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The Fossil Energy Interlude: Optimal Building, Maintaining and Scraping a Dedicated Capital, and the Hotelling Rule

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Abstract

It is well known that the price and consumption paths of most nonrenewable resources, including the fossil primary energies, do not follow the paths predicted by the standard Hotelling rule (Krautkraemer,1998, Gaudet, 2007). We develop a model in which a dedicated capital together with the fossil fuel are both required to produce useful energy. Starting from a state of the economy in which the fossil fuel is not yet exploited, we characterize the optimal path of the double transition: The first transition from the initial renewable energy regime to a mixed or full fossil fuel regime and later the second transition from the fossil fuel regime back to a renewable energy regime when the available stock of the fossil fuel becomes more and more rare. We show that, absent any technical progress, the useful energy price must first decrease, next be constant during the phase of maximum expansion of the fossil fuel energy consumption before entering the phase of decreasing use of the fossil energy. Only this third phase of decreasing fossil fuel consumption looks like a standard Hotelling path.

Keywords:Nonrenewable resources, Renewable resources, Energy transition, Hotelling rule.

JEL Classification: Q00, Q32, Q43, Q54.

Abstract

Il est bien connu que les sentiers de prix et de consommation de la plupart des resources non renouvelables, y compris les ressources énergétiques primaires, diffèrent des sentiers prédits par la règle d'Hotelling (Krautkramer, 1998; Gaudet, 2007). Nous proposons un modèle dans lequel, pour produire de l'énergie utile deux facteurs sont nécessaires, une ressource primaire et des biens capitaux dédiés. Partant d'un état de la société dans lequel l'énergie fossile n'est pas encore exploitée, nous caractérisons le sentier optimal d'une double transition. La première au cours de laquelle l'énergie fossile se substitue énergie renouvelable, la seconde au cours de laquelle l'énergie fossile est progressivement abandonnée au profit de l'énergie renouvelable. Nous montrons qu'il existe un plateau d'exploitation maximale de la ressource fossile plutôt qu'un pic et que le sentier ne suit ce que prédit la règle d'Hotelling qu'au cours de la phase de régression de l'utilisation de la ressource non renouvelable.

Keywords: Ressources non renouvelables, Ressources renouvelables, Transition énergétique, Règle d'Hotelling.

JEL Classification: Q00, Q32, Q43, Q54.

1 Introduction

A recurrent theme in the nonrenewable resource literature is that the price and the consumption paths of most non renewable resources, including the fossil energy primary resources, do not follow the paths predicted by the most simple formulations of the Hotelling rule (Hotelling, 1931). A large sample of resources the price paths of which do not apparently comply with the rule is given in Krautkraemer, (1998) and Gaudet, (2007), for instance, and different possible explanations of the divergence are listed (see also Farzin, (1992), and Livernois, (2009)). Amongst the sources of divergence or the reasons justifying a reformulation of the arbitrage conditions underlying the rule, is the necessity to take into account the heavy investments required in many mining industries (Puu, (1977), Crémer, (1979), Campbell, (1980), Lasserre, (1982), (1985-a) and (1985-b), Cairns and Lasserre, (1986), Olsen, (1989), Cairns and Lasserre, (1991), Lozada, (1993), Cairns (1998) and (2001), Holland (2003-a)). A related literature is the literature on the set-up costs which can be seen as necessary investment costs, and the corresponding existence problems of competitive equilibria in nonrenewable resource markets (Hartwick *et al.* (1986), Holland, (2003-b), Vu and Im, (2011), Bommier *et al.* (2018)).[1]

Whatever the kind of marginal extraction cost modelling to be chosen for a right reformulation of the Hotelling rule, a closely linked problem is the way how the primary resource Hotelling rule translates into prices paths of goods whose the primary resource is a substantial input. Amongst these resources a special attention must be devoted to the primary energy resources. In the transformation process of underground fossil energy into useful energy, that is the energy consumed by the final users, the main part of the necessary capital is not in the mining industry. For example to transform underground oil into useful transportation energy services, the extracted petroleum must be refined, in some cases the output of the refinery industry is transformed into another form of energy like electricity and maybe one time more transformed in between, like hydrogen to feed fuel cells, and next used together with cars, buses, trucks, trains, ships or planes and the necessary infrastructures allowing to put into operation these transportation devices and finally obtain

[1]A larger acceptation of the capital concept would lead to also include the literature on the exploration and R&D costs.

useful mechanical energy. [2]

A characteristic of a large part of these capital goods is that they are strongly adapted to the use of fossil fuels although some of which could be retrofitted for the use of renewable energy but at some cost.

In this paper we explore the way how starting from a pure renewable energy system, the fossil fuel industry should expand and finally decline, and the energy sector come back to a pure renewable system, when the production of the fossil useful energy requires both a nonrenewable resource and dedicated capital, what we call the fossil fuel interlude or the twofold energy transition. Equivalently we explore the way how the price or full marginal cost of the primary nonrenewable energy and the price of the useful energy are linked together through time assuming that they are optimally exploited and produced.

To keep the analysis tractable we assume that there exist only two primary resources, a nonrenewable one that we call 'coal' and a renewable one that we call 'solar', both from which can be produced a unique kind of useful energy. To produce useful energy from coal both coal and a dedicated capital are required. We do not model the capital in the mining sector. Coal is extracted at a constant marginal cost, without fixed cost, so that the optimal price path of the extracted coal is the most simple form of an Hotelling path. The production function of useful energy from coal is a three inputs Leontief function: coal, capital services and another input, e.g labor. Although capital services and coal are strictly complementary inputs at each date, this is not the case in the long run for coal and capital itself. For example with a constant operating capital stock, more or less coal can be processed according to the length of the period during which the capital is maintained into operation at some maintenance cost. The same holds during the period of capital building since for a given capacity to be erected at the end of the investment period, more or less coal can be processed depending upon the speed at which the capital is accumulated. Last the same holds also during the decline period. From the same capital at the beginning of the decline period more or less coal can be processed according to the speed at which the capital is scraped, the scraping rate being a command variable in our

[2]See Fouquet (2008) for a well documented record over several centuries of the price paths of the main useful energies, what Fouquet calls “energy services”.

model. Thus although strictly complementary at each point of time, capital and coal may be seen as inter-temporal substitutes. As for the useful energy produced from the solar primary resource, we do not detail the production process and assume that the useful energy is produced from the solar energy at an increasing marginal cost.

Assuming an increasing marginal cost of investment, a constant unitary capital maintenance cost and absent any technical progress, we show that, facing a stationary useful energy demand, the optimal production path should be a four phases path, first develop the coal useful energy industry, the first transition phase or decline phase of the solar useful energy production, next maintain the coal useful energy production rate at its maximum level during a second phase before entering the third phase of coal useful energy decline, the second transition now from nonrenewable energy back to renewable energy, the fourth and last phase being the restoration of the initial pure solar energy regime. Thus there does not exist a peak of fossil fuel extraction at some date but rather a plateau of maximum production. The reason is that the cost of the capital invested in the production of useful energy must be recovered and this is not feasible along a single peaked production path since the operational lifetime of the last built pieces of equipment would be too short. Thus the result should hold under more general assumptions on the demand and production sides.

During the three first phases of coal exploitation, the total production of useful energy from both the coal and solar sectors, is larger than the production under the pure solar regime. The price path of useful energy is first decreasing during the first phase, next constant during the second phase and increasing during the third one to come back to its initial level. But the price path of the extracted coal, the full marginal cost path of the resource (marginal cost + mining rent) is a standard Hotelling path along which the mining rent increases at a proportional rate equal to the social discount rate. Thus during the two first phases the two price paths are negatively correlated, the decreasing discrepancy corresponding to the decrease of the marginal shadow value of the capital. As time goes on the resource is more and more rare while the accumulated capital is less and less rare, hence the contrasted moves of the two shadow marginal values up to the time at which the accumulated capital is no more rare. Initially, during the investment phase of capital expansion, the decrease of the shadow value of capital dominates the increase of the shadow value of the resource, the mining rent, hence the decrease of the price of the useful energy. During the second phase at the maximum expansion of the coal sector both moves annihilate each other, the increase of the mining rent exactly balancing the decrease of the rental value of capital and the price of useful energy is constant. At the end of this phase the marginal shadow value of capital is nil and this is the time at which begins the decline of the coal exploitation. Then while the coal continues to be more and more rare the accumulated capital is potentially in excess so that its shadow marginal value will stay nil. However to exploit this contracting stock of coal some part of the initially available capital must be kept into operation at a marginal cost equal to the marginal maintenance cost. Then the discrepancy between the useful energy price and the extracted coal price is no more no less than the marginal maintenance cost and now both prices are positively correlated. Under the constant unitary maintenance cost assumption the useful energy price path mimics the Hotelling price path of the extracted coal.

The paper is organized as follows. The model in laid down in the next section 2. In section 3 we formulate the optimality problem and define some auxiliary functions allowing to facilitate the subsequent proofs of the paper. The qualitative properties of the optimal paths are determined in section 4. We conclude in section 5.

2 The model

We consider an economy producing useful energy (U.E) from either a nonrenewable primary energy resource (coal) or from a renewable resource (solar). Coal useful energy (C.U.E) and solar useful energy (S.U.E) are perfect substitutes for the final users. We denote by q_x the instantaneous production rate of C.U.E, by q_y the instantaneous production rate of S.U.E and by q the instantaneous production rate of U.E. Under the perfect substitutability assumption $q(t) = q_x(t) + q_y(y)$, $t \geq 0$. Without useful energy storage possibilities, $q(t)$ is also the U.E consumption rate.

User surplus

Let $u(q(t))$ denote the instantaneous gross surplus of the U.E users. The function u is assumed to satisfy the below standard assumption.[3]

Assumption A. 1 *$u : \mathbb{R}_{++} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and strictly concave with $u'(0^+) = +\infty$ and $u'(\infty) = 0$.* [4]

We sometimes denote by $p(q)$ the marginal surplus function $u'(q)$, the inverse demand function, and by $q^d(p)$ the direct demand function, the inverse of $p(q)$, where p is the U.E price.

Coal U.E production

Producing U.E from coal requires capital and other inputs together with coal. We assume that the C.U.E production function is a Leontief one and without loss of generality that there is only one input other than capital and coal, hence the following Assumption A.2.

Assumption A. 2 *The C.U.E production function reads:*

$$q_x = \min\{K, v, \bar{r}x\}, \;\; K, v, x \geq 0 \textit{ and } 1 > \bar{r} > 0 \;, \tag{2.1}$$

where x, K, v are respectively the coal, the capital and the other input, all measured in energy units.

The assumption $\bar{r} < 1$ means that some energy is lost in the transformation of coal into U.E. In what follows we mainly use its inverse $r = 1/\bar{r} > 1$, the quantity of coal energy required to produce one unit of U.E.[5]

[3]For any function $f(x)$ defined on $X \subseteq R$ we denote by $f(\bar{x}^+)$ and $f(\bar{x}^-), \bar{x} \in X$, respectively, the limits $\lim_{x\downarrow\bar{x}} f(x)$ and $\lim_{x\uparrow\bar{x}} f(x)$ when such limits exist.

[4]Admittedly we need only that $u'(0^+)$ be "sufficiently" high.

[5]It is well known that the energy is constant. Thus by energy loss we mean that some part of the chemical energy of coal is transformed into energy improving the surplus of the final users the remaining being mostly dissipated in useless heat.

We denote by c_v the unitary cost of the input v, assumed to be constant through time. The capital K is dedicated to the production of C.U.E without valuable use outside the C.U.E industry and requires an unitary maintenance cost m assumed to be constant through time. This capital dies by lack of maintenance and cost-free scraping. Let $k(t)$ be the production rate of new capital, then absent any "abrupt" reduction of the installed capacity $K(t)$, the capital stock dynamics satisfies the following condition:[6]

$$\dot{K}(t) = k(t) - \delta(t)K(t), \quad \delta(t) \geq 0, \quad t > 0 \ , \tag{2.2}$$

where $\delta(t)$ is the instantaneous proportional scraping rate.

Let $c_k(k)$ be the production cost function of new capital. This function satisfies the following standard Assumption A.3 where $c_k'(k)$ and $ac_k(k)$ denote respectively the marginal and the average cost functions.

Assumption A. 3 $c_k : \mathbb{R}_+ \to \mathbb{R}_+$ *is a twice continuously differentiable function, strictly increasing and strictly convex, with* $c_k(0) = 0$ *and* $c_k'(0^+) = ac_k(0^+) > 0$.

The assumption $c_k'(0^+) > 0$ means that nothing can be built without some costly input. In what follows we use the more compact notation $\underline{c}_k'$ for $c_k'(0^+)$.

Coal mining

Let X_0 be the initial coal endowment and $X(t)$ what has not yet been extracted at time t, measured in energy units. Denoting by $x(t)$ the extraction rate at time t, the dynamics of $X(t)$ reads:

$$\dot{X}(t) = -x(t) = -rq_x(t) \ . \tag{2.3}$$

To simplify we assume that the coal extraction and delivery costs to the C.U.E industry, $c_x(x)$, are linear, hence:

[6] By "abrupt" reduction at a time t we mean that $K(t^-) > K(t^+)$.

Assumption A. 4 $c_x : \mathbb{R}_+ \to \mathbb{R}_+$ *is the linear function* $c_x(x) = \underline{c}_x x$, $\underline{c}_x > 0$.

Solar U.E production

The sites devoted to the production of solar U.E are exploited by merit order that is by increasing marginal opportunity costs including the loss in net surplus generated by their allocation to the S.U.E production rather than to other net surplus generating uses, for example food production when S.U.E is biofuel. However for some S.U.E production processes the opportunity cost is nil, for example the S.U.E production via photovoltaic cells in desert land. Taking care that other costs than the pure opportunity loss must be supported, we assume, denoting by $c_y(q_y)$ the full cost of S.U.E and by $c'_y(q_y)$ and $ac_y(q_y)$ respectively the marginal and average costs, that:

Assumption A. 5 $c_y : \mathbb{R}_+ \to \mathbb{R}_+$ *is a twice continuously differentiable function, strictly increasing and strictly convex, with* $c_y(0) = 0$ *and* $c'_y(0^+) = ac_y(0^+) > 0$.

The rationale for $c'_y(0^+) > 0$ is the same as for $c'_k(0^+) > 0$, and we use from now the more compact notation $\underline{c}'_y$ for $c'_y(0^+)$.

When the S.U.E industry is the only supplier of U.E, then the marginal surplus $u'(q_y)$ must be equal to its marginal cost $c'_y(q_y)$. Under A.1 and A.5 the solution of $u'(q_y) = c'_y(q_y)$ is unique and strictly positive. We denote by $\tilde{q}_y$, equivalently by $\tilde{q}$, this solution and by $\tilde{p}$ the corresponding U.E price: $\tilde{p} = u'(\tilde{q})$. This is the state of the energy sector at the beginning and at the end of the fossil fuel interlude.

In order that the C.U.E production be a competitive option we must assume that its lowest marginal cost be lower than the marginal cost of the S.U.E production when S.U.E is the only supplier of U.E, hence:

Assumption A. 6 $c_v + r\underline{c}_x + \rho\underline{c}'_k + m < c'_y(\tilde{q})$, *where* $\rho\underline{c}'_k$ *is the rental cost of the least costly piece of C.U.E equipment valued at the social rate of discount* ρ.

The welfare W is the sum of the net surplus discounted at a social rate of discount $\rho > 0$, constant through time.

3 The social planner problem and preliminary results

The social planner determines a path $\{(q_x(t), q_y(t), k\,(t), \delta(t))\}_{t=0}^{\infty}$ maximizing the social welfare, that is solves the following problem (S.P):

$$\text{(S.P)} \max_{q_x,q_y,k,\delta} \int_0^\infty \{u(q_x(t)+q_y(t)) - (c_v + r\underline{c}_x)q_x(t) - mK(t)$$
$$- c_k(k(t)) - c_y(q_y(t))\}e^{-\rho t}dt \tag{3.1}$$

$$s.t. \quad \dot{X}(t) = -rq_x(t), X(0) = X_0 > 0 \text{ given}, \ X(t) \geq 0 \tag{3.2}$$
$$\dot{K}(t) = k(t) - \delta(t)K(t), K(0) = 0, K(t) \geq 0 \tag{3.3}$$
$$K(t) \geq q_x(t), \ q_x(t) \geq 0, \ q_y(t) \geq 0, k(t) \geq 0, \ \delta(t) \geq 0. \tag{3.4}$$

3.1 Optimality conditions

For the dual variables we denote by $\lambda's$ the co-state variables, by $\nu's$ the Lagrange multipliers associated to the constraints on the state variables, by $\gamma's$ the multipliers associated to the constraints on the command variables and by η the multiplier associated to the constraint involving both a state and a command variable.

The current value Hamiltonian, $\mathcal{H}$, and Lagrangian, $\mathcal{L}$, read:[7]

$$\begin{aligned}\mathcal{H}=&u(q_x+q_y)-(c_v+r\underline{c}_x)q_x-mK-c_k(k)-c_y(q_y)-\lambda_X r q_x\\&+\lambda_K(k-\delta K);\\\mathcal{L}=&\mathcal{H}+\nu_X X+\nu_K K+\eta(K-q_x)+\gamma_x q_x+\gamma_y q_y+\gamma_k k+\gamma_\delta\delta.\end{aligned}$$

The F.O.C's are:

$$\frac{\partial\mathcal{L}}{\partial q_x}=0\Longrightarrow u'(q_x+q_y)=c_v+r(\underline{c}_x+\lambda_X)+\eta-\gamma_x \tag{3.5}$$

$$\frac{\partial\mathcal{L}}{\partial q_y}=0\Longrightarrow u'(q_x+q_y)=c'_y(q_y)-\gamma_y \tag{3.6}$$

$$\frac{\partial\mathcal{L}}{\partial k}=0\Longrightarrow \lambda_K=c'_k(k)-\gamma_k \tag{3.7}$$

$$\frac{\partial\mathcal{L}}{\partial \delta}=0\Longrightarrow \lambda_K K=\gamma_\delta\ , \tag{3.8}$$

together with the usual complementary slackness conditions.

The co-state variables satisfy the following conditions when time differentiable:

$$\dot\lambda_X=\rho\lambda_X-\frac{\partial\mathcal{L}}{\partial X}\Longrightarrow\dot\lambda_X=\rho\lambda_X-\nu_X,\nu_X\geq 0 \text{ and } \nu_X X=0, \tag{3.9}$$

$$\dot\lambda_K=\rho\lambda_K-\frac{\partial\mathcal{L}}{\partial K}\Longrightarrow\dot\lambda_K=(\rho+\delta)\lambda_K+m-\eta-\nu_K,\nu_K\geq 0 \text{ and } \nu_K K=0\ . \tag{3.10}$$

The transversality condition at infinity is:

$$\lim_{t\uparrow\infty} e^{-\rho t}[\lambda_X(t)X(t)+\lambda_K(t)K(t)]=0\ . \tag{3.11}$$

3.2 Some properties of the optimal plans

Mining rent

Under the constant average extraction cost assumption A.4 and the C.U.E industry competitiveness assumption A.6 the coal initial endowment X_0 must

[7] We drop the time argument as far as no confusion is possible.

be exhausted in finite time.[8] Then (3.9) with $\nu_X = 0$ is the standard Hotelling rule under constant marginal extraction cost prevailing up to the exhaustion date t_X:

$$\lambda_X(t) = \lambda_{X_0} e^{\rho t}, \quad \lambda_{X_0} = \lambda_X(0), \quad t \leq t_X \tag{3.12}$$

Gross and net margins of the C.U.E industry and shadow value of the C.U.E production capacity

The multiplier η is the shadow marginal current value of the C.U.E production capacity. At any time t at which the C.U.E industry is active $q_x(t) > 0$ and $\gamma_x(t) = 0$ so that, from (3.5):

$$q_x(t) > 0 \Longrightarrow \eta(t) = u'(q(t)) - [c_v + r(\underline{c}_x + \lambda_X(t))] \ . \tag{3.13}$$

Since $u'(q(t)) = p(t)$, then $\eta(t)$ appears as the current gross margin of the C.U.E industry when the price of the coal input is equal to its full marginal cost $\underline{c}_x + \lambda_X$.

Deducing the unitary maintenance cost of capital from the gross margin we get the net operation margin, $\beta(t) \equiv \eta(t) - m$. Then for any time period during which the C.U.E industry does not scrap production capacity, $\delta(t) = 0$ and (3.10) may be rewritten as follows:

$$\dot{\lambda}_K(t) = \rho \lambda_K(t) - \beta(t) \ , \tag{3.14}$$

where $\rho\lambda_K$ is the rental price of a piece of equipment valued at λ_K. What (3.14) states is that the shadow marginal value of the installed capacity increases or decreases according to its rental cost be higher or lower than the net margin.

C.U.E production capital ends its active life when coal is exhausted. Hence at the time t_X, $\lambda_K(t_X) = 0$ since now this dedicated capital is useless. However as shown in Section 4, C.U.E production capacity begins to be scraped before the exhaustion of coal. During the scraping phase $\delta(t) > 0$ and $\gamma_\delta = 0$, and because $K(t) > 0$ (scraping requires that there exist equipments to be scraped), (3.8) can be satisfied if and only if $\lambda_K(t) = 0$. Thus

[8] See Appendix A.3 for a proof in the present context. The proof uses properties of the optimal paths proven in Section 4.

denoting by t_δ the time at which the scraping phase begins and integrating (3.14) over $[t, t_\delta]$, $0 \leq t \leq t_\delta$, results in :

$$\lambda_K(t) = \int_t^{t_\delta} \beta(\tau) e^{-\rho(\tau - t)} d\tau \ . \tag{3.15}$$

The shadow marginal value of equipment is equal to the sum of the discounted future net margins of the C.U.E industry.

Full marginal cost of C.U.E and marginal cost of capital

In (3.5) the full cost of C.U.E appears as the sum of monetary costs $c_v + r\underline{c}_x$ in both C.U.E and mining industries, the shadow marginal cost or mining rent of the coal input, $r\lambda_X$, and the multiplier associated to the capacity constraint, η. The alternative expression of η given by (3.10) allows to interpret η as the shadow marginal cost of the capital use.

Using (3.10) for the time $t's$ at which the capital is not scraped, $\delta(t) = 0$, we get $\eta = \rho\lambda_K - \dot{\lambda}_K + m$. Substituting for η in (3.10) results into:

$$u'(q_x + q_y) = c_v + r(\underline{c}_x + \lambda_X) + m + \rho\lambda_K - \dot{\lambda}_K \ ,$$

where $m + \rho\lambda_K - \dot{\lambda}_K$ is the full marginal cost of capital use. Using the capital requires that it must be maintained at the marginal cost m. To this maintenance cost must be added its rental cost $\rho\lambda_K$ from which must be deduced its instantaneous variation $\dot{\lambda}_K$. This is a local arbitrage condition as far as the price of equipment is equal to λ_K and ρ is the interest rate to take into account. Assuming that the maintenance cost m is supported by the user, would the rental price paid at $t+dt$, $dt > 0$, for the use of a piece of equipment during the time interval $[t, t+dt]$, be some amount $g(t+dt)$ lower than $[\rho\lambda_K - \dot{\lambda}_K]dt$, then the best would be for the owner to sell the equipment at time t for the price $\lambda_K(t)$ and obtain a return $\rho\lambda_K dt$. With the first option the owner asset profit would amount to $\lambda_K + \dot{\lambda}_K dt + g(t + dt)$ at time $t + dt$ and with the second one to $(1 + \rho)\lambda_K(t)$, and if $g(t + dt) < (\rho\lambda_K - \dot{\lambda}_K)dt$, a price higher than $\lambda_K(t)$ could be asked from selling the equipment at time t.

We prove in the next section that $\dot{\lambda}_K(t)$ is initially negative up to the time at which begins the decline of the C.U.E industry. Thus the full marginal cost of the capital use is first larger than $m+\rho\lambda_K(t)$. The capital is relatively rare, given the coal stock not yet exploited and having to be transformed into U.E, but less and less. Once the underground coal stock is sufficiently low, then the capital is no more rare, hence its shadow price, λ_K, must be nil and the marginal cost of its use reduces to its maintenance cost m.

Benchmarks and useful auxiliary functions

The following functions summarize some necessary relations between the C.U.E industry capacity, K, the production of S.U.E, q_y, the total production, q, and the sum of the shadow value components of the C.U.E industry marginal cost that we denote by $\mu : \mu \equiv r\lambda_X + \eta$. They will be repeatedly used to characterize the optimal paths.

First note that there exists a critical level of the C.U.E production capacity below which the S.U.E industry is active and above which it is no more competitive. Let us denote by K_y this critical level defined as the solution of $u'(K) = \underline{c}'_y$, hence $K_y > \tilde{q}_y$.

Let us denote by $\hat{q}_y(K)$ and $\hat{q}(K)$ respectively the optimal production of the S.U.E industry and the total U.E production as functions of the C.U.E production $q_x = K : \hat{q}(K) = K + \hat{q}_y(K)$. From the above definition of K_y and the F.O.C (3.6) relative to q_y, we get:

$$\hat{q}_y(K) : \begin{cases} = \tilde{q}_y, & K = 0 \\ \in (0, \tilde{q}_y), & 0 < K < K_y \\ = 0, & K_y \leq K \end{cases}, \frac{d\hat{q}_y}{dK} = \begin{cases} \frac{u''}{c''_y - u''} \in (-1, 0), & 0 < K < K_y \\ 0, & K_y \leq K \ . \end{cases} \qquad (3.16)$$

$$\hat{q}(K) : \begin{cases} = \tilde{q}_y, & K = 0 \\ \in (\tilde{q}_y, K_y), & 0 < K < K_y \\ = K, & K_y \leq K \end{cases}, \frac{d\hat{q}}{dK} = \begin{cases} \frac{c''_y}{c''_y - u''} \in (0, 1), & 0 < K < K_y \\ 1, & K_y \leq K \ . \end{cases} \qquad (3.17)$$

Consider now the critical level of K denoted by K_μ for which the F.O.C

(3.5) relative to q_x is satisfied with $\mu = 0$ given that q_y is optimal, that is $q_y = \hat{q}_y(K)$. Then K_μ solves $u'(K + \hat{q}_y(K)) = c_v + r\underline{c}_x$. Since μ must be non-negative, the function $\hat{\mu}(K)$, the value of μ satisfying (3.5), is a decreasing function of K on $[0, K_\mu)$:

i) either $c_v + r\underline{c}_x > \underline{c}'_y$, that is $K_\mu < K_y$ and

$$\hat{\mu}(K) > 0, 0 \leq K < K_\mu \text{ and } \frac{d\hat{\mu}}{dK} = \frac{u''c''_y}{c''_y - u''} < 0 \; ; \tag{3.18}$$

ii) or $\underline{c}'_y > c_v + r\underline{c}_x$, that is $K_y < K_\mu$ and

$$\hat{\mu}(K) > 0, 0 \leq K < K_\mu \text{ and } \frac{d\hat{\mu}}{dK} = \begin{cases} \frac{u''c''_y}{c''_y - u''} < 0, & 0 < K < K_y \\ \\ u'' < 0, & K_y < K < K_\mu \; . \end{cases} \tag{3.19}$$

Note that $\hat{q}_y(K)$, $\hat{q}(K)$ and $\hat{\mu}(K)$ if $K_y < K_\mu$, are not differentiable at K_y (see Appendix A.1):

$$\frac{d\hat{q}_y}{dK}|_{K=K_y^-} < \frac{d\hat{q}_y}{dK}|_{K=K_y^+} \text{ and } \frac{d\hat{q}}{dK}|_{K=K_y^-} < \frac{d\hat{q}}{dK}|_{K=K_y^+} \tag{3.20}$$

$$\frac{d\hat{\mu}}{dK}|_{K=K_y^-} > \frac{d\hat{\mu}}{dK}|_{K=K_y^+} \; . \tag{3.21}$$

Let us show now how to use these functions to determine the qualitative properties of the optimal paths.

4 Optimal paths

All the optimal paths include four main phases: An initial phase of C.U.E production capacity building up to some maximum $\bar{K}$, followed by a phase of constant C.U.E capacity at this maximum $\bar{K}$ before entering a third phase of scraping induced by the increasing scarcity of the coal resource, ending when coal is exhausted and the energy system comes back to its initial regime, the fourth and the last phase of the path. According to $\bar{K}$ be larger or smaller than K_y, the S.U.E sector must be either temporarily closed or permanently kept active when the C.U.E sector is at its maximum development. In the former alternative the initial phase of C.U.E industry expansion includes two sub-phases, a first one during which the S.U.E industry is active but declining and a second one during which the S.U.E production rate is nil. Symmetrically the third phase, the scraping phase, includes also two sub-phases, a first one during which the S.U.E sector is still at rest and a second one of S.U.E production revival. The sequence of phases and sub-phases and the date notations are summarized in the below Figure 1.

Figure 1 here.

We first characterize the three phases of active C.U.E industry and next we determine the way they succeed in the (K, X)-plane and show how the paths depend upon the coal initial endowment.

4.1 Expansion, stabilization and decline phases of the C.U.E industry

Under the assumption A.6 relative to the competitiveness of the C.U.E industry the initial phase must be a phase of investment in C.U.E production capital.[9]Proposition 1 states that this phase is a phase of decreasing investment rate which must be followed by a phase of constant capacity. Proposition 2 states that the second phase of constant C.U.E production capacity must extend up to the time at which $\lambda_K(t) = 0$ when there is no new phase of capacity investment. Proposition 3 characterizes the decline of the production capacity of the C.U.E industry during the C.U.E industry capital scraping phase.

Proposition P. 1 ***Expansion phase of the C.U.E industry***

Let $[0, t_k)$ *be the initial time interval of investment in C.U.E production capacity:* $k(t) > 0$, $t \in [0, t_k)$ *and* $k(t) = 0$, $t \in [t_k, t_k + \Gamma)$ *for some* $\Gamma > 0$.

[9] If not, given the stationarity assumption of the model, the C.U.E industry would never be developed, a contradiction since it is assumed to be competitive.

Then:

$$\dot{\lambda}_K(t) < 0, \quad t \in (0, t_k) \text{ and } \lim_{t \uparrow t_k} \lambda_K(t) = \underline{c}'_k \ . \tag{4.1}$$

Furthermore this initial phase is followed by a phase $[t_k, t_k + \Delta)$*,* $\Delta > 0$*, of constant industry capacity:*

$$K(t) = K(t_k), \quad t \in [t_k, t_k + \Delta]. \tag{4.2}$$

Proof: Assume that $[0, t_k)$ is followed by a scraping phase during which $\delta(t) > 0$. Then some part of the last investment built at time $t_k - \theta$, $\theta > 0$ and sufficiently small, at a marginal cost at least equal to $\underline{c}'_k > 0$ would have an infinitely short operation life and its cost could never be recovered.

Consider first the case $K(t_k) < K_y$ so that $d\hat{\mu}/dK$ is continuous over $[0, K(t_k)]$ and assume that $\dot{\lambda}_K(t_1) \geq 0$ at some $t_1 \in (0, t_k)$, then we first show that $\dot{\lambda}_K(t) \geq 0$ for all $t \in (t_1, t_k)$ and next that such an inequality implies a contradiction.

At time t_1, because $\delta(t_1) = 0$, then since (3.14) holds: $\dot{\lambda}_K(t_1) = \rho\lambda_K(t_1) - \beta(t_1) \geq 0$. Since $\beta(t_1) = \eta(t_1) - m$, then $\dot{\beta}(t_1) = \dot{\eta}(t_1)$.

Because $\eta(t) = \mu(t) - r\lambda_{X_0}e^{\rho t}$, then $\dot{\eta}(t) = \dot{\mu}(t) - r\rho\lambda_{X_0}e^{\rho t}$, and since $\dot{K}(t_1) = k(t_1) > 0$, then $(d\hat{\mu}/dK)\, \dot{K}(t_1) < 0$ by (3.18), implying that $\dot{\mu}(t_1) = (d\hat{\mu}/dK)\, \dot{K}(t_1) < 0$ and in turn also that $\dot{\beta}(t_1) = \dot{\eta}(t_1) < 0$. Thus for $dt > 0$:

$$\lambda_K(t_1 + dt) = \lambda_K(t_1) + \dot{\lambda}_K(t_1)dt \geq \lambda_K(t_1) \tag{4.3}$$

$$\beta(t_1 + dt) = \beta(t_1) + \dot{\beta}(t_1)dt \leq \beta(t_1) \ . \tag{4.4}$$

Hence

$$\rho\lambda_K(t_1 + dt) \geq \beta(t_1) + \dot{\beta}(t_1)dt \Longrightarrow \dot{\lambda}_K(t_1 + dt) \geq 0 \ .$$

Repeating the argument we conclude that $\dot{\lambda}_K(t) \geq 0$ for all $t \in (t_1, t_k)$ implying together with $k(t_1) > 0$ and (3.7) hence $\lambda_K(t_1) = c'_k(k(t_1)) > \underline{c}'_k$, that:

$$\lim_{t \uparrow t_k} \lambda_K(t) > \underline{c}'_k.$$

However, because $k(t) = 0$, $t \in (t_k, t_k + \Delta)$, then by (3.7), $\lambda_K(t) \leq \underline{c}'_k, t \in (t_k, t_k + \Delta)$ and $\lambda_K(t)$ should jump downward at $t = t_k$, a contradiction since $\lambda_K(t)$ must be continuous (Seierstad and Sydsaeter, 1987, chapter 3, Theorem 16, p 244).

Consider now the case $K_y < K(t_k)$ and let $\underline{t}_y$ be the time at which $K(t) = K_y$, $0 < \underline{t}_y < t_k$. At time $\underline{t}_y$, $d\hat{\mu}/dK$ jumps downwards (c.f (3.21)). Firstly if $t_1 \in (\underline{t}_y, t_k)$ at which $\dot{\lambda}_K(t_1) \geq 0$, then the argument is exactly the same since $d\hat{\mu}/dK$ is continuous on $(K_y, K(t_k))$. Secondly, if $0 < t_1 < \underline{t}_y$, then by using the same argument over $(t_1, \underline{t}_y)$, clearly $\lambda_K(\underline{t}_y^-) \geq \lambda_K(t_1)$ and $\beta(\underline{t}_y^-) \leq \beta(t_1)$. Because both $\lambda_K(t)$ and $\beta(t)$ are continuous then $\dot{\lambda}_K(\underline{t}_y)$ is well defined hence $\dot{\lambda}_K(\underline{t}_y) \geq 0$ and the same argument may be repeated over $(\underline{t}_y, t_k)$ resulting in $\lambda_K(t_k^-) \geq c_k'(k(t_k^-)) > \underline{c}_k'$ and the same kind of contradiction.[10] ■

Corollary 1 is a direct implication of Proposition 1.

Corollary 1 *During the initial phase* $[0, t_k)$ *of investment in C.U.E production capacity, the speed of capital accumulation,* $\ddot{K}(t) = \dot{k}(t) < 0$*, decreases from some initial positive level at the beginning of the phase down to* 0 *at the end.*

Let us turn now to the phase of maximal expansion of the C.U.E industry.

Proposition P. 2 ***Phase of maximal expansion and stabilization of the C.U.E industry***

There exists an extension $[t_k, t_\delta)$ *of the time interval* $[t_k, t_k + \Delta)$ *during which* $K(t) = K(t_k)$*, such that the net operation margin* $\beta(t)$ *decreases within the interval, down to zero at* t_δ*. Provided that* $\beta(t) = 0$*,* $t \in (t_\delta, t_X)$*,* $\lambda_K(t)$ *decreases within the interval down to zero at time* t_δ*.*

Proof: During the interval $q(t) = \hat{q}(K(t_k))$ and $\beta(t) = \eta(t) - m = u'(\hat{q}(K(t_k)) - [c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m]$. Hence $\dot{\beta}(t) < 0$ and there exists t_δ such that $\beta(t_\delta) = 0$. We now show that $\dot{\lambda}_K(t)/\lambda_K(t) < 0$.

[10] The jump of $d\hat{\mu}/dK$ at $K = K_y$ implies a jump of $\ddot{\lambda}_K(t)$ at $t = \underline{t}_y$, not a jump of $\dot{\lambda}_K(t)$.

Under the condition $\beta(t) = 0, t > t_\delta$, then (3.15) holds within the interval because $\lambda_K(t_\delta) = 0$, and since $\beta(t)$ decreases we get for all $t \in (t_k, t_\delta)$

$$\begin{aligned}\lambda_K(t) &= \int_t^{t_\delta} \beta(\tau) e^{-\rho(\tau - t)} d\tau < \beta(t) \int_t^{t_\delta} e^{-\rho(\tau - t)} d\tau \\ &= \frac{\beta(t)(1 - e^{-\rho(t_\delta - t)})}{\rho} \equiv h(t) > 0 \ .\end{aligned}$$

Now from (3.14):

$$\frac{\dot{\lambda}_K(t)}{\lambda_K(t)} = \rho - \frac{\beta(t)}{\lambda_K(t)} < \rho - \frac{\beta(t)}{h(t)} = -\frac{\rho e^{-\rho(t_\delta - t)}}{1 - e^{-\rho(t_\delta - t)}} < 0. \tag{4.5}$$

Therefore $\dot{\lambda}_K(t) < 0$. ■

Last, for the Hotelling phase.

Proposition P. 3 ***Hotelling phase of capital scraping***

If $\beta(t) = 0, t \in (t_\delta, t_X)$, then:

a. The last phase of C.U.E production is the Hotelling path corresponding to a constant marginal extraction and transformation cost $c_v + r\underline{c}_x + m$ and a coal endowment $X(t_\delta)$, hence a U.E price equal to its full marginal cost, $p(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m$, with $p(t_X) = \tilde{p}$;

b. The phase is a phase of scraping of the C.U.E production capacity, the capacity $K(t)$ decreasing from $K(t_k)$ at time t_δ down to zero at time t_X, with $\dot{K}(t) = \dot{q}_x(t)$. Thus $K(t_k)$ appears as the maximum capacity $\bar{K}$ of the C.U.E industry along the optimal path. Simultaneously the coal stock $X(t)$ decreases from $X(t_\delta)$ at time t_δ down to zero at time t_X, with $\dot{X}(t) = -rq_x(t)$.

Proof If $\beta(t) = 0$ then $\eta(t) = m$ and the F.O.C (3.5) reduces to

$$u'(q(t)) \equiv p(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m \ .$$

Time differentiating results in:

$$\dot{q}(t) = \frac{r\rho\lambda_{X_0} e^{\rho t}}{u''(q(t))} < 0 \ , \quad t \in (t_\delta, t_X) \ . \tag{4.6}$$

From $K(t) = q_x(t) = q(t) - \hat{q}_y(K(t))$ we get, taking (3.17) into account:

$$\dot{q}_x(t) = \dot{K}(t) = \begin{cases} \dot{q}(t) < 0 & , \quad K(t) > K_y \\ \\ \dfrac{\dot{q}(t)}{1 + d\hat{q}_y/dK} < 0 & , \quad K(t) < K_y \end{cases} \quad \text{and } \delta(t) = -\frac{\dot{q}_x(t)}{q_x(t)} > 0 \tag{4.7}$$

$$\dot{q}_y(t) = \begin{cases} 0 & , \quad K(t) > K_y \\ \\ \dfrac{d\hat{q}_y}{dK}\dot{K}(t) & , \quad K(t) < K_y \; . \end{cases} \tag{4.8}$$

Denote by $\bar{t}_y$ the time at which $K(t) = K_y$ if $\bar{K} > K_y$, then:

$$\dot{q}_x(\bar{t}_y^-) = \dot{K}(\bar{t}_y^-) > \dot{K}(\bar{t}_y^+) = \dot{q}_x(\bar{t}_y^+), \;\; \delta(\bar{t}_y^-) < \delta(\bar{t}_y^+) \text{ and } \dot{q}_y(\bar{t}_y^-) = 0 < \dot{q}_y(\bar{t}_y^+). \tag{4.9}$$

■

Let us turn now to the condition that the pairs (K, X) must satisfy at the phase switching dates in order that the sequence Expansion-Stabilization-Hotelling (E.S.H) be an optimal path candidate. To determine these conditions we build a phase diagram in the (K, X) plane.

4.2 The phase diagram in the (K, X) plane

To build the diagram we proceed backwards and characterize first the Stabilization-Scraping frontier and next the Expansion-Stabilization one.

4.2.1 The Stabilization-Scraping or Hotelling frontier

Let K_δ and X_δ be respectively the production capacity of the C.U.E industry and the coal stock available at the beginning of the Hotelling phase. We denote by $K_H(X_\delta)$ the frontier function: $K_H(X_\delta)$ is the capacity of the C.U.E industry required to follow the Hotelling path starting at the time t_δ with a coal endowment X_δ. Because $q_x(t)$ is decreasing during the Hotelling phase, maintaining a capacity larger than the C.U.E production is unnecessarily costly at all times during the phase. Hence $K(t) = q_x(t),\ t \in (t_\delta, t_X)$ and the capacity $K_H(X_\delta)$ is the C.U.E production rate at the beginning of the Hotelling path starting with the coal endowment X_δ. Thus would K_δ be larger than $K_H(X_\delta)$, then $K_\delta - K_H(X_\delta)$ would have to be scraped immediately before starting the progressive scraping process implied by the Hotelling policy. Clearly such a state of the system is never attained along an optimal path.

Note that the graph of the function $K_H(X_\delta)$ is nothing but than the Hotelling path itself. The reason is that any point $(K_H(X'_\delta), X'_\delta)$ on the frontier attained at some time t' is the beginning of an Hotelling path. After t', $\{(K_H(X(t)), X(t)),\ t > t'\}$ is an Hotelling path. But this path does not depend upon the way $(K_H(X'_\delta), X'_\delta)$ has been attained at time t', either as the end of a stabilization phase closed at time t' or as a point of an Hotelling trajectory posterior to a state $(K_H(X''_\delta), X''_\delta),\ X''_\delta > X'_\delta$, attained at some time $t'' < t'$.

Let λ_{X_δ} be the value of λ_X at time t_δ and $\Gamma_H(\lambda_{X_\delta})$ be the duration of the Hotelling phase. Given that at the coal exhaustion date the U.E price must be equal to $\tilde{p}$ then $\Gamma_H(\lambda_{X_\delta})$ is this value of Γ_H solving $c_v + r[\underline{c}_x + \lambda_{X_\delta} e^{\rho\Gamma_H}] + m = \tilde{p}$, hence:

$$\frac{d\Gamma_H}{d\lambda_{X_\delta}} = -\frac{1}{\rho\lambda_{X_\delta}} < 0,\ \ \lambda_{X_\delta} \in (0, \tilde{\lambda}_X),\ \ \lim_{\lambda_{X_\delta}\downarrow 0} \Gamma_H(\lambda_{X_\delta}) = +\infty\ , \tag{4.10}$$

and

$$\lim_{\lambda_{X_\delta}\uparrow\tilde{\lambda}_X} \Gamma_H(\lambda_{X_\delta}) = 0\ , \tag{4.11}$$

where $\tilde{\lambda}_X = [\tilde{p} - (c_v + r\underline{c}_x + m)]/r$.

For $\lambda_{X_\delta} \in (0, \tilde{\lambda}_X)$ and $t \in [t_\delta, t_\delta + \Gamma_H(\lambda_X)]$, let us define $p(t, \lambda_{X_\delta})$ as the Hotelling price, $p(t, \lambda_{X_\delta}) \equiv c_v + r[\underline{c}_x + \lambda_{X_\delta} e^{\rho(t-t_k)}] + m$, and denote respectively by $q(t, \lambda_{X_\delta})$, $q_x(t, \lambda_{X_\delta})$ and $q_y(t, \lambda_{X_\delta})$ the corresponding U.E, C.U.E and S.U.E production rates. Since $K(t) = q_x(t, \lambda_{X_\delta})$, then the F.O.C (3.5) relative to q_x may be rewritten as

$$\begin{aligned} u'(q(t, \lambda_{X_\delta})) = u'(q_x(t, \lambda_{X_\delta}) + q_y(t, \lambda_{X_\delta})) = u'\left(q_x(t, \lambda_{X_\delta}) + \hat{q}_y(q_x(t, \lambda_{X_\delta}))\right) \\ = c_v + r[\underline{c}_x + \lambda_{X_\delta} e^{\rho(t-t_\delta)}] + m. \end{aligned}$$

Differentiating, taking (4.6) into account while noting that $\partial q(t, \lambda_{X_\delta})/\partial t = \dot{q}(t)$, we obtain:

$$\frac{\partial q(t, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}} = \frac{1}{\rho \lambda_{X_\delta}} \frac{\partial q}{\partial t} \ , \quad t \in (t_\delta, t_\delta + \Gamma_H(\lambda_{X_\delta})) \ , \tag{4.12}$$

from which, taking (4.7) and (4.8) into account, we get:

$$\frac{\partial q_x(t, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}} = \frac{1}{\rho \lambda_{X_\delta}} \frac{\partial q_x}{\partial t} < 0 \ , \quad t \in (t_\delta, t_\delta + \Gamma_H(\lambda_{X_\delta})) \\ \text{and } t \neq \bar{t}_y \text{ if } q_x(t_\delta, \lambda_{X_\delta}) > K_y \ , \tag{4.13}$$

with

$$\frac{\partial q_x(\bar{t}_y^-, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}} > \frac{\partial q_x(\bar{t}_y^+, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}}, \quad \text{if } q_x(t_\delta, \lambda_{X_\delta}) > K_y \ , \tag{4.14}$$

and

$$\frac{\partial q_y}{\partial \lambda_{X_\delta}} = \frac{1}{\rho \lambda_{X_\delta}} \frac{\partial q_y}{\partial t} \begin{cases} = 0 \ , & t \in (t_\delta, \bar{t}_y) \qquad\qquad\qquad \text{if } q_x(t_\delta, \lambda_{X_\delta}) > K_y \\ > 0 \ , & t \in (\bar{t}_y, t_\delta + \Gamma_H(\lambda_{X_\delta})) \quad\ \ \text{if } q_x(t_\delta, \lambda_{X_\delta}) > K_y \\ & \text{or } t \in (t_\delta, t_\delta + \Gamma_H(\lambda_{X_\delta})) \ \ \text{if } q_x(t_\delta, \lambda_{X_\delta}) < K_y \ . \end{cases} \tag{4.15}$$

Hence

$$\frac{\partial q_y(\bar{t}_y^-, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}} < \frac{\partial q_y(\bar{t}_y^+, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}}, \quad \text{if } q_x(t_\delta, \lambda_{X_\delta}) > K_y \ . \tag{4.16}$$

Determination of the optimal value of λ_{X_δ}, given X_δ, and of the slope of $K_H(X_\delta)$.

The optimal value of λ_{X_δ}, given X_δ, is this value which solves the coal exhaustion condition:

$$r \int_{t_\delta}^{t_\delta + \Gamma_H(\lambda_{X_\delta})} q_x(t, \lambda_{X_\delta}) dt = X_\delta \ . \tag{4.17}$$

By a slight abuse of notation let us denote by $\lambda_{X_\delta}(X_\delta)$ the value of the mining rent at the beginning of the Hotelling phase as a function of the coal stock X_δ available at the same date. Differentiating (4.17) and taking into account that $q_x(t_\delta + \Gamma_H(\lambda_{X_\delta}), \lambda_{X_\delta}) = 0$, we obtain:

$$\frac{d\lambda_{X_\delta}}{dX_\delta} = \frac{1}{r \int_{t_\delta}^{t_\delta + \Gamma_H(\lambda_{X_\delta})} \frac{\partial q_x(t, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}} dt} < 0 \ . \tag{4.18}$$

Hence

$$\frac{dK_H}{dX_\delta} = \frac{\partial q_x(t_\delta, \lambda_{X_\delta})}{\partial \lambda_{X_\delta}} \frac{d\lambda_{X_\delta}}{dX_\delta} > 0, \quad X_\delta \neq X_{\delta y} \ \text{ if } K_y < \bar{K}_{\sup} \ , \tag{4.19}$$

and

$$\lim_{X_\delta \downarrow 0} K_H(X_\delta) = 0 \ \text{ and } \lim_{X_\delta \uparrow 0} K_H(X_\delta) = \bar{K}_{\sup} < +\infty, \tag{4.20}$$

where $X_{\delta y}$ is this value of X_δ for which $K_H(X_\delta) = K_y$ if $K_y < \bar{K}_{\sup}$, and $\bar{K}_{\sup}$ solves $u'(K + \hat{q}_y(K)) = c_v + r\underline{c}_x + m$, that is $\bar{K}_{\sup}$ is the limit of the constant production level of C.U.E when λ_{X_δ} tends towards 0, alternatively when X_δ tends towards $+\infty$. Clearly K_y may be either larger or smaller than $\bar{K}_{\sup}$. Note that in the case $K_y < \bar{K}_{\sup}$ the function $K_H(X_\delta)$ is not differentiable at $X_\delta = X_{\delta y}$, (4.14) and (4.19) together imply that:

$$\frac{dK_H(X_{\delta y}^+)}{dX_\delta} < \frac{dK_H(X_{\delta y}^-)}{dX_\delta} \ . \tag{4.21}$$

The frontier $K_H(X_\delta)$ is illustrated in Figure 2 for the case $K_y < \bar{K}_{\sup}$.

Figure 2 here.

Would the initial state of the system be some pair located above the frontier $K_H(X_\delta)$ like (K', X') in Figure 2 then the optimal policy would be to scrap initially the excess of capital $K' - K_H(X')$ and next to move along the Hotelling frontier from the point $(K_H(X'), X')$ down to $(0, 0)$. But clearly such state (K', X') will never be attained starting from $K(0) = 0$, whatever X_0.

4.2.2 The Expansion-Stabilization frontier

We proceed backward and start from the state $(K_\delta, X_\delta, \lambda_{X_\delta})$ of the C.U.E industry at the end of the stationary phase: $K_\delta = K_H(X_\delta), \lambda_{X_\delta} = \lambda_{X_\delta}(X_\delta)$ and $K_\delta = \bar{K}$ where $K_k \equiv K(t_k)$. Since the capacity of the C.U.E industry is constant during the phase, $K(t) = \bar{K}$, $t \in [t_k, t_\delta]$, then the coal input consumption of the C.U.E industry during the stabilization phase is proportional to its duration that we denote by Γ_S: $\Gamma_S \equiv t_\delta - t_k$.

Let X_k be the coal stock available at the beginning of the phase, then: $X_k = X_\delta + r\bar{K}\Gamma_S$. Denoting by $K_S(X_k)$ the frontier function we should have:

$$K_H(X_\delta) = \bar{K} = K_S(X_\delta + rK_H(X_\delta)\Gamma_S) \ .$$

We show that the duration of phase, Γ_S, is an increasing function of $\bar{K}$ so that in the (K, X) plane the horizontal distance between $K_H(X_\delta)$ and $K_S(X_k)$ is an increasing function of $\bar{K} = K_\delta = K_k$ as illustrated in Figure 2, provided that $\bar{K}$ be lower than some upper limit $\bar{K}_{\max}$ itself lower than $\bar{K}_{\sup}$.

Arbitrage condition for the last built piece of C.U.E industry equipment

In order that the C.U.E industry capital be kept constant during the phase succeeding to the investment phase, the sum of the discounted profit margins of the phase of constant capital in value at the beginning of the phase must be equal to the cost of the last piece of equipment put into operation at the same date, $\underline{c}'_k$ by Proposition 1. Thus for the optimal duration Γ_S of the stabilization phase, given the optimal value λ_{X_δ} of the mining rent at the end of the phase and the optimal production capacity $\bar{K}$, the arbitrage condition reads:

$$\int_0^{\Gamma_S} \{u'(\hat{q}(\bar{K})) - (c_v + r[\underline{c}_x + \lambda_{X_\delta} e^{-\rho(\Gamma_S - \tau)}] + m)\} e^{-\rho\tau} d\tau = \underline{c}'_k \ . \qquad (4.22)$$

Next remember that λ_{X_δ} is the function of $\bar{K}$ determined by the optimality condition which must hold at the beginning of the Hotelling phase, that is the end of the stabilization phase. This condition states that at time t_δ, $\lambda_K = 0$, hence:[11]

$$r\lambda_{X_\delta} = u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + m] \Longrightarrow r\frac{d\lambda_{X_\delta}}{d\bar{K}} = u''(\hat{q}(\bar{K}))\frac{d\hat{q}}{dK}\,|_{K=\bar{K}} < 0\ . \tag{4.23}$$

In the arbitrage condition (4.22), let us substitute for λ_{X_δ} the function $\lambda_{X_\delta}(\bar{K})$ and rearrange to get:

$$\int_0^{\Gamma_S} \{u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + m]\}e^{-\rho\tau}d\tau - re^{-\rho\Gamma_S}\lambda_{X_\delta}(\bar{K})\Gamma_S = \underline{c}_k'\ . \tag{4.24}$$

Differentiating, we obtain:

$$\begin{aligned}
0 &= \{u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + m]\}e^{-\rho\Gamma_S} - re^{-\rho\Gamma_S}\lambda_{X_\delta}(\bar{K})\}d\Gamma_S \\
&\quad + r\rho e^{-\rho\Gamma_S}\lambda_{X_\delta}(\bar{K})\Gamma_S d\Gamma_S \\
&\quad + \{\int_0^{\Gamma_S}\left[\{e^{-\rho\tau}u''(\hat{q}(\bar{K}))\frac{d\hat{q}}{dK}|_{K=\bar{K}} - r\frac{d\lambda_{X_\delta}}{d\bar{K}}e^{-\rho\Gamma_S}\right]d\tau\}d\bar{K}.
\end{aligned}$$

In the above expression the first term is nil by the condition (4.23;l.h.s.), and given the value of $d\lambda_{X_\delta}/d\bar{K}$ given also by (4.23;r.h.s.) the third term may be rewritten as:

$$\{u''(\hat{q}(\bar{K}))\frac{d\hat{q}}{dK}|_{K=\bar{K}}\int_0^{\Gamma_S}(e^{-\rho\tau} - e^{-\rho\Gamma_S})d\tau\}d\bar{K}\ . \tag{4.25}$$

Hence:

$$\frac{d\Gamma_S}{d\bar{K}} = \frac{u''(\hat{q}(\bar{K}))\frac{d\hat{q}}{dK}|_{K=\bar{K}}\int_0^{\Gamma_S}(e^{-\rho\tau} - e^{-\rho\Gamma_S})d\tau}{-r\rho e^{-\rho\Gamma_S}\lambda_{X_\delta}(\bar{K})\Gamma_S} > 0. \tag{4.26}$$

From $d(X_k - X_\delta)/d\bar{K} = r[\bar{K}d\Gamma_S/d\bar{K} + \Gamma_S]$ and the above inequality, and given that $dX_\delta/d\bar{K} > 0$ (c.f (4.19) with $\bar{K} = K_H$) we conclude:

$$\frac{d(X_k(\bar{K}) - X_\delta(\bar{K}))}{d\bar{K}} > 0 \ \text{ and } \ \frac{dX_k(\bar{K})}{d\bar{K}} > 0, \tag{4.27}$$

provided that $\bar{K} \neq K_y$.

[11] Note that (4.23), $d\lambda_{X_\delta}/d\bar{K} < 0$, may also be obtained from (4.18), $d\lambda_{X_\delta}/dX_\delta < 0$, and (4.19), $dK_H/dX_\delta > 0$, because $K_H = \bar{K}$.

However (4.27) holds as far as the derivatives of (4.24) leading to (4.27) hold, that is as far as there exists a pair $(\bar{K}, \Gamma_S)$ for which (4.24) is satisfied. Let us show now that (4.24) may be satisfied only for $\bar{K}'s$ lower than an upper bound $\bar{K}_{\max}$, itself lower than the upper bound $\bar{K}_{\sup}$ under which start the Hotelling phases.

Let us assume that $e^{-\rho\Gamma_S}\lambda_{X_\delta} = \lambda_{X_k} \simeq 0$ because Γ_S is very large so that the instantaneous unitary net margin in the C.U.E industry is approximately equal to $u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + +m]$ during a large part of the stationary phase, hence a capitalized value at t_k of any piece of equipment at most equal to:

$$\int_0^{\Gamma_S} \{u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + m]\}e^{-\rho\tau}d\tau \ .$$

At the limit when $\Gamma_S \to \infty$, the arbitrage condition (4.24) reads:

$$u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + m] = \rho\underline{c}'_k \ . \tag{4.28}$$

Let us denote by $\bar{K}_{\max}$ the solution of (4.28) , then $\bar{K}_{\max} < \bar{K}_{\sup}$ and the arbitrage equation (4.24) may be satisfied i.f.f $\bar{K} < \bar{K}_{\max}$.[12] To conclude:[13]

$$K_S(X_k) < \bar{K}_{\max} \text{ and } \frac{dK_S}{dX_k} > 0, X_k \neq X_{ky} \text{ if } K_y < \bar{K}_{\max}, \tag{4.29}$$

$$\lim_{X_k \downarrow 0} K_S(X_k) = 0 \text{ and } \lim_{X_k \uparrow \infty} K_S(X_k) = \bar{K}_{\max}, \tag{4.30}$$

where X_{ky} solves $K_S(X_k) = K_y$ when $K_y < \bar{K}_{\max}$. In this last case:

$$\frac{dK_S(X_{ky}^+)}{dX_k} < \frac{dK_S(X_{ky}^-)}{dX_k} \ . \tag{4.31}$$

[12]Remember that $\bar{K}_{\sup}$ solves $u'(\hat{q}(\bar{K})) - [c_v + r\underline{c}_x + +m] = 0$, hence the inequality $\bar{K}_{\max} < \bar{K}_{\sup}$ and the distance between the two limits is an increasing function of $\underline{c}'_k$: $d\bar{K}_{\max}/d\underline{c}'_k < 0$.

[13]For very small X_δ, by lengthening the duration of the phase that is by choosing a sufficiently thin C.U.E production rate during the stationary phase the energy price can be kept approximatively equal to $\tilde{p} > u'(\hat{q}(\bar{K}_{\max}))$. Repeating the argument put forward to determine $\bar{K}_{\max}$, the arbitrage condition (4.24) can be satisfied provided that $\tilde{p} - [c_v + r\underline{c}_x + +m] \geq \rho\underline{c}'_k$. By Assumption A.6, substituting $\tilde{p}$ for $c'_y(\tilde{q})$, this inequality is strict. By the same token we get the existence of $\bar{K}_{\max} > 0$, hence (4.30) below. However note that the above arguments suggest that $\lim_{X_k \downarrow 0} \Gamma_S(X_k) > 0$.

Would the initial state of the system be located under the Hotelling frontier $K_H(X_\delta)$ but above the horizontal line $\bar{K}_{\max}$ like (K'', X'') in Figure 2, the optimal policy would be first to use the available capital K'' and extract $x(t) = rK''$ up to the time at which $X(t)$ solves $K_H(X) = K''$. At this time the system is on the Hotelling frontier and the best is to follow the frontier down to $(0, 0)$. During the first phase of this scenario the shadow marginal value of the capital K'' is positive, starting from some level lower than $\underline{c}'_k$ and decreasing down to 0 once the Hotelling frontier is attained. But clearly such a state (K'', X'') is never attained along an optimal path starting from $K(0) = 0$ however large is the initial coal endowment.

4.2.3 Critical coal endowment when $K_y < \bar{K}_{\max}$

If $K_y < \bar{K}_{\max}$, among the paths of the first phase during which the C.U.E industry accumulates capital, there exists a unique path joining the $K_\delta(X_k)$ frontier at K_y. Let X_{0y} be the initial coal endowment corresponding to this path (see Figure 2). This endowment is the critical endowment of coal below which the optimal paths are such that the S.U.E industry is always active and above which it is temporarily closed during a time interval $(\underline{t}_y, \bar{t}_y)$ including the phase at maximum capacity: $\underline{t}_y < t_k$ and $t_\delta < \bar{t}_y < t_X$.

The following proposition summaries our results.

Proposition P. 4 ***Optimal path***

Starting from a pure renewable energy state in which $q(0) = q_y(0) = \tilde{q}$ and $p(0) = \tilde{p}$, the optimal energy mix evolves as follows.

1. During a first phase $[0, t_k)$, the capacity of the C.U.E industry is developed, $\dot{K}(t) = k(t) > 0$, at a decreasing speed, $\ddot{K}(t) = \dot{k}(t) < 0$. The C.U.E production $q_x(t)$ increases, the S.U.E production $q_y(t)$ decreases however the total U.E production $q(t)$ expands and the U.E price decreases.

If the S.U.E industry is sufficiently competitive, $K_y > \bar{K}_{\max}$, it must remain active however large is the initial coal endowment X_0, and is reduced to its historical minimum size at the end of the phase. On the contrary, if *potentially uncompetitive, $K_y < \bar{K}_{\max}$, the S.U.E industry survives or not the C.U.E industry expansion according to the coal initial endowment be smaller or larger than a critical size X_{0y}. For small endowments the S.U.E sector is reduced to some minimum size like in the competitive case: The lack of coal hampers the competitiveness of the C.U.E industry and compensates for the lack of competitiveness of the S.U.E industry. For larger coal endowments the C.U.E industry is no more sufficiently hampered and the S.U.E industry must be closed before the end of the C.U.E industry expansion.*

2. During a second phase $[t_k, t_\delta)$ the production rates of both C.U.E and S.U.E are constant and equal to the production rates of the end of the preceding phase. This phase is the plateau of the maximum C.U.E and U.E production rates and the bottom of the S.U.E production flow. This is also the phase of cheapest U.E price.

3. The third phase $[t_\delta, t_X)$, the last phase of coal exploitation, is a standard Hotelling phase. Both C.U.E and total U.E productions decrease, the C.U.E production down to 0, hence the C.U.E industry capital is accordingly scraped. The C.U price increases. If the S.U.E industry remains active its production increases and if it has been closed its revival occurs within the phase. However the increase of the S.U.E production fails to compensate for the decrease of the C.U.E production. The phase ends when the coal endowment is exhausted and the economy comes back and forever to its pure renewable initial state. The coal interlude is gone.

The price and quantities paths for the case of a potentially uncompetitive S.U.E industry and large coal endowment are illustrated in Figure 3.

Figure 3 here

4.3 Determination of the optimal path

Let us show now how to use the frontier function $K_S(X_k)$ and the phase duration functions $\Gamma_S(\bar{K})$ and $\Gamma_H(\lambda_{X_\delta})$ to determine the optimal path.[14]

Consider first the cases $\bar{K}_{\max} < K_y$ whatever X_0 and $K_y < \bar{K}_{\max}$ but $X_0 < X_{0y}$ for which the S.U.E industry is kept active along the whole optimal path.

Expansion phase $[0, t_k)$:

From (3.5) with $q_x(t) = K(t), q_x(t) + q_y(t) = \hat{q}(K(t))$ and $\gamma_x(t) = 0$ (since $q_x(t) > 0$) we get

$$u'(\hat{q}(K(t))) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + \eta(t). \tag{4.32}$$

From (3.10) with $\delta(t) = 0$ and $\nu_K(t) = 0$ (since $K(t) > 0$), we obtain:

$$\eta(t) = -\dot{\lambda}_K(t) + \rho\lambda_K(t) + m,$$

and substituting for $\eta(t)$ in (4.32) results in:

$$u'(\hat{q}(K(t))) + \dot{\lambda}_K(t) - \rho\lambda_K(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m. \tag{4.33}$$

Next from (3.7) with $\gamma_k(t) = 0$ (since $k(t) > 0$), and taking care that $k(t) = \dot{K}(t)$, we get

$$\lambda_K(t) = c_k'(\dot{K}(t)) \Rightarrow \dot{\lambda}_K(t) = c_k''(\dot{K}(t))\ddot{K}(t),$$

so that substituting for $\lambda_K(t)$ and $\dot{\lambda}_K(t)$ in (4.33), we obtain:

$$u'(\hat{q}(K(t))) - \rho c_k'(\dot{K}(t)) + c_k''(\dot{K}(t))\ddot{K}(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m. \tag{4.34}$$

[14] The frontier $K_S(X_k)$ and the optimal phase durations $\Gamma_S(\bar{K})$ and $\Gamma_H(\lambda_{X_\delta})$ have been determined backwards, starting from the state of the system at the time at which the coal stock is exhausted, and do not depend upon what happened within the initial expansion phase. Thus to determine the optimal path what remains to determine is what must happen during the initial expansion phase taking as given the functions $K_S(X_k), \Gamma_S(\bar{K})$ and $\Gamma_H(\lambda_{X_\delta})$.

For a given λ_{X_0} let us denote by $K(t, \lambda_{X_0})$ a solution of (4.34) satisfying the initial condition $K(0) = 0$ and by $t_k(\lambda_{X_0})$ the time at which $\dot{K}(t, \lambda_{X_0}) = 0$ (equivalently the time at which $\lambda_K(t) = \underline{c}'_k$). In order that $K(t, \lambda_{X_0})$, $t \in [0, t_k(\lambda_{X_0})]$, be the optimal expansion phase of the C.U.E production capacity, the state of the system at time $t_k(\lambda_{X_0})$, $(K(t_k(\lambda_{X_0}), \lambda_{X_0}), X(t_k(\lambda_{X_0}))$, must be located on the expansion-stabilization frontier, that is the optimal value of λ_{X_0} must solve the following equation (4.35):

$$K(t_k(\lambda_{X_0}), \lambda_{X_0}) = K_S[X_0 - r \int_0^{t_k(\lambda_{X_0})} K(t, \lambda_{X_0}) dt] \ . \tag{4.35}$$

Let $\lambda^*_{X_0}$ denote the solution, then during the phase the optimal shadow value of the C.U.E industry capital stock, $\lambda^*_K(t)$, is given by (3.7) with $\gamma_k(t) = 0$, that is $\lambda^*_K(t) = c'_k(\dot{K}(t, \lambda^*_{X_0}))$.

Stabilization phase $[t_k, t_\delta]$:

During the stabilization phase the C.U.E industry capacity is constant: $K(t) = \bar{K} = K(t_k(\lambda^*_{X_0}), \lambda^*_{X_0})$. The optimal duration of the phase is equal to $\Gamma_S(\bar{K})$ solving (4.22) with $\lambda_{X_\delta} = \lambda^*_{X_0} e^{\rho[t_k(\lambda^*_{X_0}) + \Gamma_S]}$.[15] $\lambda_K(t)$ decreases according to (4.5) starting from $\lambda_K(t_k(\lambda^*_{X_0})) = \underline{c}'_k$ at the beginning of the phase down to zero at the end, at time $t_\delta = t_k(\lambda^*_{X_0}) + \Gamma_S(\bar{K})$. At this date the available stock of coal, X_δ, amounts to

$$X_\delta = X_0 - r[\Gamma_S(\bar{K})\bar{K} + \int_0^{t_k(\lambda^*_{X_0})} K(t, \lambda^*_{X_0}) dt].$$

Hotelling phase $[t_\delta, t_X]$:

Last during the Hotelling phase $K(t)$ solves:

$$u'(\hat{q}(K(t))) = c_v + r(\underline{c}_x + \lambda^*_{X_0} e^{\rho t}) + m.$$

[15]That is $\Gamma(\bar{K})$ solves:

$$\int_0^{\Gamma_S} u'(\hat{q}(\bar{K})) - (c_v + r[\underline{c}_x + \lambda^*_{X_0} e^{\rho[t_k(\lambda^*_{X_0}) + \tau]}] + m) e^{-\rho\tau} d\tau = \underline{c}'_k$$

Let us denote by $K_H(t, \lambda^*_{X_0})$ the solution of the above equation. The duration of the phase is equal to $\Gamma_H(\lambda^*_{X_\delta})$ defined in 4.2.1 supra, that is $\Gamma_H(\lambda^*_{X_0} e^{\rho[t_k(\lambda^*_{X_0})+\Gamma_S(\bar{K})]})$. At the end of the phase the initially available stock of coal is exhausted:

$$r \int_0^{t_k(\lambda^*_{X_0})} K(t, \lambda^*_{X_0}) dt + \Gamma_s(\bar{K})\bar{K} + \int_{t_\delta(\lambda^*_{X_0})}^{t_X(\lambda^*_{X_0})} K_H(t, \lambda^*_{X_0}) dt = X_0 \ , \quad (4.36)$$

where $t_\delta(\lambda^*_{X_0}) = t_k(\lambda^*_{X_0}) + \Gamma_S(\bar{K})$ and $t_X(\lambda^*_{X_0}) = t_\delta(\lambda^*_{X_0}) + \Gamma_H(\lambda^*_{X_\delta})$.

The equality (4.36) is a direct implication of the fact that $\lambda^*_{X_0}$ solves (4.35) that is $K(t_k(\lambda^*_{X_0}), \lambda^*_{X_0}), X(t_k(\lambda^*_{X_0}))$ is located on the optimal expansion-stabilization frontier so that the above path $\{K(t), X(t),\ t \in [0, t_X(\lambda^*_{X_0})]\}$ corresponds to the optimal trajectory starting from $(0, X_0)$ in the (K, X)-plane.

In the case $\bar{K}_{\max} > K_y$ and $X_0 > X_{0y}$ (see Figure 2) $\hat{q}(K)$ is not differentiable at $K = K_y$ (c.f (3.17)) and the derivative function $u'(\hat{q}(K(t))$ in (4.34) is changing when $K(t) = K_y$ during both the expansion phase at time $\underline{t}_y$ and the Hotelling phase at time $\bar{t}_y$.

The details of the optimal path determination in this case are given in Appendix A.2.

5 Conclusion

When the necessity to build an exploitation capital for providing useful energy to the final users is taken into account, the optimal exploitation path of the non renewable resource is a three phases path, with a constant production intermediate phase. Recovering the investment costs takes time, a minimum time interval during which the margin above the current operation cots must be positive. In the present model the surplus function of the final users is stationary, hence the useful energy price and the production of renewable useful energy are both constant during the intermediate phase of maximum production of non renewable useful energy. It would be no more the case with a non stationary surplus function. The drift of the surplus function would translate into a mere price increase or decrease of the useful energy, depending upon the direction of the drift, if the renewable energy is not competitive. If the renewable energy is competitive, the price increase would be damped by the increase of the renewable energy surplus, and symmetrically the price decrease would be amplified by its supply decrease. However the intermediate phase of constant maximum production of non renewable energy would not disappear from the optimal scenario.[16]

Three problems should be given additional care while staying with the same kind of simple models. The first one is concerning the scraping costs that we have assumed to be nil. Adding either scraping costs or de-pollution costs of the installation sites once they are closed, or a residual value to the no more exploited capital can complexify the Hotelling phase, for example by slowing down the decrease rate of the capacity to smooth the closing costs path. The other one is relative to the renewable energy sector the capital of which has not been modelled. During the phase of the fossil fuel expansion the capital of the renewable energy sector must shrink but like for the capital of the fossil fuel sector during the Hotelling phase, that may have a scraping cost. Next during the revival phase of renewable sector some new dedicated capital has to be built. The two transitions, from renewable to non renewable and later in the reverse direction from non renewable to renewable should receive a symmetrical treatment.[17] Last the pollution generated by the fossil fuel should be taken into account. These problems are left for further research.

[16]On the way to obtain "peak oils" in theoretical models see Holland (2008). For a skeptical appraisal of the predictive power of most models see Brandt (2010), although many models under review have a poor economic content. See also Hughes and Rudolph (2011) and for more recent and less critical reviews, see Weichtmeister *et al* (2018) and Bardi (2019).

[17]Most papers on the transition towards a clean renewable energy regime under capacity constraints and adjustment costs take as given the present capacity of the fossil sector. (See for example Amigues *et al* (2015) and Coulomb *et al* (2018)).

Appendix

A.1 Non-differentiability of $\hat{q}_y(K)$, $\hat{q}(K)$ and $\hat{\mu}(K)$ at $K = K_y$

Concerning first $\hat{q}_y(K)$ we get from (3.16):

$$\lim_{K\uparrow K_y} \frac{d\hat{q}_y}{dK} = \lim_{K\uparrow K_y} \frac{u''(K+\hat{q}_y(K))}{c''_y(\hat{q}_y(K)) - u''(K+\hat{q}_y(K))}$$
$$= \frac{u''(K_y)}{c''_y(0^+) - u''(K_y)} = \begin{cases} -1 & \text{if } c''_y(0^+) = 0 \\ \in (-1,0) & \text{if } c''_y(0^+) > 0 \end{cases}$$

while

$$\lim_{K\downarrow K_y} \frac{d\hat{q}_y}{dK} = 0,$$

hence, whatever $c''_y(0^+) \geq 0$,

$$\lim_{K\uparrow K_y} \frac{d\hat{q}_y}{dK} \quad < \quad \lim_{K\downarrow K_y} \frac{d\hat{q}_y}{dK}\ .$$

Since $\hat{q}(K) = K + \hat{q}_y(K)$, the same inequality holds for $d\hat{q}/dK$ at $K = K_y$.

As for $\hat{\mu}(K)$, we get from (3.19):

$$\lim_{K\uparrow K_y} \frac{d\hat{\mu}}{dK} = \lim_{K\uparrow K_y} \frac{u''(K+\hat{q}_y(K))c''_y(\hat{q}_y(K))}{c''_y(\hat{q}_y(K)) - u''(K+\hat{q}_y(K))} = \frac{u''(K_y)c''_y(0^+)}{c''_y(0^+) - u''(K_y)},$$

while

$$\lim_{K\downarrow K_y} \frac{d\hat{\mu}}{dK} = \lim_{K\downarrow K_y} u''(K) = u''(K_y).$$

It is easy to see that $u''(K_y) < \frac{u''(K_y)c''_y(0^+)}{c''_y(0^+)-u''(K_y)}$, hence, whatever $c''_y(0^+) \geq 0$,

$$\lim_{K\uparrow K_y} \frac{d\hat{\mu}}{dK} > \lim_{K\downarrow K_y} \frac{d\hat{\mu}}{dK}\ .$$

A.2 Determination of the optimal path. Case: $\bar{K}_{\max} > K_y$ and $X_0 > X_{0y}$

The argument parallels the one developed in the sub-section 4.3 for the other cases. In order to avoid any confusion we denote by $\hat{q}_1(K)$ the function $\hat{q}(K)$ for $K < K_y$ and by $\hat{q}_2(K)$ the same function $\hat{q}(K)$ for $K \geq K_y$: $\hat{q}_1(K) > K$, $K < K_y$ and $\hat{q}_2(K) = K$, $K \geq K_y$, with $\hat{q}_1(K_y^-) = \hat{q}_2(K_y) = K_y$ (see (3.17)).

Similarly we denote by $\hat{q}_{y1}(K)$ the function $\hat{q}_y(K)$ for $K < K_y$ and by $\hat{q}_{y2}(K)$ the same function $\hat{q}_y(K)$ for $K \geq K_y$: $\hat{q}_{y1}(K) > 0$, $K < K_y$ and $\hat{q}_{y2}(K) = 0$, $K \geq K_y$, with $\hat{q}_{y1}(K_y^-) = \hat{q}_{y2}(K_y) = 0$ (see (3.16)).

A.2.1 Expansion phase $[0, t_k)$

A.2.1.1 Sub-phase $[0, \underline{t}_y)$

From (3.5) with $q_x(t) + q_y(t) = \hat{q}_1(K(t))$ and $\gamma_x(t) = 0$ (since $q_x(t) > 0$) we get:

$$u'(\hat{q}_1(K_1(t))) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + \eta(t) \ . \tag{A.2.1}$$

From (3.10) with $\delta(t) = 0$ and $\nu_K(t) = 0$ (since $K(t) > 0$), we obtain:

$$\eta(t) = -\dot{\lambda}_K(t) + \rho\lambda_K(t) + m \ ,$$

and substituting for $\eta(t)$ in (A.2.1) results in:

$$u'(\hat{q}_1(K(t))) + \dot{\lambda}_K(t) - \rho\lambda_K(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m \ . \tag{A.2.2}$$

Next from (A.2.2) with $\gamma_k(t) = 0$ (since $k(t) > 0$), and taking into account that $k(t) = \dot{K}(t)$, we get:

$$\lambda_K(t) = c_k'(\dot{K}(t)) \Rightarrow \dot{\lambda}_K(t) = c_k''(\dot{K}(t))\ddot{K}(t) \ ,$$

so that, substituting for $\lambda_K(t)$ and $\dot{\lambda}_K(t)$ in (A.2.2), we obtain:

$$u'(\hat{q}_1(K(t))) - \rho c_k'(\dot{K}(t)) + c_k''(\dot{K}(t))\ddot{K}(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m \ . \tag{A.2.3}$$

For a given λ_{X_0} let us denote by $K_1(t, \lambda_{X_0})$ a solution of (A.2.3) satisfying the initial condition $K(0) = 0$ and by $\underline{t}_y(\lambda_{X_0})$ the time at which $K_1(t, \lambda_{X_0}) = K_y$, equivalently the time at which $\hat{q}_{y1}(K_1(t, \lambda_{X_0})) = 0$.

A.2.1.2 Sub-phase $[\underline{t}_y, t_k)$

During this sub-phase the dynamics of $K(t)$ is given by an equation similar to (A.2.3) with $\hat{q}_2(K)$ substituted for $\hat{q}_1(K)$, that is:

$$u'(\hat{q}_2(K(t))) - \rho c_k'(\dot{K}(t)) + c_k''(\dot{K}(t))\ddot{K}(t) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t}) + m \ . \quad \text{(A.2.4)}$$

For the same λ_{X_0} that in (A.2.3) and the function $K_1(t, \lambda_{X_0})$, let us denote by $K_2(\ t, \lambda_{X_0})$ the solution of (A.2.4) satisfying the initial condition $K_2(\underline{t}_y(\lambda_{X_0}), \lambda_{X_0}) = K_y = K_1(\underline{t}_y(\lambda_{X_0}), \lambda_{X_0})$ preserving the continuity of $K(t)$ at $\underline{t}_y(\lambda_{X_0})$ the time at which $\dot{K}_2(\ t, \lambda_{X_0}) = 0$.

In order that $K(t, \lambda_{X_0})$, $t \in [0, t_k(\lambda_{X_0})]$:

$$K(t, \lambda_{X_0}) = \begin{cases} K_1(t, \lambda_{X_0}), & 0 \le t < \underline{t}_y(\lambda_{X_0}) \\ \\ K_2(t, \lambda_{X_0}), & \underline{t}_y(\lambda_{X_0}) \le t < t_k(\lambda_{X_0}), \end{cases} ,$$

be the optimal expansion phase of the C.U.E production capacity, the state of the system at time $t_k(\lambda_{X_0}), (K(t_k(\lambda_{X_0}), \lambda_{X_0}), X(t_k(\lambda_{X_0}),$ must be located on the expansion-stabilization frontier, that is the optimal value of λ_{X_0} must satisfy the following equation (A.2.5):

$$K(\underline{t}_k(\lambda_{X_0}), \lambda_{X_0}) = K_S(X_0 - r[\int_0^{\underline{t}_y(\lambda_{X_0})} K_1(t, \lambda_{X_0})dt + \int_{\underline{t}_y(\lambda_{X_0})}^{t_k(\lambda_{X_0})} K_2(t, \lambda_{X_0})dt]). \quad \text{(A.2.5)}$$

Let us denote by $\lambda_{X_0}^*$ the solution. Then during the expansion phase the optimal shadow marginal value of the C.U.E industry capital stock, $\lambda_K^*(t)$, is given by (3.7) with $\gamma_k(t) = 0$, that is

$$\lambda_K^*(t) = \begin{cases} c_k'\left(\dot{K}_1(t, \lambda_{X_0}^*)\right) & , \ 0 \le t < \underline{t}_y(\lambda_{X_0}^*) \\ \\ c_k'\left(\dot{K}_2(t, \lambda_{X_0}^*)\right) & , \ \underline{t}_y(\lambda_{X_0}^*) \le t < t_k(\lambda_{X_0}^*) \ . \end{cases}$$

A.2.2 Stabilization phase $[t_k, t_\delta)$

During the expansion phase the capacity of the C.U.E industry is constant: $K(t) = \bar{K} = K_2(t_k(\lambda^*_{X_0}), \lambda^*_{X_0})$. The optimal duration of the phase is equal to $\Gamma(\bar{K})$ solving (4.22) with $\lambda_{X_\delta} = \lambda^*_{X_0} e^{\rho[t_k(\lambda^*_{X_0})+\Gamma_S]}$. At time $t_\delta = t_k(\lambda^*_{X_0}) + \Gamma_S(\bar{K})$, the end of the phase:

$$
\begin{aligned}
X_\delta &= X_0 - r[\Gamma_S(\bar{K})\bar{K} + \int_0^{\underline{t}_y(\lambda^*_{X_0})} K_1(t, \lambda^*_{X_0})dt + \int_{\underline{t}_y(\lambda^*_{X_0})}^{t_k(\lambda^*_{X_0})} K_2(t, \lambda^*_{X_0})dt], \\
\lambda^*_{X_\delta} &= \lambda^*_{X_0} e^{\rho[t_k(\lambda^*_{X_0})+\Gamma_S]} \text{ and } \lambda_k(t_\delta) = 0 \ .
\end{aligned}
$$

A.2.3 Hotelling phase $[t_\delta, t_X)$

A.2.3.1 Sub-phase $[t_\delta, \bar{t}_y)$

During the first Hotelling sub-phase the S.U.E sector is inactive and $K(t)$ solves:

$$u'(\hat{q}_2(K(t))) = c_v + r(\underline{c}_x + \lambda^*_{X_0} e^{\rho t}) + m \ .$$

Let us denote by $K_{H2}(t, \lambda^*_{X_0})$ the solution of the above equation. The sub-phase ends when the S.U.E industry becomes competitive again, at the time $\bar{t}_y$ at which $K_{H2}(t, \lambda^*_{X_0}) = K_y$, the time t solving:

$$c_v + r\left(\underline{c}_x + \lambda^*_{X_0} e^{\rho t}\right) + m = \underline{c}'_y \ .$$

A.2.3.2 Sub-phase $[\bar{t}_y, t_X)$

During the sub-phase of S.U.E industry revival, $K(t)$ solves :

$$u'(\hat{q}_1(K(t))) = c_v + r(\underline{c}_x + \lambda^*_{X_0} e^{\rho t}) + m \ .$$

The Hotelling phase ends at the time t_X solving:

$$c_v + r(\underline{c}_x + \lambda^*_{X_0} e^{\rho t}) + m = \tilde{p} = u'(\tilde{q}_y).$$

A.3 Proof that coal exploitation must end in finite time and that the initial endowment be exhausted

The proofs are by contradiction.

A.3.1 Coal exploitation ends in finite time: $t_X < +\infty$

Let us consider successively the cases $\lambda_{X_0} > 0$ and $\lambda_{X_0} = 0$.

Case $\lambda_{X_0} > 0$.

Assume that there exists an infinite sequence of dates $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n = +\infty$ and $x(t_n) > 0$, $\forall t_n$. In this case the shadow marginal operation current cost of the C.U.E industry at time t_n, that we denote by $SMOCC(t_n)$, would amount to:

$$SMOCC(t_n) = c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t_n}) + m \ ,$$

hence $\lim_{n\to\infty} SMOCC(t_n) = +\infty$ and there exists some $\underline{n}$ such that $SMOCC(t_n) > \tilde{p}$, $n \geq \underline{n}$, so that it would be better to consume the quantity $\tilde{q}_y$ of S.U.E and no C.U.E. $\blacksquare$

Case $\lambda_{X_0} = 0$.

Assume first that the expansion phase ends in in finite time: $t_k < \infty$. Then from t_k onwards the current gross margin of the C.U.E industry would amount to $u'(\hat{q}(\bar{K})) - (c_v + r\underline{c}_x + m)$ where $\bar{K}$ is the industry capacity at the end of the expansion phase. This arbitrage condition determining $\bar{K}$, which must hold at the time t_k for the last piece of invested capacity reads (cf.eq.(4.22)) with $\Gamma_S = +\infty$)

$$\int_0^{+\infty} \{u'(\hat{q}(\bar{K})) - (c_v + r\underline{c}_x + m)\} e^{-\rho\tau} d\tau = \underline{c}_k' \qquad \text{(A.3.1)}$$

Thus from t_k the amount of coal used in the C.U.E industry at each unit of time would amount to $r\bar{K}$ requiring an infinite coal stock. Assume now that $t_k = +\infty$, that is the duration of the expansion phase is infinite. Since $\dot{K}(t) > 0$ during this phase, for any ϵ , $0 < \epsilon < \bar{K}$, there exists some $\underline{t} < +\infty$ such that $K(t) \geq \bar{K} - \epsilon$, $t \geq \underline{t}$, where $\bar{K}$ solves (A.3.1). From $\underline{t}$ onwards the instantaneous coal input of the C.U.E industry would be at least equal to $r(\bar{K} - \epsilon)$, again requiring an infinite initial endowment. ■

A.3.2 The coal initial endowment must be exhausted at time t_X

Let us denote by an asterisk an assumed optimal path $\{(q_x^*(t), q_y^*(t), k^*(t), \delta^*(t)\}_{t=0}^{\infty}$ along which the coal endowment would not be exhausted. Let $\underline{X} = X^*(t_X) > 0$ denote this part of the coal stock left underground. Given the results of the preceding paragraph, if this path is optimal, it must be the case that $\lambda_{X_0} > 0$. Consider a time interval (t_d, t_X^*) within the Hotelling phase, $t_\delta^* < t_d < t_X^*$, during which $q_y^*(t) > 0$. Since $q_y^*(t)$ is increasing once positive during the Hotelling phase and $\lim_{t \uparrow t_X^*} q_y^*(t) = \tilde{q}_y > 0$, such an interval exists. Let $\{(\check{q}_x(t), \check{q}_y(t), \check{k}(t), \check{\delta}(t))\}_{t=0}^{\infty}$ be an alternative path equal to the optimal path during the initial interval $[0, t_d]$ and modified as follows from t_d onwards, up to t_X^* :

$$\check{q}_x(t) = \begin{cases} q^*(t) - q_y^*(t_d) & , \quad t_\delta < t < t_X^* \\ \\ 0 = q_x^*(t) & , \quad t_X^* \leq t \end{cases},$$
$$\check{q}_y(t) = \begin{cases} q_y^*(t_d) & , \quad t_\delta < t < t_X^* \\ \\ \tilde{q}_y = q_y^*(t) & , \quad t_X^* \leq t \end{cases},$$
$$\check{\delta}(t) = \begin{cases} \dot{q}(t)/q^*(t) - q_y^*(t_d) & , \quad t_\delta < t < t_X^* \\ \\ 0 = \delta^*(t) & , \quad t_X^* \leq t \end{cases},$$

with at time t_X^* a brutal scraping of the whole capital of the C.U.E industry, $\tilde{K}(t_X^{*-}) = \tilde{q}_y - q_y^*(t_d)$. Both paths, the assumed optimal one and the modified one are illustrated in the below Figure 4.

Figure 4 here

The modified path is deduced from the assumed optimal path by substituting the quantity $\check{q}_y - q_y^*(t_d)$ of the C.U.E for the same quantity of S.U.E at each time $t \in (t_d, t_X^*)$ so that the useful energy consumption is the same along both paths, $q^*(t) = q_x^*(t) + q_y^*(t) = \check{q}_x(t) + \check{q}_y(t) = \check{q}(t)$, hence the useful surplus $u(q^*(t)) = u(\check{q}(t))$, $t \geq 0$. The additional cumulated extraction required to sustain the modified path, $\Delta X(t_X^*)$, amounts to:

$$\Delta X(t_X^*) = r \int_{t_d}^{t_X^*} (q_y^*(t) - q_y^*(t_d))dt.$$

Clearly for $t_X^* - t_d$ sufficient small: $\Delta X(t_X^*) \leq \underline{X}$. Thus the modified path is feasible. Let us show now that the modified path is also less costly. At any time $t \in (t_d, t_X^*)$ the costs saved thank to the cutting down of the S.U.E production, that we denote by $\Delta SC(t)$, amount to:

$$\Delta SC(t) = c_y(q_y^*(t)) - c_y(q_y^*(t_d)) = \int_{q_y^*(t_d)}^{q_y^*(t)} c_y'(q_y)dq_y$$

while the additional costs induced by the increase of the C.U.E production, denoted by $\Delta CC(t)$, amount to:

$$\Delta CC(t) = [q_y^*(t) - q_y^*(t_d)](c_v + r\underline{c}_x + m).$$

Along the assumed optimal path at the time t_d within the Hotelling phase with $q_y^*(t_d) > 0:$

$$c_v + r(\underline{c}_x + \lambda_{X_0} e^{\rho t_d}) + m = u'(q^*(t_d)) = c_y'(q_y^*(t_d)).$$

On the other hand, since $c_y'(q_y)$ is an increasing function of q_y and because $\lambda_{X_0}^* > 0$, it follows that:

$$q_y > q_y^*(t_d) \Longrightarrow c_y'(q_y) > c_v + r\underline{c}_x + m,$$

hence, because $q_y^*(t) > q_y^*(t_d)$ since $q_y^*(t)$ is increasing during the interval (t_d, t_X^*), for any t within the interval:

$$\Delta CC(t) = [q_y^*(t) - q_y^*(t_d)](c_v + r\underline{c}_x + m) < \int_{q_y^*(t_d)}^{q_y^*(t)} c_y'(q_y)dq_y = \Delta SC(t).$$

To sum up the modified path is feasible, allows the same users surplus than the assumed optimal one and is less costly, a contradiction. ■

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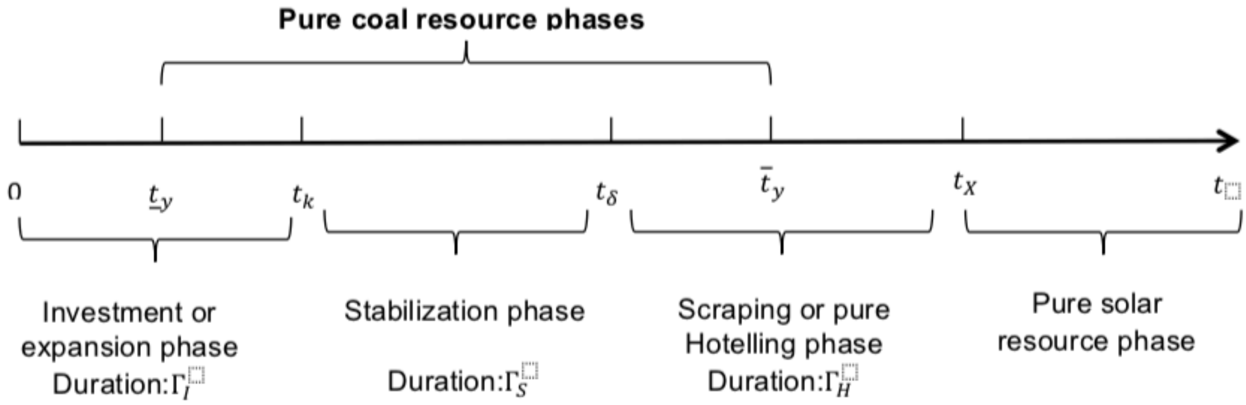

Figure 1: **Typical optimal scenario and time notations. Case $K_y < \bar{K}$.**

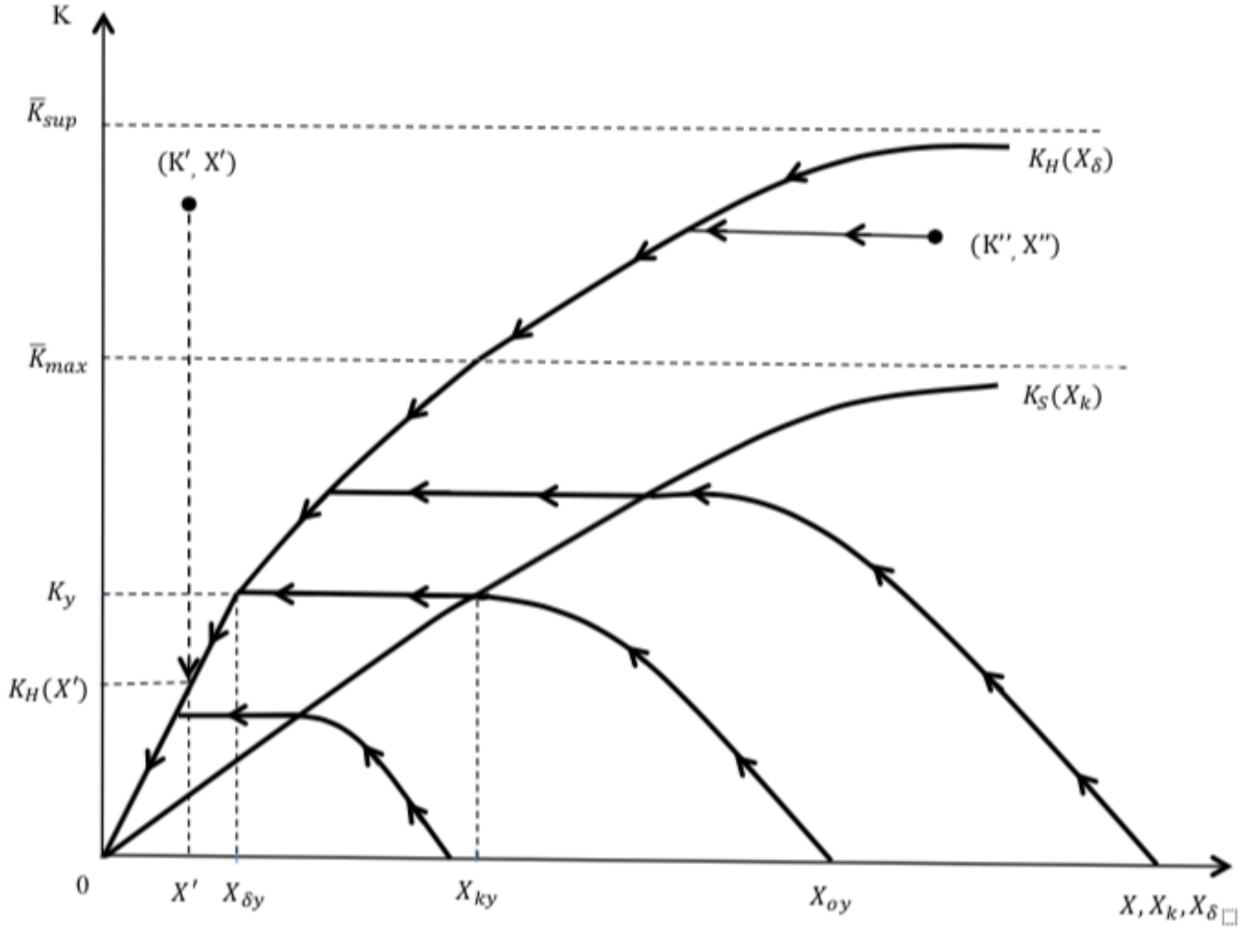

Figure 2: **Phase diagram. Case:** $K_y < \bar{K}_{max}$.

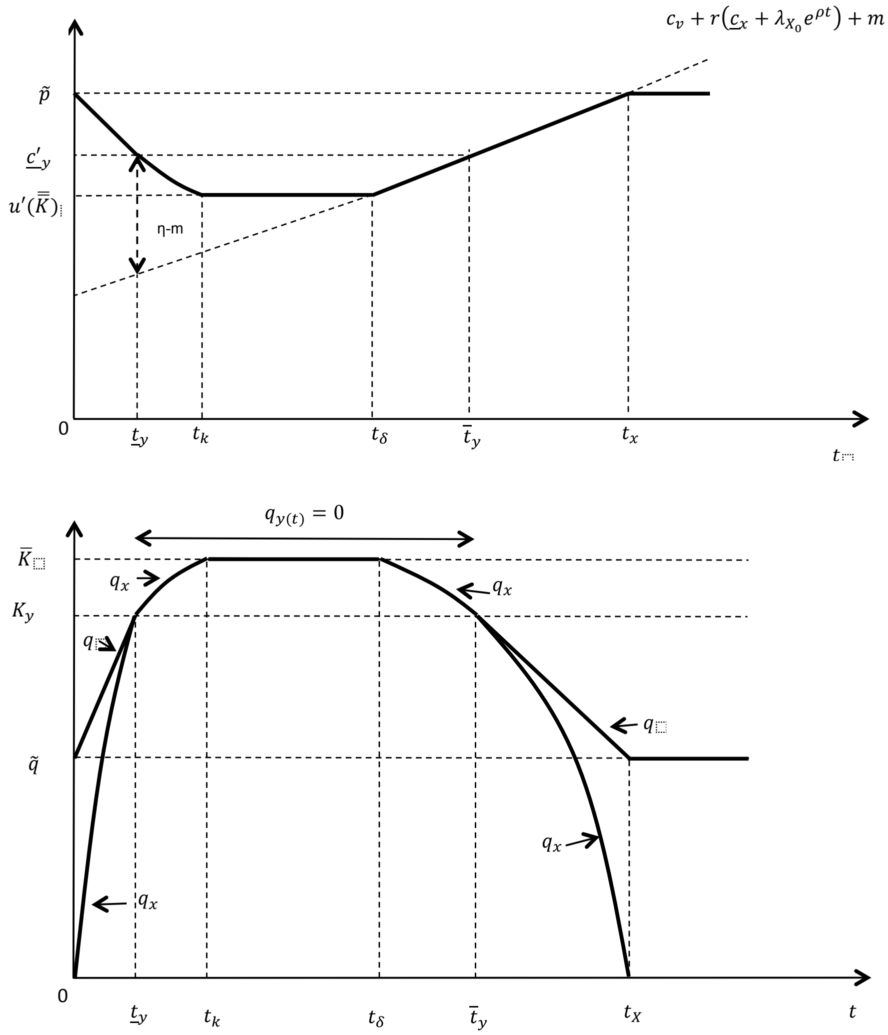

Figure 3: **Optimal price and U.E productions paths. Case:** $K_y < \bar{K}$**.**

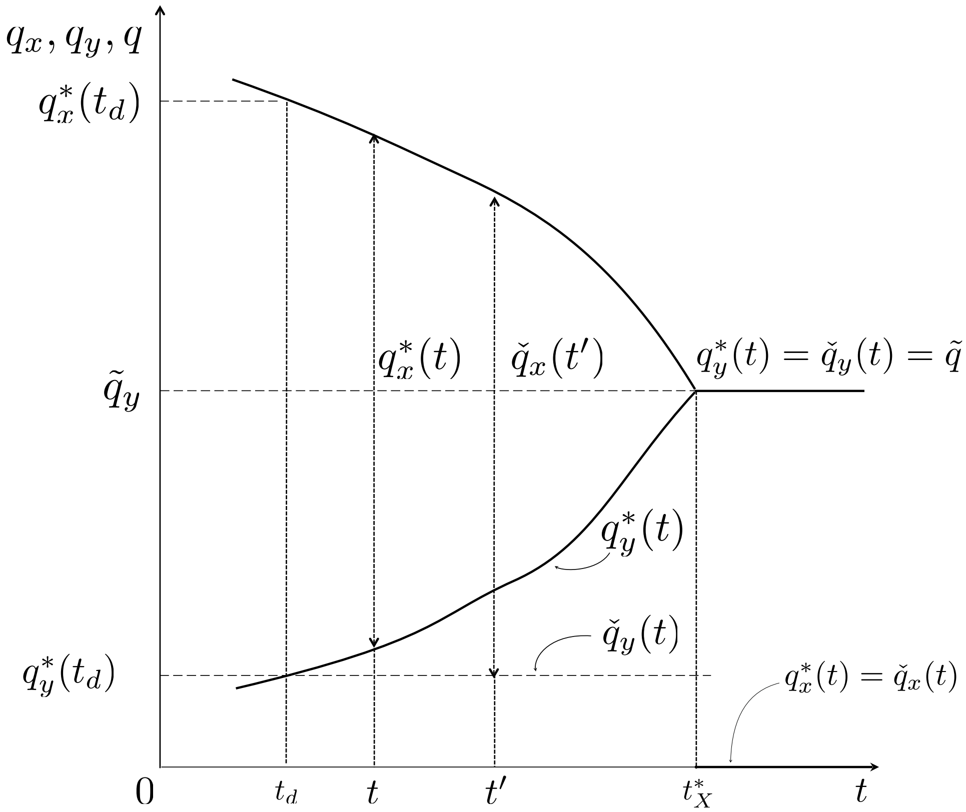

Figure 4: **Assumed optimal path and modified path.**