$ and, according to 32,

$$\varepsilon_c \frac{\tau c'(\tau_2^*)}{c(\tau_2^*)} + \varepsilon_N \frac{\tau N_2'(\tau_2^*)}{N_2(\tau_2^*)} = 0 \tag{37}$$

Replacing (18), (19) and (36) in (37), we find

$$\eta = \frac{a - b\tau_2^*}{\alpha(a-b) + (1-\tau_2^*)b} \frac{2N_2(\tau_2^*) - 1}{1 - N_2(\tau_2^*)}$$

Differentiating both the sides with respect to η and τ_2^*, we get

$$\frac{d\tau_2^*}{d\eta} = \frac{1}{\eta}\left(\frac{b(1-\alpha)(a-b)}{(a-b\tau_2^*)[\alpha(a-b)+(1-\tau_2^*)b]} + \frac{N_2'(\tau_2^*)}{[1-N_2(\tau_2^*)][2N_2(\tau_2^*)-1]}\right)^{-1}$$

We observe that $a > b$ and $a > b\tau_2^*$ (Assumption 4), that $N_2'(\tau_2^*)$ (Corollary 6) and that $1/2 < N_2(\tau_2^*) < 1$. Then, both the blocks in the RHS are positive.

Proof of Proposition 22

Solve the equation $D = ST$ for η and find $\eta = \eta_H$. Moreover,

$$S(\eta_H) = \frac{\gamma(1-\alpha)(\theta+\delta)}{\varepsilon} \frac{1-2N}{2N-\theta-1} > 0$$

iff condition (27) holds.

Proof of Proposition 23

Reconsider Proposition 17. Solve the system $D = S = 0$ for (a, η). More precisely, $D = 0$ gives $a = a^*$ or, equivalently, $N = 1/2$. Replacing $N = 1/2$ in $S = 0$ and solving for η, we get $\eta = \eta^*$.

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