FAERE
French Association
of Environmental and Resource Economists

Working papers

Energy Conversion Rate Improvements, Pollution Abatement Efforts and Energy Mix: The Transition toward the Green Economy under a Pollution Stock Constraint

Jean-Pierre Amigues - Michel Moreaux

WP 2019.14

Suggested citation:

JP. Amigues, M. Moreaux (2019). Energy Conversion Rate Improvements, Pollution Abatement Efforts and Energy Mix: The Transition toward the Green Economy under a Pollution Stock Constraint. *FAERE Working Paper, 2019.14.*

ISSN number: **2274-5556**

www.faere.fr

Energy Conversion Rate Improvements, Pollution Abatement Efforts and Energy Mix: The Transition toward the Green Economy under a Pollution Stock Constraint

Jean-Pierre Amigues[1] and Michel Moreaux[2]

February 22, 2019

[1]Toulouse School of Economics (INRA, IDEI), 21 allée de Brienne, 31000 Toulouse, France. E-mail:jean-pierre.amigues@inra.fr

[2]Toulouse School of Economics, University of Toulouse I Capitole (IDEI, IUF), Manufacture des Tabacs, 21 allée de Brienne, 31000 Toulouse, France.

Abstract

To prevent climate change, three options are currently considered: improve the energy conversion efficiency of primary energy sources, develop carbon free alternatives to polluting fossil fuels and abate potential emissions before they are released inside the atmosphere. We study the optimal mix and timing of these three mitigation options in a stylized dynamic model. Useful energy can come from two sources: a non-renewable fossil fuel resource and a carbon free renewable resource. The conversion efficiency rate of fossil energy into useful energy is open to choice but higher conversion rates are also more costly. The economy can abate some fraction of its potential emissions and a higher abatement rate incurs higher costs. The society objective is to maintain below some mandated level, or carbon cap, the atmospheric carbon concentration. In the empirically relevant case where the economy is actually constrained by the cap, at least temporarily, we show that the optimal path is a sequence of four regimes: a 'pre-ceiling' regime before the economy is actually constrained by the cap, a 'ceiling' regime at the cap, a 'post-ceiling' regime below the cap and a final regime of exclusive exploitation of renewable resources. If the abatement option has ever to be used, it should be started before the beginning of the ceiling regime, first at an increasing rate and at a decreasing rate once the cap constraint binds. The efficiency performance from any source steadily improves with the exception of a time phase under the ceiling regime when it is constant. Renewables take progressively a larger share of the energy mix but their exploitation may be delayed significantly. Absolute levels of carbon emissions drop down continuously but follow a non monotonic pattern in per useful energy unit relative terms.

Keywords: energy efficiency; carbon pollution; non-renewable resources; renewable resources; abatement.

JEL classifications: Q00, Q32, Q43, Q54.

Contents

1 Introduction

Following the first oil shock of the seventies, policymakers have expressed a strong concern for energy efficiency improvements. Numerous agencies, plans and incentives have been created and designed to alleviate the effects of high energy prices on welfare. If the economic relevance of such policy initiatives has never been put into question, the same cannot be said of their actual impacts. Many studies have shown that even given positive incentives, households or firms may not adopt more energy efficient devices.[1]

However, whatever the alleged resistance of economic agents to become more energy efficient, there is clear evidence that the overall energy efficiency of industrialized economies has constantly risen in the last decades, as an effect of dedicated policies or the mere working of market forces and the respective dynamics of costs and benefits. The Figure 1 illustrates the past and projected trend of US energy efficiency expressed in energy consumption per GDP unit. It shows that between 1950 and 2010, primary energy consumption per $ GDP has dropped more than a half and is expected to continue to decrease in the next decades at an accelerated trend after the oil shocks of the early seventies.

The fear of oil shocks has been replaced today by the concern for climate change. Fossil fuels accounts for 87% of primary energy consumption at the world scale and fossil fuel burning is the largest source of GHG emissions. Raising energy efficiency of fossil fuel exploitation stands as a top priority in carbon emissions mitigation. This is not the only option however.

Three main ways toward 'green' energy systems are today on top of the desk. The first one is energy efficiency improvements, resulting in less carbon emissions release in the atmosphere per unit of GDP. The second one is the substitution of fossil fuels use by other carbon free primary energy sources, renewable or not: nuclear power, hydropower, wind and solar power, biofuels. The transformation of these primary energy sources into useful energy raises

[1]The U.S. government has created a dedicated agency for energy efficiency, the American Council for an Energy Efficient Economy which publishes regularly reports on this topic. Similar institutions exist in UK, France or Germany. For the energy efficient devices adoption debate, see Brown, 2004, Allcott and Greenstone, 2012. For global estimates of energy efficiency trends, see Schafer, 2005, Gillingham *et al.*, 2009, IEA-OECD, 2013.

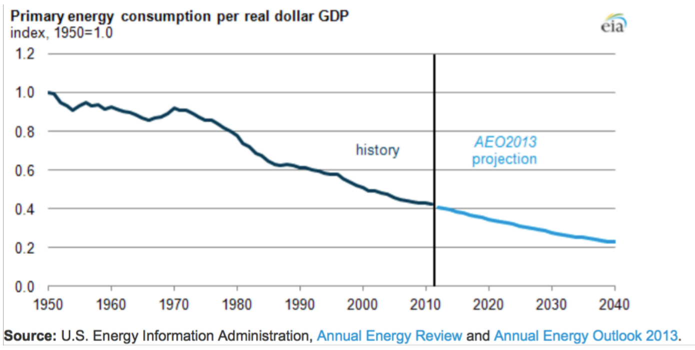

Figure 1: **US Past and Projected Energy Efficiency Trend.**

also energy efficiency problems. As an example of such difficulties, it is currently pleaded in the energy debate that renewables like wind or solar being intermittent sources with low conversion performances, the economic scope of these options should be rather limited. The third option is 'end-of-pipe' abatement of emissions and geological carbon capture and storage (CCS) is currently viewed as the most promising technique in that respect.[2] However, being still in infancy, the deployment of such abatement techniques also raises strong efficiency concerns before becoming economically relevant.

For self-evident reasons, a lot of attention has been devoted to the co-ordination failures faced by today governments trying to cope with a global externality like climate change. But even in a fully cooperative world, the design of an efficient policy agenda able to curb the current trend of GHG emissions is not en easy task. One must take care of the temporal heterogeneity between policies, usually cast in the short run, and climate dynamics, a set of complex evolutions extending well ahead the next centuries. Moreover, available options to mitigate climate change interact themselves both in the scale and time dimensions and thus should not be assessed in isolation, or on the sole basis of their relative profitability at some given time. Such 'profitabilities', whatever might be their definitions, are jointly determined by the relative competitiveness prospects of the mitigation options, themselves depending on the climate dynamics and the climate policy agenda.

[2]See for example Budinis *et al.*, 2016.

In this context, we ask the following questions. How the various mitigation options previously outlined should be combined ? What should be the right agenda of options in terms of relative scale and priorities? What will be the consequences of fossil fuel scarcity on the optimal mix and timing of mitigation options?

A lot of applied studies have explored the economic 'greening' issue. Many of them have relied on numerical simulation models at various space and time scales. As an example, the Figure 2 shows projections from the International Energy Agency about the desirable mix of different 'green' options under an atmospheric carbon concentration stabilization target in line with the $+2^0C$ objective:[3]

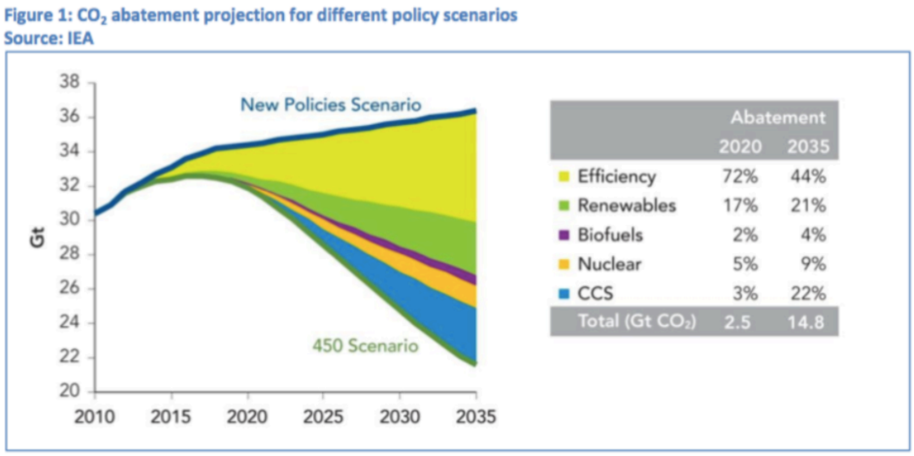

Figure 2: **Mitigation Efforts to Meet the 450 ppmV Target.**

As usual, it is hard to assess the reliability and the underlying economic rationale for such figures. They suggest however that efficiency gains should play a leading role in the mitigation efforts to meet the temperature rise stabilization goal. But if the substitution options and the abatement options have received considerable attention in the theoretical economic literature on climate change management, this is much less true for energy efficiency dynamics. Many models assume given and fixed efficiency rates for different energy sources or introduce exogenous trends of efficiency improvements.[4]

[3] IEA/OECD, 2013.

[4] On the theoretical side, main original contributions are Farzin, 1996, Farzin and Tahvo-

Efficiency gains being usually explained by technical progress, the issue is hence accommodated inside the large body of literature devoted to technological advances and innovations. This is in particular the case for all the recent works on so-called 'green' R&D. In this literature, innovation improves inputs efficiencies, in particular those of primary energy sources. But economic efficiency is a slightly different concept than energy efficiency as used in many empirical studies. One objective of this paper is to build a bridge between these two concepts.[5]

To address these questions, we develop a stylized model summarizing the main ingredients of the problem. Useful energy can be obtained from two primary sources: a polluting non-renewable resource and a carbon-free renewable resource. The exploitation of the non-renewable resource incurs extraction costs increasing with past extraction. The conversion rate of fossil primary energy into useful energy may be adjusted over time, but more efficient energy converters are more costly to manage. The global economy can also engage in emissions abatement, higher abatement rates being also more costly to achieve.

As in Chakravorty *et al.* (2006), we assume that the economy wants to maintain the atmospheric carbon concentration below a mandated level. Under this stabilization constraint, we study how the economy should manage the three options at its disposal to mitigate carbon emissions: raise the energy efficiency performance of fossils, use more clean renewables or abate some fraction of the potential carbon emissions before they are released in the atmosphere.

In the interesting case where the economy would be at least temporarily constrained by the carbon pollution mandate, we show that if the abatement option has ever to be used, it must do so around the beginning of the time phase under the carbon cap and be ended strictly before the economy

nen, 1996, Tahvonen and Withagen, 1996, Toman and Withagen, 2000. For more recent contributions see Golosov *et al.*, 2014, Van der Ploeg and Withagen, 2014. The applied literature has intensively used IAM's to assess climate policy options. Prominent contributions are Gerlagh and Van der Zwaan, 2006, Stern, 2007, Nordhaus, 2008.

[5]The issue of technical change has quite naturally attracted a lot of attention in the theoretical and empirical literature. See for example Manne and Richels, 2004, Gerlagh, 2004, Edenhofer *et al.*, 2005, Grimaud *et al.*, 2011, Hassler *et al.*, 2012, Sorrell, 2014. Recent contributions in the macro growth literature are Acemoglu *et al*, 2011, 2016, Smulders *et al.*, 2011, Van der Meijden and Smulders, 2018.

can escape the constraint, a consequence of the time increasing costs of the polluting non-renewable resource. The abatement rate should first increase before the economy is constrained by the cap, second be constantly decreasing during the constrained time period, and last be nil strictly before the end of the constrained period.

Furthermore, the fossil energy conversion performance should steadily improve with the exception of the last phase of the constrained period without abatement, a phase during which the economy should maintain at a constant level its energy conversion efficiency. The renewable option rises progressively at the expense of the polluting non renewable option. The combination of efficiency gains, substitution toward renewables and abatement induces, with a stationary useful energy demand, a permanent drop down of polluting emissions anyway insufficient in a first stage to prevent the progressive accumulation of carbon into the atmosphere.

The rising costs of extraction of the non renewable resource imply that some fraction of the initial resource endowment should remain permanently stored underground, what we call the unburnable fossil resource stock. The time increasing cost of fossil fuels extraction also induces a permanent rise of the useful energy price until the complete transition toward green energy, with the exception of the no abatement phase when constrained by the cap, during which the energy price should be constant.

Section 2 presents our model of energy use and production. The optimality problem faced by the society is laid down in Section 3. We focus on the interesting case of an economy actually constrained by the atmospheric carbon stock size, at least during some time period, a period we call the ceiling regime. Section 4 studies the unconstrained time phases before and after the ceiling regime. Section 5 describes the optimal behavior of the economy under this regime. The main features of the optimal policies are presented in Section 6. Section 7 focuses on the implications of limited carbon sequestration capacities. We examine in Section 8 the implications of alternative formalisations of the cost functions at their upper bounds. The last Section 9 concludes.

2 The model

We consider an economy in which the useful energy (U.E) can be produced from two primary resources, a nonrenewable and potentially polluting one (coal) and a clean renewable one (solar). Let respectively $q_x(t)$ and $q_y(t)$ denote the instantaneous production rates of coal and solar U.E, (C.U.E and S.U.E thereafter). Assuming that C.U.E and S.U.E are perfect substitutes we may define the aggregate U.E production, denoted by $q(t)$, as the sum of both C.U.E and S.U.E productions: $q(t) = q_x(t) + q_y(t)$.

Users gross surplus

The instantaneous gross surplus function of the final users, $u(q)$, satisfies the following standard assumption:[6]

Assumption A. 1 $u : (0, \infty) \to \mathbb{R}_+$ *is twice continuously differentiable, strictly increasing,* $u'(q) > 0$*, strictly concave,* $u''(q) < 0$*, with* $u'(0^+) = +\infty$ *and* $\lim_{q\uparrow\infty} u'(q) = 0$.

Alternatively we denote by p the marginal gross surplus u', by $p(q) \equiv u'(q)$, the inverse demand function, and by $q^d(p)$, the direct demand function, the inverse of $p(q)$ well defined under A.1.

Producing useful energy from coal

The coal sector includes two industries. The mining industry produces extracted coal from underground reserves and the coal transformation industry produces C.U.E from extracted coal. Let $X(t)$ be the stock of underground stock available at time t, X^0 be the initial endowment, $X^0 = X(0)$ and $x(t)$ be the instantaneous extraction rate: $\dot{X}(t) = -x(t)$.

Concerning its energy content, the stock of coal is assumed to be homogenous. But the deposits are more or less easily exploitable. Since inter-

[6]For any function f defined on $X \subseteq \mathbb{R}$ and for any $\bar{x} \in \bar{X}$ where $\bar{X}$ is the closure of X, we denote by $f(\bar{x}^+)$ and $f(\bar{x}^-)$ respectively the limits, $\lim_{x\downarrow\bar{x}} f(x)$ and $\lim_{x\uparrow\bar{x}} f(x)$, when such limits exist.

temporal cost minimization implies that the less costly deposit be exploited first we may assume that the unitary extraction cost is some decreasing function of the not yet exploited stock X. Let $a(X)$ be this function so that at time t, the total extraction cost amounts to $a\left(X(t)\right)x(t)$. The function $a(X)$ satisfies the now standard assumption A.2:[7]

Assumption A. 2 *$a : (0, X^0] \to \mathbb{R}_+$ is twice continuously differentiable on $(0, X^0)$, strictly decreasing, $a'(X) < 0$, strictly convex, $a''(X) > 0$, with $a(X^0) > 0$, $a'(X^{0-}) < 0$, $a(0^+) = +\infty$ and $a'(0^+) = -\infty$.*

Let η_x be this fraction of the extracted coal energy content converted into U.E by the coal transformation industry (C.T.I thereafter) so that the C.U.E production rate at time t amounts to $q_x(t) = \eta_x(t)x(t)$. Getting more U.E from a given quantity of coal requires more costly industrial processes. Let us denote by $b(\eta_x)$ the processing unitary cost of the C.T.I extracted coal input as a function of the efficiency rate η_x, hence a processing cost $b(\eta_x)/\eta_x$ per C.U.E unit, equal to the processing marginal cost.

We assume that $b(\eta_x)/\eta_x$ is increasing, hence also $b(\eta_x)$, and bounded from above by $\bar{\eta}_x < 1$, since according to the second law of thermodynamics some energy is necessarily lost in the transformation.[8]

Assumption A. 3 *$b : [0, \bar{\eta}_x) \to \mathbb{R}_+$ is twice continuously differentiable on $(0, \bar{\eta}_x)$, increasing, $b'(\eta_x) > 0$, strictly convex, $b''(\eta_x) > 0$, with $b(0) = 0$, $b'(0^+) > 0$, $b(\bar{\eta}_x^-) = +\infty$ and $b'(\bar{\eta}_x^-) = +\infty$.*

Producing U.E from coal requires not only coal but also other costly inputs hence a positive marginal processing cost at 0^+, implying, together

[7] See Solow and Wan (1976), Heal (1976), Lehvari and Liviatan (1977) and Hanson (1980) for early analysis, and Cairns *et al.* (1998) and Hart (2016) for more recent developments.

[8] Differentiating the unitary cost yields:

$$\frac{d}{d\eta_x}\frac{b(\eta_x)}{\eta_x} = \frac{1}{\eta_x}\left[b'(\eta_x) - \frac{b(\eta_x)}{\eta_x}\right] ,$$

hence $b'(\eta_x) > 0$ is necessary for $db(\eta_x)/\eta_x > 0$.

with the continuity of b at 0, that: $b'(\eta_x) > b(\eta_x)/\eta_x$, $\eta_x > 0$. Hence $b(\eta_x)/\eta_x$ is increasing, and $\lim_{\eta_x \downarrow 0} b(\eta_x)/\eta_x = b'(0^+) > 0$.

The assumption $b'(\bar{\eta}_x^-) = +\infty$ allows an easier computation of the qualitative features of the optimal paths. In the same spirit similar assumptions will be postulated for the abatement cost function, g, and the solar energy cost function, c, defined in the below paragraphs. We show in Section 8 that assuming a finite cost $b'(\bar{\eta}_x^-)$, and finite values of g and c at the upper bounds of their arguments do not alter the main qualitative characteristics of the optimal paths.

Pollution

Burning coal to produce U.E generates a potential polluting emission flow. Let ζ denote the unitary pollution content of coal so that potential emissions would amount to $\zeta x(t)$ when the C.T.I uses $x(t)$ units of coal. The C.T.I can abate some part of the potential flow. Let $\eta_z(t)$ be the fraction of the potential emissions which is captured and sequestered and $z(t)$ the effective flow feeding the atmospheric pollution stock: $z(t) = (1 - \eta_z(t))\,\zeta x(t)$.

Let us denote by $Z(t)$ the atmospheric pollution stock at time t, and by Z^0 its initial level, inherited from the past: $Z(0) = Z^0$. To simplify we assume that the atmospheric stock is self-diluting at some constant proportional rate α.[9] Thus the dynamics of $Z(t)$ is given by:

$$\dot{Z}(t) \;=\; z(t) - \alpha Z(t) = (1 - \eta_z(t))\,\zeta x(t) - \alpha Z(t)\ .$$

The atmospheric pollution stock must be kept at most equal to some upper bound, or ceiling, $\bar{Z}$, to prevent excessive damages like in Chakravorty *et al.* (2006).[10] The constraint is roughly equivalent to the $+2^0$ objective of the COP 21, would the Paris agreement be enforceable.[11] In order that the

[9]See Forster (1975), Farzin (1996), Tahvonen and Salo (1994), Tahvonen and Withagen (1996) and Toman and Withagen (2000) for theoretical models in which α depends on Z. Some such self-dilution processes give rise to strong non convexities, hence difficulties for characterizing the optimal paths.

[10]The other standard formulation is to assume that the damages increase continuously with the pollution stock. See Moreaux and Withagen (2015) for a recent study of the optimal abatement policy in such a case but without the efficiency improvement option.

[11]Actually there exists some time lags between the CO_2 concentration and temperature

model makes sense we must assume that the initial stock Z^0 is smaller than the cap $\bar{Z}$.

Abating the potential emissions is costly. Let $g(\eta_z)$ be the unitary processing cost of the potential emissions ζx to be incured by the C.T.I for abating the fraction η_z of these emissions. The total abatement cost of $\eta_z \zeta x$ amounts to $g(\eta_z)\zeta x$, hence an average cost $g(\eta_z)/\eta_z$ per unit of abated pollution, equal to the marginal cost. Higher abatement rates are technically more difficult to achieve, thus more costly. Denoting by $\bar{\eta}_z < 1$ the upper bound on feasible abatement rates, we assume that the function $g(.)$ satisfies:

Assumption A. 4 $g : [0, \bar{\eta}_z) \rightarrow \mathbb{R}_+$ *is twice continuously differentiable on* $(0, \bar{\eta}_z)$*, strictly increasing,* $g'(\eta_z) > 0$*, strictly convex,* $g''(\eta_z) > 0$*, with* $g(0) = 0$*,* $g'(0^+) > 0$*,* $g(\bar{\eta}_z^-) = +\infty$ *and* $g'(\bar{\eta}_z^-) = +\infty$*.*

The rationale underlying A.4 is the same than the rationale underlying A.3. The assumptions $b'(\bar{\eta}_x^-) = +\infty$ and $g'(\bar{\eta}_z^-) = +\infty$ are simplifying assumptions.

We first assume in Sections 3-6 that there is no sequestration cost *per se* and $g(.)$ must be understood as a capture and sequestration cost. In Section 7 we develop the model and then $g(.)$ will have to be understood as a pure capture cost to which must be added a specific sequestration cost, a cost itself depending upon the cumulative sequestered flow. The assumptions relative to this development are laid down when needed, in Section 7. We show that the main qualitative characteristics of the optimal paths are the same in both versions of the model, hence the best is to begin with the most simple one.

Solar energy

The other primary resource is the solar radiation attaining the earth surface. S.U.E production is a clean process generating no pollution to simplify, in any case less than burning fossil fuels, in competition with the abatement option to satisfy the CO_2 concentration constraint, and presently beginning

increases. See Weitzman (2010) and Mason and Wilmot (2015) for the policy implications of such lags.

to be competitive since already unfold at a significant scale contrary to the abatement device.

Concerning the S.U.E we do not detail the process of conversion of solar radiation into U.E, neither the allocation of land among its different uses.[12] We simply assume that the cost of S.U.E, denoted by $c(q_y)$, and its marginal cost are both increasing functions of the production rate. Note that this cost must include the loss incurred for diverting the land from other uses, thus include the land rent, the equivalent of the mining rent in the production of C.U.E.[13] Let us denote by $\bar{q}_y$ the upper bound on the S.U.E production rate, then:

Assumption A. 5 $c : [0, \bar{q}_y) \to \mathbb{R}_+$ *is twice continuously differentiable on* $(0, \bar{q}_y)$*, strictly increasing,* $c'(q_y) > 0$*, strictly convex,* $c''(q_y) > 0$*, with* $c(0) = 0$*,* $c'(0^+) = \underline{c} > 0$*,* $c(\bar{q}_y^-) = +\infty$ *and* $c'(\bar{q}_y^-) = +\infty$*.*

Competitiveness of the coal sector

Absent any C.U.E production, the S.U.E industry is the only supplier of U.E. Since the S.U.E is a renewable resource without stock effect, its optimal production rate is the solution of the static problem $\max\{u(q_y) - c(q_y)q_y\}$. Let $\tilde{q}_y$ be the solution of this problem, unique under A.1, A.5, then: $u'(\tilde{q}_y) = c'(\tilde{q}_y)$, and denote by $\tilde{p}$ the corresponding U.E price: $\tilde{p} = u'(\tilde{q}_y)$.

In order that the C.U.E be ever competitive we must assume that for the least costly grade of coal, X^0, free of mining rent, exploited at the minimum C.U.E average complete cost without emission levy, this minimum average cost is lower than $\tilde{p}$. The minimum average complete cost is given by the transformation rate, denoted by $\eta_x(X^0)$, solving $\min\{(a(X^0) + b(\eta_x))/\eta_x\}$. Thus $\eta_x(X^0)$ satisfies the f.o.c:

$$\frac{1}{\eta_x(X^0)}\left[b'\left(\eta_x(X^0)\right) - \frac{a(X^0) + b\left(\eta_x(X^0)\right)}{\eta_x(X^0)}\right] = 0 \ ,$$

[12]A main other use of land is the production of food. See Amigues and Moreaux (2018) for a detailed analysis of the allocation of land among food and U.E production.

[13]However there is some difference: At time t the land rent is the opportunity cost incurred for diverting the land from other uses at the same date while the mining rent is the cost incurred for diverting the coal from future uses.

that is, the marginal efficiency cost $b'(\eta_x)$ must be equal to the minimum of the average complete cost $(a(X^0) + b(\eta_x))/\eta_x$.[14]

Assumption A. 6 *The C.U.E sector is competitive, that is:*

$$\frac{a(X^0) + b(\eta_x(X^0))}{\eta_x(X^0)} < \tilde{p} \ .$$

Discounting and social welfare

The social rate of discount, ρ, is assumed to be positive and constant through time. The social welfare is the sum of the discounted net surplus provided that the CO_2 atmospheric concentration be kept at most equal to $\bar{Z}$.

3 The social planner problem

The social planner determines the path $\left\{(x^*(t), \eta_x^*(t), \eta_z^*(t), q_y^*(t))\right\}_{t=0}^{\infty}$ maximizing the social welfare, that is solves the following problem $(S.P.)$:[15]

$$\begin{aligned}
\max_{x,\eta_x,\eta_z,q_y} \quad & \int_0^\infty \{u(\eta_x(t)x(t) + q_y(t)) - a(X(t))x(t) - b(\eta_x(t))x(t) \\
& \qquad -c(q_y(t)) - g(\eta_z(t))\zeta x(t)\} e^{-\rho t} dt && (3.1) \\
s.t. \quad & \dot{X}(t) = -x(t) \ , \ X(0) = X^0 > 0 \ \text{ given} && (3.2) \\
& \dot{Z}(t) = (1 - \eta_z(t))\zeta x(t) - \alpha Z(t) \ , \ Z(0) = Z^0 < \bar{Z} \ \text{ given} \ , \bar{Z} - Z(t) \geq 0 && (3.3) \\
& x(t) \geq 0, \ \eta_x(t) \geq 0, \ \eta_z(t) \geq 0, \ q_y(t) \geq 0 \ . && (3.4)
\end{aligned}$$

Let λ_X and $-\lambda_Z$ denote the co-state variables of X and Z respectively,

[14] A similar expression holds at any time t for the grade $X(t)$ when coal is exploited. See equation (3.10) *infra*.

[15] We omit the constraints $X(t) \geq 0$, $\bar{\eta}_x - \eta_x(t) \geq 0$, $\bar{\eta}_z - \eta_z(t) \geq 0$, and $\bar{q}_y - q_y(t) \geq 0$, which are never active under A.1-A.5.

then the current value Hamiltonian reads:[16,17]

$$\begin{aligned}\mathcal{H} &= u(\eta_x x + q_y) - a(X)x - b(\eta_x)x - c(q_y) - g(\eta_z)\zeta x \\ &\quad -\lambda_X x - \lambda_Z \left[(1-\eta_z)\zeta x - \alpha Z\right] \ .\end{aligned}$$

Denote by ν the Lagrange multiplier associated to the ceiling constraint $\bar{Z} - Z \geq 0$, by γ's the multipliers associated to the non negativity constraints (3.4). Then the Lagrangian of the problem $(S.P)$ reads:

$$\mathcal{L} = \mathcal{H} + \gamma_x x + \gamma_{\eta x}\eta_x + \gamma_{\eta y}\eta_y + \gamma_y q_y + \nu(\bar{Z} - Z) \ .$$

3.1 First order conditions

The f.o.c's read:

$$\text{w.r.t. } x \ : \ u'(\eta_x x + q_y)\eta_x = a(X) + \lambda_X + b(\eta_x) + \zeta\left[g(\eta_z) + (1-\eta_z)\lambda_Z\right] - \gamma_x \tag{3.5}$$

$$\text{w.r.t. } \eta_x \ : \ u'(\eta_x x + q_y)x = b'(\eta_x)x - \gamma_{\eta x} \tag{3.6}$$

$$\text{w.r.t. } q_y \ : \ u'(\eta_x x + q_y) = c'(q_y) - \gamma_y \tag{3.7}$$

$$\text{w.r.t. } \eta_z \ : \ \lambda_Z \zeta x = g'(\eta_z)\zeta x - \gamma_{\eta z} \ , \tag{3.8}$$

together with the usual complementary slackness conditions.

The condition (3.5) states that, when $x > 0$, hence $\gamma_x = 0$, the marginal surplus generated by an additional unit of extracted coal converted into η_x units of useful energy, $u'\eta_x$, must be equal to the full marginal cost of extraction, transformation, abatement and taxation, that is the sum of:

- the full marginal cost of coal extraction, $a(X) + \lambda_X$, the unitary extraction cost, $a(X)$, augmented by the mining rent associated to the grade X;
- the marginal cost of transformation of one unit of extracted coal into η_x units of U.E, $b(\eta_x)$;

[16] By choosing $-\lambda_Z$ as the co-state variable of Z, λ_Z appears as the shadow marginal cost of the pollution stock hence as the optimal emission tax rate.

[17] We drop the time argument absent any ambiguity.

- the full marginal cost of the potential pollution ζ of this additional unit of coal input of the C.T.I when the fraction η_z is abated, that is the abatement cost, $g(\eta_z)\zeta$, augmented by the tax on the non-abated pollution, $(1-\eta_z)\zeta\lambda_Z$.

The condition (3.6) states that the marginal surplus generated by a slight increase, $d\eta > 0$, of the conversion rate of the extracted coal into U.E, $u'd\eta x$ must be equal to its marginal cost, $b'(\eta_x)d\eta x$. Note that when coal is exploited (3.6) reads more simply:

$$u'(\eta_x x + q_y) = b'(\eta_x) \implies p = b'(\eta_x) \ . \tag{3.9}$$

The condition (3.9) is nothing but than the profit maximization condition w.r.t. η_x of the C.T.I when the coal input price is equal to $a(X) + \lambda_X$, and the emission tax rate is equal to λ_Z, given an abatement rate η_z. The C.T.I profit reads:

$$\pi_c = \{p\eta_x - [a(X) + \lambda_X + b(\eta_x) + \zeta(g(\eta_z) + (1-\eta_z)\lambda_Z)]\}\, x \ ,$$

hence:

$$\frac{\partial \pi_c}{\partial \eta_x} = 0 \implies p = b'(\eta_x) \ .$$

Next substituting $b'(\eta_x)$ for $u' = p$ in (3.5), we obtain:

$$b'(\eta_x) = \frac{1}{\eta_x}\left[a(X) + \lambda_X + b(\eta_x) + \zeta\left(g(\eta_z) + (1-\eta_z)\lambda_Z\right)\right] \ . \tag{3.10}$$

The equation (3.10) is the f.o.c on η_x for the minimization of the average cost $\{a(X) + \lambda_X + b(\eta_x) + \zeta\,[g(\eta_z) + (1-\eta_z)\lambda_Z]\}\,/\eta_x$.[18] As expected, the price is equal to the marginal cost itself equal to the average cost.

The condition (3.7) states that the additional surplus generated by a small increase of the S.U.E production must be equal to its cost when the solar energy is exploited:

$$q_y > 0 \implies u'(\eta_x x + q_y) = p = c'(q_y) \ . \tag{3.11}$$

[18] Differentiating the average cost w.r.t. η_x, the minimization f.o.c results in:

$$\frac{1}{\eta_x}\left\{b'(\eta_x) - \frac{a(X) + \lambda_X + b(\eta_x) + \zeta\,[g(\eta_z) + (1-\eta_z)\lambda_Z)]}{\eta_x}\right\} = 0 \ ,$$

hence (3.10).

This is the f.o.c on q_y for the maximization of the profit of the S.U.E sector, $\pi_y = pq_y - c(q_y)$.

Last (3.8) states that for a given tax rate λ_Z, the marginal cost of abatement must be equal to the marginal cost of emissions, the non-abated potential pollution flow. Provided that $\lambda_Z > g'(0^+)$, then $\eta_z > 0$ and $\gamma_{\eta z} = 0$, hence:

$$\lambda_Z = g'(\eta_z) \iff \frac{\partial \pi_c}{\partial \eta_z} = 0 \ . \tag{3.12}$$

This is also the f.o.c of the maximization of π_c w.r.t. η_z.

3.2 Cost dynamics and unburned coal

When time differentiable, the dynamics of the co-state variables must satisfy the following conditions:

$$\dot{\lambda}_X = \rho\lambda_X - \frac{\partial \mathcal{L}}{\partial X} \quad : \quad \dot{\lambda}_X(t) = \rho\lambda_X(t) + a'\left(X(t)\right)x(t) \tag{3.13}$$

$$\begin{aligned} \dot{\lambda}_Z = \rho\lambda_Z + \frac{\partial \mathcal{L}}{\partial Z} \quad : \quad & \dot{\lambda}_Z(t) = (\rho+\alpha)\lambda_Z(t) - \nu(t),\ \nu(t) \geq 0,\ \bar{Z} - Z(t) \geq 0, \\ & \text{and } \nu(t)[\bar{Z} - Z(t)] = 0 \ . \end{aligned} \tag{3.14}$$

Furthermore, λ_X, X, λ_Z and Z must satisfy the transversality condition:

$$\lim_{t\uparrow\infty} \left[\lambda_X(t)X(t) + \lambda_Z(t)Z(t)\right] e^{-\rho t} = 0 \ . \tag{3.15}$$

Since $a'(X) < 0$ then the current mining rent, λ_X, may be either increasing or decreasing. However the full marginal cost of the extracted coal, that from now we denote by ω, $\omega(t) \equiv a\left(X(t)\right) + \lambda_X(t)$, increases during the period of coal exploitation. Substituting for $\dot{\lambda}_X(t)$ its expression (3.13) in the time derivative of $\omega(t)$, we get:[19]

$$\dot{\omega}(t) = -a'\left(X(t)\right)x(t) + \rho\lambda_X(t) + a'\left(X(t)\right)x(t) = \rho\lambda_X(t) > 0 \ . \tag{3.16}$$

[19]However note that the discounted mining rent is decreasing:

$$\frac{d}{dt}\left\{\lambda_X(t)e^{-\rho t}\right\} = a'\left(X(t)\right)x(t) < 0 \ .$$

Let (t_1, t_2), $t_1 < t_2$, be a time interval during which the ceiling constraint does not bind. Since $Z(t) < \bar{Z}$, then, from (3.14) with $\nu = 0$, the shadow marginal cost of the CO_2 stock must grow at the proportional rate $\rho + \alpha$:

$$\bar{Z} - Z(t) > 0 \implies \lambda_Z(t) = \lambda_Z(t_1^+)e^{(\rho+\alpha)(t-t_1)} \ , \ t \in (t_1, t_2) \ . \quad (3.17)$$

For any optimal path along which the constraint binds, let us denote by $\underline{t}_Z$ and $\bar{t}_Z$ respectively the first and the last date at which the constraint is tight. We show later (Section 6, Proposition P.5) that the time interval during which the constraint binds is unique, that is the whole interval $[\underline{t}_Z, \bar{t}_Z]$. Furthermore the ceiling regime must end before the time t_X at which the extraction is closed.[20] Since $Z^0 < \bar{Z}$, then $\underline{t}_Z > 0$, and since $\bar{t}_Z < t_X$, then from $\bar{t}_Z$ on, the shadow marginal cost of the CO_2 stock must be nil since the constraint is no more active and forever so that abatement is superfluous:[21]

$$\lambda_Z(t) = \lambda_Z^0 e^{(\rho+\alpha)t} \ , \ t \leq \underline{t}_Z \text{ and } \lambda_Z(t) = 0 \ , \implies \eta_z(t) = 0 \ , \ t \geq \bar{t}_Z \ ,$$
$$\text{where } \lambda_Z^0 \equiv \lambda_Z(0) \ . \quad (3.18)$$

At time t_X, or asymptotically if $t_X = +\infty$, $\lambda_X(t)$ must be nil by the transversality condition (3.15). Because the solar U.E becomes the only available resource, the U.E price is equal to $\tilde{p} = u'(\tilde{q}_y)$ where $\tilde{q}_y$ solves $u'(q_y) = c'(q_y)$. Then using (3.5) and (3.6) with $u' = \tilde{p}$, $\lambda_Z = 0$, $\eta_z = 0$, $g(\eta_z) = 0$, and γ_x and $\gamma_{\eta x}$ both nil, we may determine the unburned stock of coal left underground, denoted by $\tilde{X}$, as the stock which together with the conversion rate denoted by $\tilde{\eta}_x$ solve the following two equations system (3.19):

$$\tilde{p} = \frac{a(X) + b(\eta_x)}{\eta_x} \quad \text{and} \quad \tilde{p} = b'(\eta_x) \ . \quad (3.19)$$

The sequential determination of first $\tilde{\eta}_x$, and next $\tilde{X}$, is illustrated in the below figure 3.

In Figure 3, $\tilde{p}$ is given by the useful energy demand-supply matching in the green economy, thus does not depend upon the costs in the extraction and

[20]Note that $t_X = +\infty$ is not excluded *a priori* since the marginal extraction cost depends on the cumulated extraction, as pointed out by Salant *et al.* (1983).

[21]However we show also (see Proposition P.5) that the abatement of emissions ends within the period at the ceiling, that is before $\bar{t}_Z$.

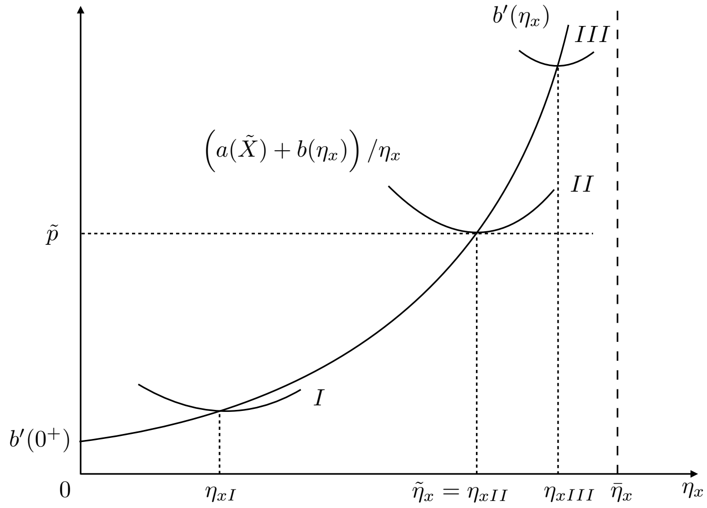

Figure 3: **Determination of the Optimal Terminal Transformation Rate of Coal into Useful Energy, $\tilde{\eta}_x$, and the Unburned Stock of Coal, $\tilde{X}$.**

coal transformation sectors. The optimal values of η_x is first determined as the value $\tilde{\eta}_x$ at which the horizontal $\tilde{p}$ intersects the curve $b'(\eta_x)$ (eq. (3.5)). Next consider the average cost curves I, II and III with respective minima at η_{xI}, η_{xII} and η_{xIII}, corresponding respectively to $X_I > X_{II} = \tilde{X} > X_{III}$, hence $\eta_{xI} < \eta_{xII} = \tilde{\eta}_x < \eta_{xIII}$. The curve II is the only one for which the minimum is located at $\eta_{xII} = \tilde{\eta}_x$, hence the corresponding underground stock $X_{II} = \tilde{X}$ is the only stock for which the minimum of the average cost is equal to $\tilde{p}$ (c.f eq. (3.5)) . The following characteristics of $\tilde{X}$ is worth to be pointed out.

Proposition P. 1 ***Stranded coal.***

Along an optimal path solving the social planner problem $(S.P)$ *within which the ceiling constraint is binding on some time interval, the coal stock* $\tilde{X}$ *to be left unexploited underground does not depend upon the unitary pollution content,* ζ*, nor upon the cap* $\bar{Z}$ *on the pollution stock, and nor upon* α*, the self-dilution rate of the pollution stock.*

4 Dynamics during the unconstrained periods

During the unconstrained periods the dynamics of the flows are easy to determine because either $\lambda_Z = 0$ or $\dot{\lambda}_Z > 0$ when $\lambda_Z > 0$, contrary to what happens within the ceiling regime, during which the sign of $\dot{\lambda}_Z$ is *a priori* ambiguous (c.f. (3.14) with $\nu_Z(t) > 0$).

Consider first a pre-ceiling phase or a post-ceiling phase with abatement and S.U.E production. Since (3.6) holds with $\gamma_{\eta x} = 0$, we may substitute $b'(\eta_x)$ for u' in (3.5) to get:

$$b'(\eta_x(t))\eta_x(t) \;\; = \;\; \omega(t) + b(\eta_x(t)) + \zeta\left[g(\eta_z(t)) + (1-\eta_z(t))\lambda_Z(t)\right] \; . \quad (4.1)$$

Time differentiating the last term of the r.h.s of (4.1) and substituting $g'(\eta_z(t))$ for $\lambda_Z(t)$ (from (3.8) with $\gamma_{\eta z} = 0$), we obtain:

$$\frac{d}{dt}\left[g(\eta_z(t)) + (1-\eta_z(t))\lambda_Z(t)\right] \;\; = \;\; (1-\eta_z(t))\dot{\lambda}_Z(t) \; , \quad (4.2)$$

so that time differentiating (4.1) results in:

$$\dot{\eta}_x(t) \;\; = \;\; \frac{\dot{\omega}(t) + \zeta(1-\eta_z(t))\dot{\lambda}_Z(t)}{b''(\eta_x(t))\eta_x(t)} \; > 0 \; . \quad (4.3)$$

Next, substituting $b'(\eta_x)$ for u' in (3.7) and time differentiating, we obtain:

$$\dot{q}_y(t) \;\; = \;\; \frac{b''(\eta_x(t))\dot{\eta}_x(t)}{c''(q_y(t))} \; > 0 \; . \quad (4.4)$$

Now restore $u'(q(t))$ instead of $b'(\eta_x(t))$ in the l.h.s of (4.1) and once again time differentiate while taking (4.2) into account to determine $\dot{q}(t)$:

$$\dot{q}(t) = \frac{\dot{\omega}(t) + \zeta(1-\eta_z(t))\dot{\lambda}_Z(t)}{u''(q(t))\eta_x(t)} < 0 \iff \dot{p}(t) > 0 \; , \quad (4.5)$$

and from (4.3)-(4.5) we get:

$$\dot{q}_x(t) = \dot{q}(t) - \dot{q}_y(t) < 0 \quad \text{and} \quad \dot{x}(t) = \frac{\dot{q}_x(t)}{\eta_x(t)} - \frac{q_x(t)}{\eta_x^2(t)}\dot{\eta}_x(t) < 0 \; . \quad (4.6)$$

Last from (3.12), (3.17) and (4.6):

$$\dot{\eta}_z(t) = \frac{\dot{\lambda}_Z(t)}{g''(\eta_z(t))} \geq 0 \quad \text{and} \quad \frac{d}{dt}\left\{\zeta(1-\eta_z(t))x(t)\right\} \leq 0 \; , \quad (4.7)$$

with the strict inequality within a pre-ceiling period and an equality within a post-ceiling period. The same qualitative results hold without S.U.E production. To sum up:

Proposition P. 2 ***Dynamics during the unconstrained phases of coal exploitation***

Along an optimal path, during the unconstrained phases:

a. *The transformation rate of the extracted coal into useful energy, $\eta_x(t)$, increases, the extracted coal input of the transformation industry, $x(t)$, decreases and the compound of both effects results into a decrease of the coal useful energy production.*

b. *When the solar energy must be exploited, its production $q_y(t)$ increases once started but this increase does not balance the decrease of the solar useful energy production $q_x(t)$, hence the consumption of useful energy $q(t)$ decreases and its price, $p(t)$, increases.*

c. *When pollution must be abated, the abatement rate, $\eta_z(t)$, increases once abatement starts and since the potential emissions, $\zeta x(t)$, decrease, then the flow of pollution released in the atmosphere, $\zeta(1 - \eta_z(t))x(t)$, decreases.*

d. *The shadow cost of pollution, λ_Z, is either nil or either positive and time increasing.*

5 The ceiling regime

As pointed out in the beginning of the above section 4 the problem during the ceiling regime is that the sign of $\dot{\lambda}_Z(t)$ is not known. Thus the direct method developed for the unconstrained periods consisting in time differentiating the f.o.c's, next proceed to some substitutions to express the command variables as functions of $\dot{\omega}$ and $\dot{\lambda}_Z$ and last deduce from $\dot{\omega} > 0$ and $\dot{\lambda}_Z > / = 0$ the signs of the time derivatives of these command variables, cannot be used any more. However when the abatement option is too costly we can exploit other structural characteristics of the problem.

In what follows we denote by $\bar{x}(\eta_z)$ the maximum quantity of coal that the coal transformation industry may use when the ceiling constraint is binding:

$$Z(t) = \bar{Z} \implies \dot{Z}(t) = (1 - \eta_z(t))\zeta x(t) - \alpha\bar{Z} = 0 \ ,$$

hence:

$$\bar{x}(\eta_z) = \frac{\alpha\bar{Z}}{\zeta(1-\eta_z)} \ , \ \bar{x}'(\eta_z) = \frac{\alpha\bar{Z}}{\zeta(1-\eta_z)^2} > 0 \quad \text{and} \quad \frac{\bar{x}(\eta_z)}{\bar{x}'(\eta_z)} = 1 - \eta_z \ . \tag{5.1}$$

5.1 Phases of the ceiling regime without abatement

Let us assume that the abatement is too costly so that $\eta_z(t) = 0$ hence $x(t) = \bar{x}(0) = \alpha\bar{Z}/\zeta$. Then we may determine η_x and whether the S.U.E must be used or not.

Would the S.U.E not be used, η_x would be determined as the solution of (3.6) with $q_y = 0$, that is:

$$u'\left(\eta_x\bar{x}(0)\right) = b'(\eta_x) \ . \tag{5.2}$$

Let η_x^c denote the solution, then either $u'\left(\eta_x^c\bar{x}(0)\right) \leq c'(0^+)$ and $q_y = 0$ is optimal or $u'\left(\eta_x^c\bar{x}(0)\right) > c'(0^+)$ and it is optimal to exploit the solar option.

In the case $u'\left(\eta_x^c\bar{x}(0)\right) \leq c'(0^+)$, then the optimal U.E production amounts to $q_x^c = \eta_x^c\bar{x}(0)$ and from (3.5) with $\eta_z = 0$, we obtain:

$$u'\left(\eta_x^c\bar{x}(0)\right)\eta_x^c = \omega + b(\eta_x^c) + \zeta\lambda_Z \implies \dot{\lambda}_Z = -\dot{\omega}/\zeta < 0 \ . \tag{5.3}$$

In the opposite case both the transformation rate η_x and the S.U.E production rate q_y are simultaneously determined as the solution of (3.6) and (3.7):

$$u'\left(\eta_x\bar{x}(0) + q_y\right) = b'(\eta_x) \quad \text{and} \quad u'\left(\eta_x\bar{x}(0) + q_y\right) = c'(q_y) \ . \tag{5.4}$$

Let (η_x^c, q_y^c) denote the solution of (5.4) with $\eta_z = 0$:

$$u'\left(\eta_x^c \bar{x}(0) + q_y^c\right)\eta_x^c = \omega + b(\eta_x^c) + \zeta\lambda_Z \implies \dot{\lambda}_Z = -\dot{\omega}/\zeta < 0\ . \quad (5.5)$$

The Proposition P.3 sums up the results.

Proposition P. 3 ***Dynamics of the ceiling regime under uncompetitive abatement***

Within a phase of the ceiling regime during which it is optimal to not abate, hence during which the coal input of the coal transformation industry is constant, $x(t) = \bar{x}(0) = \alpha\bar{Z}/\zeta$:

a. *The transformation rate of coal into useful energy, $\eta_x(t)$, must be kept constant hence also the production of coal useful energy $q_x(t)$.*

b. *The production of solar useful energy, $q_y(t)$ (possibly nil), must also be kept constant, hence the total production of useful energy, $q(t)$.*

c. *Although the atmospheric CO_2 stock is constant at the ceiling level $\bar{Z}$, its shadow marginal cost, $\lambda_Z(t)$, decreases.*

What remains to determine is the value of the constant sum $\omega + \zeta\lambda_Z$. But for that we need additional characteristics of the optimal path and must wait up to the Proposition P.5 in the next Section 6.

5.2 Phases of the ceiling regime with abatement

Now we have to take into account that $\bar{x}(\eta_z)$ is evolving through time. To overcome the difficulty we reformulate the problem that the society faces as a static problem in which only the full marginal cost of coal, ω, is taken as given, like in the preceding formulation of the Section 3, and next express the command variables and λ_Z as functions of ω. Last, since $\dot{\omega} > 0$ even when the ceiling constraint is active (c.f. (3.16)) we determine the signs of the time derivatives of the command variables and λ_Z during the phase of the ceiling regime with abatement.

5.2.1 Alternative formulation of the optimality problem

To determine its U.E consumption the society must simultaneously choose the C.U.E-S.U.E mix, the transformation rate of coal into U.E and the proportion of potential emissions to abate, under a maximum processed coal constraint. Let $(S.S.P)$ be the static social planner problem to solve:[22]

$$\begin{array}{lll}(S.S.P) & \max_{x,\eta_x,\eta_y,\eta_z} & u(\eta_x x + q_y) - \omega x - b(\eta_x)x - c(q_y) - g(\eta_z)\zeta x \\ & s.t. & \bar{x}(\eta_z) - x \geq 0 \ , \ \eta_z \geq 0 \text{ and } q_y \geq 0 \ .\end{array}$$

Let us denote by μ_x the multiplier associated to the coal constraint and by $\mathcal{L}_c$ the Lagrangian of $(S.S.P)$:

$$\begin{aligned}\mathcal{L}_c &= u(\eta_x x + q_y) - \omega x - b(\eta_x)x - c(q_y) - g(\eta_z)\zeta x + \mu_x\left[\bar{x}(\eta_z) - x\right] \\ &\quad + \gamma_{\eta z}\eta_z + \gamma_y q_y \ .\end{aligned}$$

The f.o.c's are:

$$\text{w.r.t. } x \ : \ u'(\eta_x x + q_y)\eta_x = \omega + b(\eta_x) + \zeta g(\eta_z) + \mu_x \tag{5.6}$$
$$\text{w.r.t. } \eta_x \ : \ u'(\eta_x x + q_y)x = b'(\eta_x)x \tag{5.7}$$
$$\text{w.r.t. } q_y \ : \ u'(\eta_x x + q_y) = c'(q_y) - \gamma_y \tag{5.8}$$
$$\text{w.r.t. } \eta_z \ : \ g'(\eta_z)\zeta x = \mu_x \bar{x}'(\eta_z) - \gamma_{\eta z} \ , \tag{5.9}$$

together with the standard complementary slackness conditions.

The condition (5.7) relative to η_x is the same than the condition (3.6) in the initial formulation of the $(S.P.)$ problem for $\gamma_{\eta x} = 0$ and the condition (5.8) relative to q_y is the same than (3.7).

The conditions (5.6) and (5.8) are the new formulations of the initial conditions (3.5) and (3.7) respectively. In (5.6) appears now μ_x instead of $\zeta(1-\eta_z)\lambda_Z$ and in (5.9), $\mu_x\bar{x}'(\eta_z)$ is substituted for $\lambda_Z\zeta x$ in (3.8).

Note that in (5.6)-(5.9), $\bar{x} = \bar{x}(\eta_z)$ since the ceiling constraint is active by assumption, hence $\mu_x > 0$. The multiplier μ_x is the current shadow marginal

[22] Since the ceiling constraint is tight then $x > 0$ and $\eta_x > 0$, and we omit the corresponding non-negativity constraints.

cost of the ceiling constraint expressed here as a constraint over the maximum coal input that the C.T.I may process.

5.2.2 Marginal cost of useful energy

a. Arbitrage between efficiency and abatement in the coal sector

Let us assume a fixed S.U.E production q_y (possibly $q_y = 0$). Then to increase its U.E consumption by increasing its C.U.E production, the society can resort to two policies: An increase of the efficiency rate η_x and/or an increase of the abatement rate η_z to relax the constraint on the maximum use of coal combined with an increase of the coal input.

Marginal cost of q_x through increases of η_x

An increase $dq_x > 0$ requires an increase $d\eta_x = dq_x/\bar{x}(\eta_z)$ of the transformation rate and since the increase of the unitary transformation cost, $b'(\eta_x)d\eta - x$, applies to the whole amount of processed coal, we get:

$$\text{Marginal cost of } q_x = \bar{x}(\eta_z)b'(\eta_x)d\eta_x = b'(\eta_x)d\eta_x \ . \tag{5.10}$$

Marginal cost of q_x through increases of η_z

With increases of the abatement rate η_z the marginal cost of q_x includes two components: The cost of extraction, processing and partial pollution abatement of the additional coal input allowed by the increase of η_z, $\bar{x}(\eta_z)d\eta_z$, and the increase of the cost of the abated pollution, $\eta_z\zeta\bar{x}(\eta_z)$, induced by the increase of the unitary abatement cost, $g'(\eta_z)d\eta_z$.

An increase $dq_x > 0$ requires an increase $d\eta_z = dq_x/\eta_x\bar{x}'(\eta_z)$ of the abatement rate, hence:

- the cost of the additional coal $\bar{x}'(\eta_z)d\eta_z$ to be transformed into U.E amounts to:

$$[\omega + b(\eta_x) + \zeta g(\eta_z)]\,\bar{x}'(\eta_z)d\eta_z \quad = \quad \frac{1}{\eta_x}\,[\omega + b(\eta_x) + \zeta g(\eta_z)]\,dq_x \ ;$$

- the additional abatement cost of the abated pollution amounts to:

$$\zeta \bar{x}(\eta_z) g'(\eta_z) d\eta_z \quad = \quad \frac{1}{\eta_x} \zeta \frac{\bar{x}(\eta_z)}{\bar{x}'(\eta_z)} g'(\eta_z) d\eta_z = \frac{1}{\eta_x} \left[\zeta (1-\eta_z) g'(\eta_z) \right] d\eta_z \ .$$

Summing up the two components, we get:

$$\text{Marginal cost of } q_x = \frac{1}{\eta_x} \left\{ \omega + b(\eta_x) + \zeta \left[g(\eta_z) + (1-\eta_z) g'(\eta_z) \right] \right\} dq_x \ . \tag{5.11}$$

If it is optimal to abate, both marginal costs must be equalized so that, denoting the optimal values by an asterisk, there exists η_z^* such that:[23]

$$b'(\eta_z^*) \quad = \quad \frac{1}{\eta_x^*} \left\{ \omega^* + b(\eta_x^*) + \zeta \left[g(\eta_z^*) + (1-\eta_z^*) g'(\eta_z^*) \right] \right\} \ . \tag{5.12}$$

b. Arbitrage between coal and solar energies

The marginal cost of solar energy is equal to $c'(q_y)$, hence:

- either the solar energy is too costly, that is:

$$b'(\eta_x^*) \quad \leq \quad c'(0^+) \ , \tag{5.13}$$

- or the solar energy is competitive, that is there exists q_y^* such that:

$$b'(\eta_x^*) \quad = \quad c'(q_y^*) \ . \tag{5.14}$$

[23]Note that $\eta_z^* > 0$ implies $\gamma_{\eta z} = 0$, hence (5.9) reduces to $g' = \mu_x$, and $u' = b'$ by (5.7), hence substituting g' for μ_x and b' for u' in (5.6), we obtain (5.12).

5.2.3 Dynamics at the ceiling with abatement

Assuming that it is optimal to abate and that the solar sector is competitive while the ceiling constraint is binding, then (5.6)-(5.9) may be rewritten as:[24]

$$\begin{aligned} b'(\eta_x)\eta_x - b(\eta_x) - \zeta g(\eta_z) - \mu_x &= 0 & (5.15)\\ u'\left(\eta_x \frac{\alpha\bar{Z}}{\zeta(1-\eta_z)} + q_y\right) - b'(\eta_x) &= 0 & (5.16)\\ b'(\eta_x) - c'(q_y) &= 0 & (5.17)\\ \zeta(1-\eta_z)g'(\eta_z) - \mu_x &= 0\ . & (5.18) \end{aligned}$$

Differentiating, and after some substitutions detailed in Appendix A.1, we get the following reduced system in $d\eta_x$, $d\eta_z$ and $d\omega$:

$$\begin{bmatrix} b''\eta_x & -\zeta(1-\eta_z)g'' \\ \dfrac{\alpha\bar{Z}}{\zeta(1-\eta_z)} + \dfrac{b''}{c''} - \dfrac{b''}{u''} & \dfrac{\eta_x\alpha\bar{Z}}{\zeta(1-\eta_z)^2} \end{bmatrix} \begin{bmatrix} d\eta_x \\ d\eta_z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\omega, \tag{5.19}$$

hence, after simple calculations:

$$\frac{d\eta_x}{d\omega} > 0 \text{ and } \frac{d\eta_z}{d\omega} < 0 \implies \frac{dx}{d\omega} = \frac{d\bar{x}}{d\eta_z}\frac{d\eta_z}{d\omega} < 0\ . \tag{5.20}$$

From (5.16): $u''dq = b''d\eta_x$, (5.17): $b''d\eta_x = c''d\eta_y$, and (5.20) we get:

$$\frac{dq}{d\omega} = \frac{b''}{u''}\frac{d\eta_x}{d\omega} < 0\ ,\ \frac{dq_y}{d\omega} = \frac{b''}{c''}\frac{d\eta_x}{d\omega} > 0 \text{ and } \frac{dq_x}{d\omega} = \frac{dq}{d\omega} - \frac{dq_y}{d\omega} < 0\ . \tag{5.21}$$

Last from (3.8): $\lambda_Z = g'(\eta_z)$, and $d\eta_z/d\omega < 0$, we obtain:

$$\frac{d\lambda_Z}{d\omega} = g''\frac{d\eta_z}{d\omega} < 0\ . \tag{5.22}$$

It is easy to check that similar results hold for all the variables, q_y excepted, when only coal is exploited. To conclude, since $\dot{\omega} > 0$:

[24] (5.15)-(5.17) are obtained from (5.6)-(5.8) by substituting b' for u' (thank to (5.7)) together with $x = \bar{x}(\eta_z) = \alpha\bar{Z}/\zeta(1-\eta_z)$ in the arguments of u'. Next (5.18) is obtained from (5.9) noting that $x = \bar{x}(\eta_z)$ and $\bar{x}(\eta_z)/\bar{x}'(\eta_z) = 1 - \eta_z$ (see (5.1)).

Proposition P. 4 ***Dynamics of the ceiling regime under competitive abatement***

Within a phase of the ceiling regime during which it is optimal to abate, hence during which the coal input of the coal transformation industry amounts to $x(t) = \bar{x}(\eta_z(t))$, $\eta_z(t) > 0$*:*

a. *The transformation rate of coal into useful energy,* $\eta_x(t)$*, must increase while the abatement rate,* $\eta_z(t)$*, must decrease inducing a decrease of the quantity of coal processed by the coal transformation industry,* $\bar{x}(\eta_z(t))$*. On the whole this second effect dominates the first one and the production of coal useful energy,* $q_x(t)$*, decreases.*

b. *When the solar energy sector is competitive, its production rate,* $q_y(t)$*, must increase, but this increase is smaller than the decrease of the coal useful energy production,* $q_x(t)$*, so that the consumption of useful energy,* $q(t)$*, decreases.*

c. *Although the atmospheric pollution is constant,* $Z(t) = \bar{Z}$*, its shadow marginal cost,* $\lambda_Z(t)$*, decreases.*

6 Optimal paths

Combining the results of the Propositions P.1-P.4 together with the continuity of $\omega(t)$ and $\lambda_Z(t)$ we conclude that all the optimal paths along which abating the potential emissions is required have the following common structure.[25]

Proposition P. 5 ***Structural characteristics of the optimal paths with abatement***

Along any path:

[25]Under A.1-A.6 both $\lambda_X(t)$ and $\lambda_Z(t)$ are continuous according to Seierstad and Sydsæter (1987, Th 16, p 244), hence also all the paths $x(t)$, $q_x(t)$, $q_y(t)$, $\eta_z(t)$ on $[0, \infty)$ and $\eta_x(t)$ on $[0, t_X)$.

a. *If it is optimal to abate during some time interval* $[\underline{t}_a, \bar{t}_a)$*, then the ceiling constraint must bind during an interval* $[\underline{t}_Z, \bar{t}_Z)$*.*

Assuming that the ceiling constraint binds and that it is optimal to abate:

b. *It is never optimal to start, next stop and last restart pollution abatement. Abatement must be carried on during a single time interval* $[\underline{t}_a, \bar{t}_a)$*,* $\underline{t}_a < \bar{t}_a$*.*

c. *The ceiling constraint binds during a single time interval* $[\underline{t}_Z, \bar{t}_Z)$*,* $0 \leq \underline{t}_Z < \bar{t}_Z$*.*

d. *Abatement must be started before the beginning of the ceiling period and must be closed within the period:* $0 \leq \underline{t}_a < \underline{t}_Z < \bar{t}_a < \bar{t}_Z$*.*

e. *There exists a post-ceiling period* $[\bar{t}_Z, t_X)$*,* $\bar{t}_Z < t_X$ *(possibly* $t_X = +\infty$*), during which coal exploitation must go on before the advent of the green regime of exclusive solar energy consumption.*

The proofs are given in Appendix A.2.

The only structural differences amongst the possible optimal paths lie:

- in the time $\underline{t}_a$ at which the abatement begins, either immediately at time $t = 0$, or later within the pre-ceiling period $[0, \underline{t}_Z)$, according to the abatement costs are low or high;

- and in the time t_y at which begins the exploitation of the solar energy. According to its costs be low or high, the solar energy production must be started either before $\bar{t}_a$, the end of the abatement phase (possibly $t_y = 0$), or after $\bar{t}_Z$ the end of the ceiling regime and then clearly before t_X, but never within the second sub-period at the ceiling, $[\bar{t}_a, \bar{t}_Z)$, during which all the command variables are constant, including the solar energy production, nil or positive.

With the Proposition P.5 we are in position to determine the values of of $\omega(\bar{t}_Z)$ and $\lambda_Z(\bar{t}_Z)$, hence the constant value of $\omega(t) + \zeta\lambda_Z(t)$ during the no-abatement phase of the ceiling regime. During this phase of the ceiling regime, $x(t) = \bar{x}(0)$. Immediately before the end of the preceding phase also at the ceiling, at time $\bar{t}_a^-$, $\lambda_Z(\bar{t}_a^-) = g'(0^+)$ by (3.8). Thus given that $\eta_x(t) = \eta_x^c$ and $q_y(t) = q_y^c$ (possibly $q_y^c = 0$), $t \in [\bar{t}_a, \bar{t}_Z)$, determined in the sub-section 5.1, then by continuity $\omega(\bar{t}_Z)$ must solve (3.5) with $\eta_z = 0$ and $g(0) = 0$, that is:

$$u'(\eta_x^c \bar{x}(0) + q_y^c)\eta_x^c \quad = \quad \omega(\bar{t}_Z) + b(\eta_x^c) + \zeta g'(0^+) \ , \tag{6.23}$$

and:

$$\omega(t) + \zeta\lambda_Z(t) \quad = \quad \omega(\bar{t}_Z) + \zeta g'(0^+) \ , \quad t \in [\bar{t}_a, \bar{t}_Z] \ . \tag{6.24}$$

An example of optimal scenario is illustrated in the Figures 4-7, in which neither the abatement nor the production of solar energy begin immediately, $0 < \underline{t}_a$ and $0 < t_y$, but the solar energy production begins before the abatement, hence also before the ceiling regime: $0 < t_y < \underline{t}_a < \underline{t}_Z$.

The U.E price path is illustrated in Figure 4 together with the shadow marginal cost of pollution.

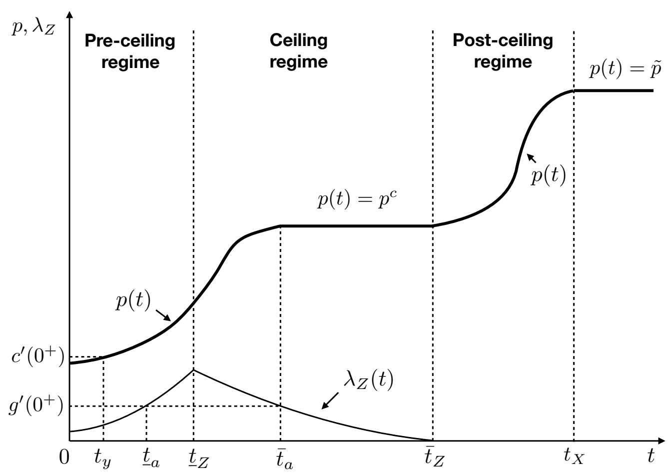

Figure 4: **Optimal Paths of Useful Energy Price and Shadow Marginal Cost of the Pollution Stock. Case $0 < t_y < \underline{t}_a$ and $t_X < \infty$.**

The U.E price increases till the time t_X at which ends the coal exploitation, except for the second sub-period at the ceiling without abatement, $(\bar{t}_a, \bar{t}_Z)$, during which it is constant and denoted by p^c in the figure. The shadow marginal cost of pollution increases from an initially positive level, $\lambda_Z(0) > 0$, up to a peak $\lambda_Z(\underline{t}_Z) > g'(0^+)$, at the beginning of the ceiling period and next decreases during the ceiling period down to 0 at the end of the period.

The production paths are illustrated in Figure 5.

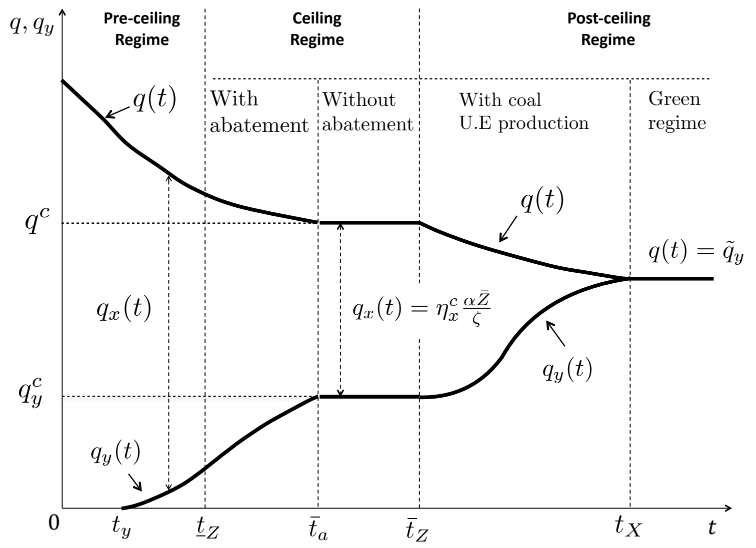

Figure 5: **Useful Energy Production Rates. Case** $0 < t_y < \underline{t}_a$ **and** $t_X < \infty$**.**

Except for the second sub-period at the ceiling, up to the time t_X, the beginning of the green energy regime, the production of coal U.E declines. The consumption of the coal input by the C.T.I decreases because its price increases and compensating improvements of the transformation rate η_x are not sufficient to balance the input cutting down because higher transformation rates are also more costly. Once started the production of solar U.E increases, except during the no-abatement episode at the ceiling, but not sufficiently to wholly compensate for the coal U.E production fall, hence the

U.E consumption decreases down to $\tilde{q}$ in the green regime.

The paths of potential, abated and released pollution flows are illustrated in Figure 6.

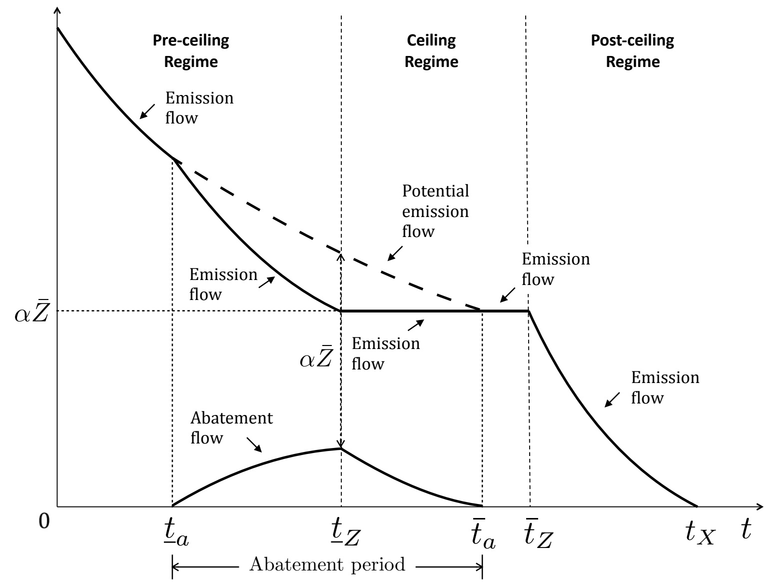

Figure 6: **Potential Emissions, Released Emissions and Abatement Flows.**

The path of potential emissions, $\zeta x(t)$, mimics the path of the coal input use, that is decreases up to time t_X except within the no abatement interlude at the ceiling during which it is constant. The released flow, $(1 - \eta_z(t))\zeta x(t)$, is also decreasing except during the whole ceiling regime period. Before the beginning of the abatement phase the released flow is equal to the potential flow which is decreasing. With abatement before the ceiling regime, the potential flow decreases whereas the abatement rate increases, thus the decrease of the released emissions accelerates. Next during the first ceiling sub-period with abatement, the potential flow decreases but the abatement rate also decreases. The result is that the released flow is constant and exactly balances the self-dilution flow $\alpha\bar{Z}$. Within the second sub-period at the ceiling without abatement, the released flow is again equal to the potential flow itself equal to the self-dilution flow.

Before the ceiling period, the released flow is larger than $\alpha\bar{Z}$. If not, since $Z^0 < \bar{Z}$, and the potential flow is time decreasing, the ceiling constraint could never bind. After the ceiling period the emission flow is lower than $\alpha\bar{Z}$ and without abatement, so that $Z(t)$ never comes back to the critical level $\bar{Z}$.

Although the total emissions decrease, the emissions per unit of useful energy follow a non-monotonic path as illustrated in Figure 7.

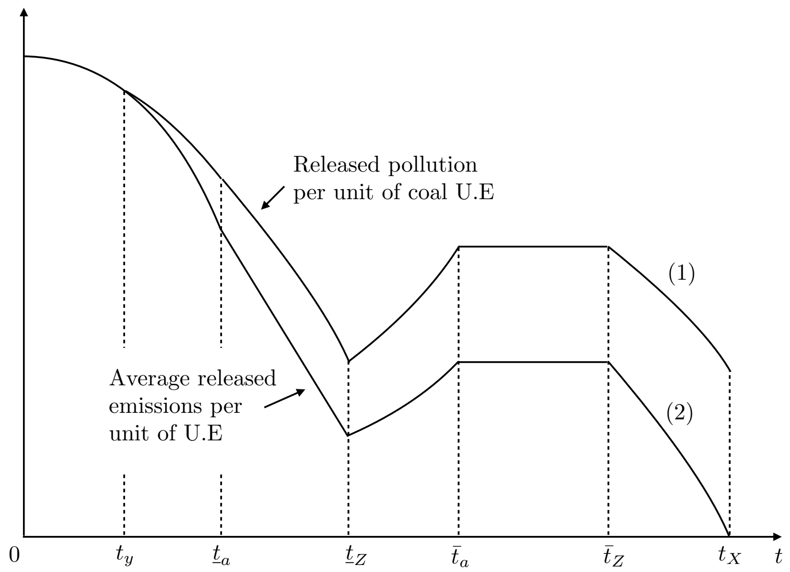

Figure 7: **Released Emissions.**
(1): per unit of coal useful energy
(2): per unit of useful energy

In the Figure 7 the path of emissions per unit of C.U.E is the top one. During the initial phase without abatement, $[0, \underline{t}_a)$, the ratio "released pollution/ C.U.E production" amounts to $\zeta/\eta_x(t)$ and since $\eta_x(t)$ increases then the emission content of coal U.E decreases.

Within the second phase with abatement and before the ceiling regime, $(\underline{t}_a, \underline{t}_Z)$, the unitary emission content of C.U.E is now given by $\zeta(1-\eta_z(t))/\eta_x(t)$, and:

$$\frac{d}{dt}\left(\frac{\zeta(1-\eta_z(t))}{\eta_x(t)}\right) = -\frac{\zeta\left[\eta_x(t)\dot{\eta}_z(t)+(1-\eta_z(t))\dot{\eta}_x(t)\right]}{\eta_x(t)^2} < 0 ,$$

so that the decrease of the released pollution per unit of C.U.E accelerates. But within the next phase, the sub-period at the ceiling with abatement, $(\underline{t}_Z, \bar{t}_a)$, the released flow is constant, equal to $\alpha\bar{Z}$, while $q_x(t)$ decreases resulting in an increasing unitary pollution level. The following phase $(\bar{t}_a, \bar{t}_Z)$ is characterized by a constant production $q_x(t) = q_x^c$ and a constant pollution flow $\alpha\bar{Z}$, hence a constant unitary pollution content of the coal U.E. During the post-ceiling period of coal exploitation, $(\bar{t}_Z, t_X)$, the ratio is again equal to $\zeta/\eta_x(t)$, hence is decreasing down to some positive level.

The profile of the path of released pollution per unit of U.E (C.U.E + S.U.E) is the same because $\dot{q}(t)$ and $\dot{q}_x(t)$ have the same signs during the different phases of the scenario. It is lower once the solar sector is active and the discrepancy increases through time since the proportion of S.U.E in the energy mix increases, except during the second sub-period at the ceiling, and falls down to 0 at t_X.

We may conclude from the non-monotonicity of the optimal polluting content of C.U.E and U.E paths that mandates prescribing a decreasing (possibly by step) unitary released pollution content of the useful energy production are not a first best policy in the present model.

On the whole what could maybe seem surprising is that, as the time is nearer the advent of the ceiling constrained regime, the abatement effort must grow but, once the constraint binds, the effort must be progressively relaxed down to 0 before the end of the ceiling regime. Hence the dominant logic seems to be a sequentially dominant prospective one: As far as the ceiling constraint is not yet hurting, the urgent problem is to prepare a not too difficult constrained episode and, once constrained, the problem is also to prepare the following unconstrained episode during which abatement is no more useful, while maintaining a not too low U.E consumption.

These locally dominant effects translate to the global arbitrage scheme in which the society is embedded. As time goes on, the increasing full marginal cost of the non-renewable resource (extraction + mining rent) becomes the globally dominant effect and progressively the acuteness of the ceiling problem vanishes. However what remains surprising is that, during the second phase of the ceiling regime the extraction rate of the non-renewable resource together with its conversion rate must be kept constant, and also the production of renewable energy, so that the energy mix must also be kept unmodified.

7 Limited storage capacity of the captured pollution

A well known limit of the abatement option is the disposability of favourable sequestration sites, a highly debated issue.[26] To take into account these limited availability of the sequestration sites and the fact that they are more or less easily accessible we extend the model by introducing a specific sequestration cost h distinct from the capture cost g, as in Lafforgue *et al.* (2008-a, 2008-b).

Let us denote by $\bar{S}$ the upper bound of the sum of the storage capacities of the different sites, and by $S(t)$ the not yet used sequestration capacity at time t. Given that the captured pollution flow must be sequestered, then $S(t) = \bar{S} - \zeta \int_0^t \eta_z(\tau)x(\tau)d\tau$. Like for the coal mines let us order the storage sites by decreasing order of accessibility costs and denote by $h(S)$ the unitary sequestration cost of the abated pollution when the part S of the storage capacity has not yet be filled. The function $h(S)$ is assumed to satisfy the following assumption A.7.

Assumption A. 7 $h : [0, \bar{S}) \to \mathbb{R}_+$ *is twice continuously differentiable on* $(0, \bar{S})$*, strictly decreasing,* $h'(S) < 0$*, strictly convex,* $h''(S) > 0$*, with* $h(\bar{S}) > 0$*,* $h'(\bar{S}^-) < 0$*,* $h(0^+) = +\infty$ *and* $h'(0^+) = -\infty$.[27]

[26] See Leung *et al.* (2014).

[27] Clearly the simple version of the model may be seen as a model in which the average sequestration cost is constant, independent from the cumulative sequestration, and the storage capacity is sufficiently large to not be saturated along the optimal path so that the scarcity rent (constant in discounted terms in this case) is nil, that is: $g(\eta_z) = \hat{g}(\eta_z) + \eta_z h$ where $\hat{g}(.)$ satisfies the Assumption A.4 and h is some positive constant, so that the total abatement cost amounts to $[\hat{g}(\eta_z) + \eta_z h]\, \zeta x$. If $\hat{g}(.)$ satisfies A.4 then $g(.)$ also satisfies A.4.

Producing dirty coal U.E requires two natural resource inputs: a nonrenewable one, coal, and a renewable one, the pollution stock which is self-diluting. Producing clean coal U.E by capturing and sequestering the potential pollution requires now the use of two nonrenewable inputs, coal and sequestration capacity. The extra monetary cost induced by producing clean coal U.E rather than dirty one amounts to $(g(\eta_z(t))/\eta_z(t) + h(S(t))$ per unit of abated pollution at time t, to which must be added the rent corresponding to the sequestration capacity.

The social planner problem reads now:[28]

$$
\begin{aligned}
\max_{x,\eta_x,\eta_z,q_y} \quad & \int_0^{\infty} \{u\left(\eta_x(t)x(t) + q_y(t)\right) - a(X(t))x(t) - b(\eta_x(t))x(t) - c(q_y(t)) \\
& \qquad - \left[g(\eta_z(t) + h(S(t))\right]\zeta\eta_z(t)x(t)\}\, e^{-\rho t}dt \qquad (7.1) \\
s.t. \quad & (3.2),\ (3.3),\ (3.4) \text{ and:} \\
& \dot{S}(t) = -\zeta\eta_z(t)x(t)\ , \quad S(0) = \bar{S} > 0 \text{ given}\ . \qquad (7.2)
\end{aligned}
$$

Let us denote by λ_S the co-state variable of S, then the Lagrangian of the problem is:

$$
\begin{aligned}
\mathcal{L} \;=\; & u(\eta_x x + q_y) - a(X)x - b(\eta_x)x - c(q_y) - g(\eta_z)\zeta x - h(S)\zeta\eta_z x - \lambda_X x \\
& -\lambda_Z[(1-\eta_z)\zeta x - \alpha Z] - \lambda_S\zeta\eta_z x + \nu[\bar{Z} - Z] + \gamma_x x + \gamma_{\eta x}\eta_x + \gamma_{\eta z}\eta_z + \gamma_y q_y\ .
\end{aligned}
$$

The f.o.c's (3.6) and (3.7) relative to η_x and q_y respectively still hold, and the f.o.c's relative to x and η_z read:

$$
\begin{aligned}
\text{w.r.t. } x \ :\ & u'(\eta_x x + q_y)\eta_x = a(X) + \lambda_X + b(\eta_x) \\
& +\zeta\left[q(\eta_z + (1-\eta_z)\lambda_Z + \eta_z(h(S) + \lambda_S)\right] - \gamma_x \qquad (7.3) \\
\text{w.r.t. } \eta_z \ :\ & \lambda_Z\zeta x = \left[g'(\eta_z) + h(S) + \lambda_S\right]\zeta x - \gamma_{\eta z}\ . \qquad (7.4)
\end{aligned}
$$

The dynamics of λ_X and λ_Z are unmodified, that is (3.13) and (3.14) respectively still hold, and:

$$
\dot{\lambda}_S = \rho\lambda_S - \frac{\partial\mathcal{L}}{\partial S} \implies \dot{\lambda}_S = \rho\lambda_S + h'(S)\zeta\eta_z x\ . \qquad (7.5)
$$

[28]We omit the capacity constraint $S(t) \geq 0$ which is never active under A.7.

The transversality condition becomes:

$$\lim_{t\uparrow\infty}\left[\lambda_X(t)X(t)+\lambda_S(t)S(t)+\lambda_Z(t)Z(t)\right]e^{-\rho t} \;=\; 0\;. \tag{7.6}$$

Like for the coal mines rent, the current sequestration sites rents, λ_S, may be either increasing or decreasing during the period of pollution abatement since in (7.5) the term $h'(S)\zeta\eta_z x$ is negative. However denoting by θ the full marginal sequestration cost, $\theta \equiv h(S)+\lambda_S$, we get:

$$\begin{aligned}\dot{\theta}(t)=\dot{h}(S(t))+\dot{\lambda}_S(t) \;&=\; -h'(S(t))\zeta\eta_z(t)x(t)+\rho\lambda_S(t)+h'(S(t))\zeta\eta_z(t)x(t)\\ &=\; \rho\lambda_S(t)\;. \end{aligned}\tag{7.7}$$

The marginal sequestration cost increases when the society produces clean coal U.E.[29] Thus we may rewrite (7.4) as:

$$\eta_z>0 \implies \lambda_Z(t)=g'(\eta_z(t))+\theta(t)\;. \tag{7.8}$$

The cost of a slight abatement increase, $\zeta x d\eta_z$, which amounts to $[g'(\eta_z)+\theta]\zeta x d\eta_z$, must be equal to the saving of the shadow marginal cost of the same pollution release, $\lambda_Z\zeta x d\eta_z$.

Also, substituting $b'(\eta_x)$ for $u'(\eta_x x+q_y)$ thank to (3.6) which still holds, we may rewrite (7.3) as:

$$b'(\eta_x) \;=\; \frac{\omega+b(\eta_x)+\zeta\left[g(\eta_z)+\eta_z\theta+(1-\eta_z)\lambda_Z\right]}{\eta_x}\;, \tag{7.9}$$

a condition similar to (3.10) stating that the full average cost of the coal transformation industry must be minimized.

Note that during the last part of the coal exploitation period it is no more necessary to abate the pollution. Hence η_z, $g(\eta_z)$ and λ_Z are all nil, and the conditions determining the stock of unburned coal are the same than the conditions (3.19) *supra*.

Like for the original model without specific sequestration costs we have to take care of the indetermination of the sign of $\dot{\lambda}_Z$ during the period at the ceiling since (3.14) is still commanding the dynamics of λ_Z.

[29] Like for the coal mines, the discounted sequestration rent is decreasing:

$$\frac{d}{dt}\left\{\lambda_S(t)e^{-\rho t}\right\} \;=\; h'(S(t))\zeta\eta_z(t)x(t)e^{-\rho t}<0\;.$$

7.1 Dynamics during the unconstrained periods

Substituting $b'(\eta_x)$ for u' in (7.3) thank to (3.6) gets:

$$b'(\eta_x)\eta_x \quad = \quad \omega + b(\eta_x) + \zeta\left[g(\eta_z) + (1-\eta_z)\lambda_Z + \eta_z\theta\right] \tag{7.10}$$

Time differentiating, we obtain after simplifications:

$$\dot{\eta}_x(t) \quad = \quad \frac{\dot{\omega}(t) + \zeta\left[(1-\eta_z(t))\dot{\lambda}_Z(t) + \eta_z(t)\dot{\theta}(t)\right]}{b''(\eta_x(t))\eta_x(t)} > 0\ , \tag{7.11}$$

since during an unconstrained period, $\dot{\omega}(t)$, $\dot{\lambda}_Z(t)$ and $\dot{\theta}(t)$ are all strictly positive if $\eta_z(t) > 0$, or the two firsts are positive and the third one is nil if $\eta_z(t) = 0$ but the ceiling constraint is still binding, or last the first one is positive and $\lambda_Z(t)$ is nil if $\eta_z(t)$ is itself nil if the ceiling constraint never binds thereafter.

Since the f.o.c. (3.7) relative to $q_y(t)$ does not change, we get the same expression than (4.4) for $\dot{q}_y(t)$ when $q_y(t) > 0$;

$$q_y(t) > 0 \implies \dot{q}_y(t) = \frac{b''(\eta_x(t))\,\dot{\eta}_x(t)}{c''(q_y(t))} > 0\ . \tag{7.12}$$

Next restore $u'(q(t))$ instead of $b(\eta_x(t))$ in the l.h.s. of (7.10) and time differentiate to obtain:

$$\dot{q}(t) = \frac{\dot{\omega}(t) + \zeta\left[(1-\eta_z(t))\dot{\lambda}_Z(t) + \eta_z(t)\dot{\theta}(t)\right]}{u''(q(t))\eta_x(t)} < 0 \iff \dot{p}(t) > 0\ , \tag{7.13}$$

and from (7.11)-(7.13) we get the same relation than (4.6) without sequestration costs:

$$\dot{q}_x(t) = \dot{q}(t) - \dot{q}_y(t) < 0 \quad \text{and} \quad \dot{x}(t) = \frac{\dot{q}_x(t)}{\eta_x(t)} - \frac{\dot{q}_x(t)}{\eta_x^2(t)}\dot{\eta}_x(t) < 0\ . \tag{7.14}$$

Last let us show that the fraction of the potential emissions which is abated increases during such a phase preceding the ceiling regime while the flow of pollution released in the atmosphere decreases. From (7.8), $\dot{\lambda}_Z = (\rho+\alpha)\lambda_Z$ and $\dot{\theta} = \rho\lambda_S$, we get:

$$\begin{aligned}
\eta_z(t) > 0 \Longrightarrow \dot{\eta}_z(t) &= \frac{\dot{\lambda}_Z(t) - \dot{\theta}(t)}{g''(\eta_z(t))} = \frac{\rho(\lambda_Z(t) - \lambda_S(t)) + \alpha\lambda_Z(t)}{g''(\eta_z(t))} \\
&= \frac{\rho\left[g'(\eta_z(t)) + h(S(t))\right] + \alpha\lambda_Z(t)}{g''(\eta_z(t))} > 0 \; , \qquad (7.15)
\end{aligned}$$

and

$$\frac{d}{dt}\left\{\zeta(1-\eta_z(t))x(t)\right\} = \zeta\left[-\dot{\eta}_z(t)x(t) + (1-\eta_z(t))\dot{x}(t)\right] < 0 \; . \qquad (7.16)$$

Note that (7.15) implies that during the unconstrained period preceding the ceiling, once it is optimal to abate, the instantaneous increase rate of the shadow marginal cost of the atmospheric pollution stock $\dot{\lambda}_Z$ is larger than the instantaneous increase rate of the full sequestration cost $\dot{\theta}$.

To sum up, all the qualitative properties of the dynamics of the unconstrained phases without sequestration costs are preserved when these last costs are distinct of the pure capture costs. Let us show now that it is also the case for the phases of the ceiling regime.

7.2 Dynamics during the ceiling regime

Phase at the ceiling with abatement

For this phase the analogous of the reduced system (5.19) is the following system (7.17) (see Appendix A.3 for the details):

$$\begin{bmatrix} b''\eta_x & -\zeta(1-\eta_z)g'' \\ \dfrac{\alpha\bar{Z}}{\zeta(1-\eta_z)} + \dfrac{b''}{c''} - \dfrac{b''}{u''} & \dfrac{\eta_x\alpha\bar{Z}}{\zeta(1-\eta_z)^2} \end{bmatrix} \begin{bmatrix} d\eta_x \\ d\eta_z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\omega + \begin{bmatrix} \zeta \\ 0 \end{bmatrix} d\theta \; . \qquad (7.17)$$

The matrix of the system (7.17) is the same than the matrix of the system (5.19), hence the derivatives with respect to ω and the signs of the derivatives with respect to ω and θ are the same:[30]

$$\frac{d\eta_x}{d\theta} = \zeta \frac{d\eta_x}{d\omega} > 0 \quad \text{and} \quad \frac{d\eta_z}{d\theta} = \zeta \frac{d\eta_z}{d\omega} < 0 \ . \tag{7.18}$$

Hence:

$$\frac{dx}{d\omega} = \frac{d\bar{x}}{d\eta_z}\frac{d\eta_z}{d\theta} = \zeta \frac{dx}{d\omega} < 0 \ , \tag{7.19}$$

so that:

$$\frac{dq}{d\theta} = \frac{b''}{u''}\frac{dq_x}{d\theta} = \zeta \frac{dq}{d\omega} < 0 \quad , \quad \frac{dq_y}{d\theta} = \frac{b''}{c''}\frac{d\eta_x}{d\theta} = \zeta \frac{d\eta_z}{d\omega} > 0 \ , \tag{7.20}$$

and

$$\frac{dq_x}{d\omega} = \frac{dq}{d\theta} - \frac{dq_y}{d\theta} = \zeta \frac{dq_x}{d\omega} < 0 \ . \tag{7.21}$$

Last from (7.4): $d\lambda_Z = g''(\eta_z)d\eta_z$, and together with (7.17), we get:

$$\frac{d\lambda_Z}{d\theta} = g''\frac{d\eta_z}{d\theta} = \zeta \frac{d\lambda_Z}{d\omega} < 0 \ . \tag{7.22}$$

Since both $\dot{\omega} > 0$ and $\dot{\theta} > 0$ and because the signs of the derivatives with respect to ω and θ are the same we may conclude that all the decision variables moves through time in the same direction in both versions of the model during the phase at the ceiling with abatement.

Phase at the ceiling without abatement

During such a phase (η_x^c, q_y^c), the constant value of $(\eta_x(t), q_y(t))$ is determined like in Sections 5 and 6. The reason is that without abatement the f.o.c's are the same in both versions of the model, since then $\eta_z = 0$, hence $g(0) = 0$, $\lambda_S(t) = 0$ and $\omega(t)+\zeta\lambda_Z(t) = \omega(\bar{t}_a)+\zeta\lambda_Z(\bar{t}_a)$, $t \in [\bar{t}_a, \bar{t}_Z]$. The only difference is that now $\lambda_Z(\bar{t}_Z)$ is equal to (by (7.4) at $\bar{t}_a^-$ and the continuity of λ_Z):

$$\lambda_Z(\bar{t}_a) = g'(0^+) + h(S(\bar{t}_a)) \ , \tag{7.23}$$

[30]However note that the values of the arguments of the functions appearing in both matrix (5.19) and (7.17) are not the same. Thus although the signs of the derivatives $d\eta_x/d\omega$ and $d\eta_z/d\omega$ are the same their values are different.

where $S(\bar{t}_a)$ is the sequestration capacity which is left unfilled when the abatement ends. The sum $\omega(\bar{t}_a) + \zeta\lambda_Z(\bar{t}_a)$, equivalently the sum $\omega(\bar{t}_a) + \zeta[g'(0^+) + h(S(\bar{t}_a))]$ is determined as the solution of (7.24):

$$\begin{aligned} u'\left(\eta_x^c \bar{x}(0) + q_y^c\right) &= \omega(\bar{t}_a) + b(\eta_x^c) + \zeta\lambda_Z(\bar{t}_a) \\ &= \omega(\bar{t}_a) + b(\eta_x^c) + \zeta\left[g'(0^+) + h(S(\bar{t}_a))\right] . \end{aligned} \tag{7.24}$$

To sum up, we may conclude that the Propositions P.1 - P.5 are still valid for this more developed version of the model.

8 Alternative upper bounds of the efficiency, abatement and solar costs

To facilitate the analysis we have assumed that the costs, b, g and c tend to infinity when η_x, η_z, and q_y tend toward their upper bounds, respectively $\bar{\eta}_x$, $\bar{\eta}_z$ and $\bar{q}_y$. Assuming instead that they have finite values, $b(\bar{\eta}_x)$, $g(\bar{\eta}_z)$ and $c(\bar{q}_y)$, together with finite derivatives, $b'(\bar{\eta}_x^-)$, $g'(\bar{\eta}_z^-)$ and $c'(\bar{q}_y^-)$ could be empirically more relevant. Under this alternative assumption, the number of different types of phase explode, according to η_x, η_z and q_y are blockaded or not at their upper bounds.[31] However the multiplicity of phase types does not translate into a too large number of qualitatively different optimal paths because first, most phases have strong common features with the similar phases of the initial model and second, the necessary continuity of the command and co-state variables severely limits the number of phase sequences eligible to optimality.

The key features are:[32]

[31] There exist now 43 phase types: 18 before the ceiling regime, 18 during the ceiling regime, 6 after the ceiling regime before the advent of the green economy and last the green regime. For example, before the ceiling regime η_x may be blockaded or not, η_z may be either nil, or positive and not blockaded, or last blockaded at its maximum $\bar{\eta}_z$, and the same too for the solar energy production, hence $2 \times 3 \times 3 = 18$ different types of phases. There exist also 18 phase types in the ceiling regime, but after the ceiling regime η_z is necessarily nil hence only $2 \times 3 = 6$ kinds of phases.

[32] In what follows we index by the superscript $'b'$ the time at which blockading begins and ends. For abatement, $\underline{t}_a^b$ and $\bar{t}_a^b$ are the times at which the abatement effort begins

First η_x always increases except during two phases of the ceiling regime (see the next key feature) up to be possibly blockaded at $\bar{\eta}_x$ and then stay blockaded up to the end of coal exploitation.

Second, if not blockaded before $\underline{t}_Z$, the beginning of the ceiling regime, η_z is never blockaded because it begins to decrease once at the ceiling as in the preceding scenarios. But if blockaded at $\bar{\eta}_z$ before $\underline{t}_Z$ it stays blockaded up to some time $\bar{t}_a^b$, $\underline{t}_a < \bar{t}_a^b < \bar{t}_a$, at the beginning of the ceiling regime. In this case the ceiling regime includes three successive phases: A first phase $[\underline{t}_Z, \bar{t}_a^b]$ with $\eta_z(t) = \bar{\eta}_z$ and $x(t) = \bar{x}(\bar{\eta}_z)$, a second phase $(\bar{t}_a^b, \bar{t}_a)$ during which $\eta_z(t)$ decreases from its maximum level $\bar{\eta}_x$ down to 0 and as a consequence, $x(t)$ decreases from $\bar{x}(\bar{\eta}_z)$ down to $\bar{x}(0)$, and a last phase $[\bar{t}_a, \bar{t}_Z)$ during which $\eta_z(t) = 0$ and $x(t) = \bar{x}(0)$. The reason is that $\lambda_Z(t)$ increases before $\underline{t}_Z$ even if $\eta_z(t)$ is blockaded before $\underline{t}_Z$ (c.f. (3.14) and (3.17)), hence at time $\underline{t}_Z$: $\lambda_Z(\underline{t}_Z) > g'(\bar{\eta}_z^-)$.[33] Since $\lambda_Z(t)$ is continuous there must exist some time interval $[\underline{t}_Z, \bar{t}_a^b)$, $\bar{t}_a^b < \bar{t}_a$, during which $\lambda_Z(t)$ decreases down to $g'(\bar{\eta}_z^-)$, and after which $\eta_z(t)$ enter the phase of decrease.[34] Like in the last phase $[\bar{t}_a, \bar{t}_Z]$, $\eta_x(t)$ is constant during the first phase $[\underline{t}_Z, \bar{t}_a^b]$ hence also $q_x(t) = \eta_x(t)\bar{x}(\bar{\eta}_z)$.

Third $q_y(t)$ increases once the solar option becomes competitive, except during the initial phase at the ceiling with η_z blockaded if such a phase exists, and the last phase of the ceiling without abatement, phases during which $q_y(t)$ is constant. Since $q_x(t)$ too is constant then $q(t)$ is constant hence the U.E price. Thus the solar U.E may not begin to be competitive during such phases. Last, when blockaded $q_y(t)$ stays blockaded forever including in

and ends to be at its maximum rate: $\eta_z(t) = \bar{\eta}_z$, $\underline{t}_a^b \le t \le \bar{t}_a^b$. For η_x and q_y, t_x^b and t_y^b are the dates at which η_x and q_y begin to be blockaded. We may adopt these simpler notations because once blockaded, η_x and q_y never come back to unconstrained levels.

[33]With two constraints $\bar{\eta}_z - \eta_z \ge 0$ and $\eta_z \ge 0$ the Lagrangian of the problem $(S.P)$ must include two components $\bar{\gamma}_{\eta z}[\bar{\eta}_z - \eta_z]$ and $\underline{\gamma}_{\eta z}\eta_z$ so that the f.o.c (3.8) w.r.t. η_z becomes:

$$\lambda_Z \zeta x = g'(\eta_z)\zeta x + \bar{\gamma}_{\eta z} - \underline{\gamma}_{\eta z} \ , \ \bar{\gamma}_{\eta z} \ge 0 \text{ and } \bar{\gamma}_{\eta z}[\bar{\eta}_z - \eta_z] = 0 \ , \text{ and } \underline{\gamma}_{\eta z} \ge 0 \text{ and } \gamma_{\eta z}\eta_z = 0 \ .$$

Hence when η_z begins to be blockaded before $\underline{t}_Z$, then at $\underline{t}_Z$, $\underline{\gamma}_{\eta}(\underline{t}_Z) = 0$ and:

$$\lambda_Z(\underline{t}_Z)\zeta \bar{x}(\bar{\eta}_z) = g'(\bar{\eta}_z^-)\zeta \bar{x}(\bar{\eta}_z) + \bar{\gamma}_{\eta z}(\underline{t}_Z) \text{ and } \bar{\gamma}_{\eta z}(\underline{t}_Z) > 0 \ .$$

[34]We show in Appendix A.4 that $\lambda_Z(t)$ decreases when $\eta_z(t)$ is blockaded at $\bar{\eta}_z$ in the ceiling regime and that during such a phase $\eta_x(t)$ is constant, blockaded or not, and also $q_y(t)$.

the ultimate green regime.

Fourth during the pre-ceiling and post-ceiling phases the flow of pollution released within the atmosphere is always decreasing even if either η_x, or η_z, or both are blockaded because $x(t)$ decreases.

To conclude, the differences between the present optimal paths and the optimal paths of the initial specification of the model lie:

- in the possible never ending freezes of η_x and q_y at $\bar{\eta}_x$ and $\bar{q}_y$ beginning respectively at some time t_x^b and t_y^b, $t_x^b < t_X$ and $t_y < t_y^b < t_X$;
- in a possible temporary freeze of η_z at $\bar{\eta}_z$ during a time interval $[\underline{t}_a^b, \bar{t}_a^b]$ beginning before the arrival at the ceiling and ending within the ceiling regime: $0 \leq \underline{t}_a \leq \underline{t}_a^b < \underline{t}_Z < \bar{t}_a^b < \bar{t}_a < \bar{t}_Z$;
- in that, except the phase $[\underline{t}_a, \bar{t}_a^b]$ at the ceiling with η_z blockaded and the phase $[\bar{t}_a, \bar{t}_Z]$ at the ceiling without abatement, η_x and q_y may begin to be blockaded at whatever time of the coal exploitation period $[0, t_X)$ according to $b(\bar{\eta}_x)$ and $c(\bar{q}_y)$ are high or low and, for sufficiently low costs, the initial maximum rates, $\eta_x(0) = \bar{\eta}_x$ and $q_y(0) = \bar{q}_y$, may be proved optimal;
- in that the abatement effort may begin to be blockaded at whatever date of the interval $[0, \underline{t}_Z)$, including the initial date $t = 0$, provided that $g(\bar{\eta}_z)$ be sufficiently low.

An example of optimal path along which η_z is blockaded at its maximum $\bar{\eta}_z$ while both η_x and q_y are free, is illustrated in Figure 8. The S.U.E production rate, q_y, (not illustrated) becomes positive only during the post-ceiling period, $(\bar{t}_Z, t_x)$, of coal exploitation.

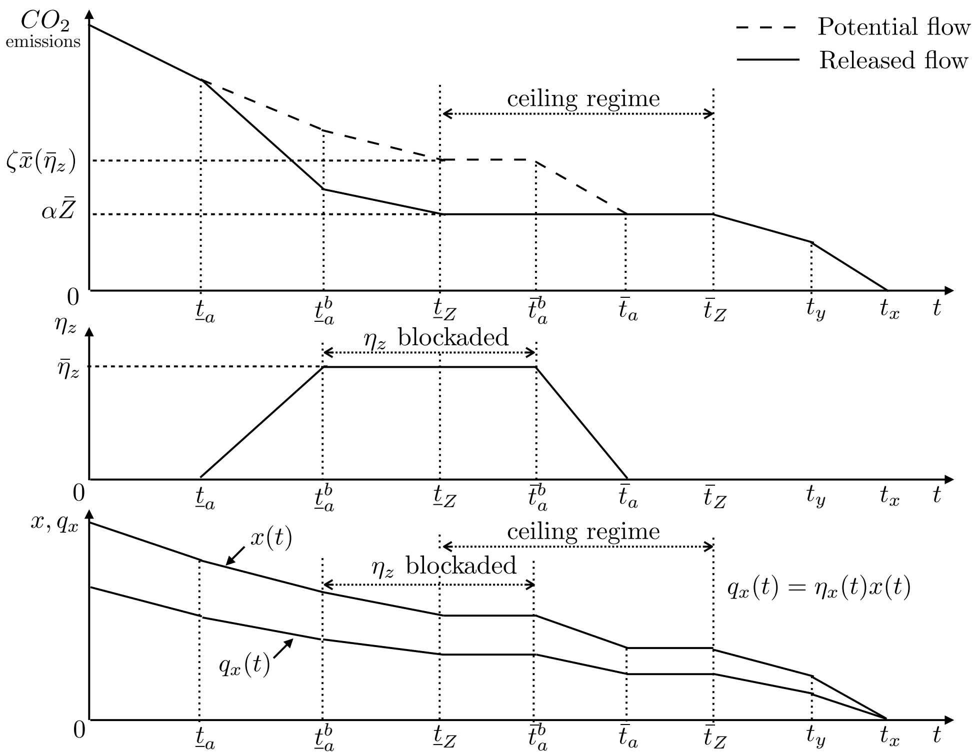

Figure 8: **Optimal Path with η_z Blockaded, η_x and q_y Free.**

9 Concluding remarks

Among the three options commonly considered to mitigate the harmful impacts of fossil energy use on climate change, two of them: fossil energy efficiency improvements and energy transition to clean renewables, appear clearly complement. All along the transition from fossil fuels to a green energy regime, the fossil fuel energy efficiency increases while renewables take an increasing share in the energy mix, with the exception of the last phase of the ceiling regime during which they stay constant.

The history of CCS deployment is more complex. If competitive, the CCS option is used at an increasing rate before the ceiling regime but begins to decline once this regime begins, the pollution abatement rate falling to zero strictly before the economy can escape the carbon constraint. CCS stands as an interim option most valuable when the CO_2 atmospheric concentration approaches the cap and during the beginning of the ceiling regime.

The released flow of emissions decreases before and after the ceiling regime, a time period during which it is maintained at the constant level allowed by the carbon cap constraint. The fall of emissions is amplified by the abatement activity before the ceiling regime. This is not the case in terms of potential pollution per fossil energy units or total energy production units. Although declining before the ceiling regime, the potential carbon pollution per fossil and total energy units rises after the beginning of the ceiling regime, next stabilises at the end of this regime before falling once again after the end of the carbon constrained episode. This is a consequence of the progressive fading out of the emissions abatement efforts during the first part of the ceiling regime.

Our findings, initially obtained in a highly stylised setting are robust to alternative formulations. We have shown that they remain qualitatively valid after taking into account the limited availability of carbon sequestration sites both in size and cost accessibility or if technical limitations apply to the energy efficiency gains or to the abatement rates of emissions. We have assumed a stationary energy demand but an expanding demand would mainly amplify the time evolutions of energy efficiency, energy transition and abatement with the possible elimination of the constant phase before the end of the ceiling regime.

The predictable difficulties to meet the objectives of the Paris agreement suggest that the climate policy mix will combine carbon mitigation and adaptation to climate change. The adaptation option should thus also be included into the description of an optimal climate change management policy. It is currently claimed that the cost gap of CCS could be filled by sufficient learning-by-doing or technical progress in abatement techniques.[35] More generally the innovation literature has largely emphasised the role of technical progress in the transition toward a non-fossil based energy system. We leave these issues for further research.

[35] See Amigues *et al.* (2016) for an analysis of the dynamic implications of learning-by-doing in carbon pollution abatement.

References

Acemoglu D., Aghion P., Bursztyn L. and D. Hemous, (2012), The environment and directed technical change, *American Economic Review*, 102(1), 131-66.

Acemoglu D., Akcigit U., Kerr W. and D. Hanley, (2016). Transition to clean technology, *Journal of Political Economy*, 124(1), 52-104.

Allcott, H., and M. Greenstone, (2012), Is there an energy efficiency gap? *Journal of Economic Perspectives*, 26, 3-28.

American Council for an Energy-Efficient Economy, (2014), *The best value for America's energy Dollar: A national review of the cost of utility energy efficiency programs.*

Amigues J. P. and M. Moreaux, (2016), From primary resources to useful energy: The pollution ceiling efficiency paradox, *Toulouse School of Economic*, W.P. 624.

Amigues J. P., Lafforgue G. and M. Moreaux, (2016), Optimal timing of carbon capture and storage policies under learning-by-doing, *Journal of Environmental Economics and Management*, 78. 23-37.

Budinis S., Crever S., Mac Bowell N., Brandon N. and A. Hawkes, (2016), Can technology unlock 'un-burnable carbon'?, Sustainable Gas Institute, Imperial College, London.

Brown M., (2004), Obstacles to energy efficiency. *Encyclopedia of Energy*, Elsevier.

Chakravorty U., Magné B., and M. Moreaux, (2006), A Hotelling model with a ceiling on the stock of pollution, *Journal of Economic Dynamics and Control*, 30, 2875-2904.

Chakravorty U., Moreaux M., and M. Tidball, (2008). Ordering the extraction of polluting nonrenewable resources, *American Economic Review*, 98, 1128-1144.

Edenhofer O., Bauer N., and E. Kriegler, (2005), The impact of technological change on climate protection and welfare: Insights from the model MIND, *Ecological Economics*, 54, 277-292.

Farzin Y. H., (1996), Optimal pricing of environmental and natural resource use with stock externalities, *Journal of Public Economics*, vol. 62(1-2), 31-57.

Farzin Y. H. and O. Tahvonen, (1996), Global carbon cycle and the optimal time path of a carbon tax, *Oxford Economic Papers*, 48(4), 515-536.

Gerlagh R., (2006), ITC in a global growth-climate model with CCS. The value of induced technical change for climate stabilization, *Energy Journal*, Special Issue, 55-72.

Gerlagh R., and B. C. van der Zwaan, (2006), Options and instruments for a deep cut in CO2 emissions: carbon capture or renewable, taxes or subsidies?, *Energy Journal*, 27, 25-48.

Gillingham K.,Newell R. G. and K. Palmer, (2009). Energy efficiency economics and policy, *Annual Review of Resource Economics*, 1, 597-620.

Grimaud A., Lafforgue G., and B. Magné, (2011), Climate change mitigation options and directed technical change: A decentralized equilibrium analysis, *Resource and Energy Economics*, 33, 938-962.

Golosov M., Hassler J., Krusell P. and A. Tsyvinski, (2014), Optimal tax on fossil fuel in general equilibrium, *Econometrica*, 82(1), 41-88.

Hassler J., Krusell P. and C. Olovsson, (2012), Energy-saving technical change, WP NBER 18456.

Heal, G., (1976), The relationship between price and extraction cost for a resource with a backstop technology, *Bell Journal of Economics*, 7(2), 371-

378.

IEA/ OECD. (2013). *Energy efficiency market report.*

Lafforgue G., Magné B., and M. Moreaux, (2008-a), Energy substitutions, climate change and carbon sinks, *Ecological Economics*, 67, 589-597.

Lafforgue G., Magné B., and M. Moreaux, (2008-b), The optimal sequestration policy with a ceiling on the stock of carbon in the atmosphere, in: Guesnerie, R., Tulkens, H. (eds), *The Design of Climate Policy.* The MIT Press, Boston, 273-304.

Lasserre, P. (1985-a), Capacity choice by mines, *Canadian Journal of Economics*, 18(4), 831-842.

Lasserre, P (1985-b), Exhaustible resource extraction with capital, in A.D. Scott (ed), *Progress in Natural Resource Economics*, Oxford University Press, Oxford.

Lehvari D. and N. Liviatan, (1977). Notes on Hotelling's economics of exhaustible resources, *Canadian Journal of Economics*, 10, 177-192.

Leung D., Caramanna G. and M. Maroto-Valer, (2014), An overview of current status of carbon dioxide capture and storage technologies, *Renewable and Sustainable Energy Reviews*, 39, 426-443.

Manne A. and R. Richels, (2004), The impact of learning-by-doing on the timing and costs of CO2 abatement, *Energy Economics*, 26, 603-619.

Mason C. F. and N. Wilmot, (2015), Modeling damages in climate policy models: Temperature-based or carbon-based?, CESifo WP 5287.

McFarland J.R., Herzog H.J., and J. M. Reilly, (2003), Economic modeling of the global adoption of carbon capture and sequestration technologies. In: Gale, J. and Y. Kaya (eds.), *Proceedings of the sixth international conference on greenhouse gas control technologies*, Vol.2. Elsevier Science, Oxford, 1083-1089.

Moreaux M. and C. Withagen, (2015), Optimal abatement of carbon emissions flows, *Journal of Environmental Economics and Management*, 74, 55-70.

Nordhaus W., (2008), *A question of balance: Weighting the options on global warming policies*, Yale University Press, New Haven, CT.

Salant S., Eswaran M. and T. Lewis, (1983). The length of optimal extraction programs when depletion affects extraction costs, *Journal of Economic Theory*, 31, 364-374.

Schafer A., (2005), Structural change in energy use, *Energy Policy*, 33, 429-437.

Smulders S., Bretschger L. and H. Egli, (2011), Economic growth and the diffusion of clean technologies: Explaining environmental Kuznet's curves, *Environment and Resource Economics*, 49(1), 79-99.

Sorrell S., (2014), Energy substitution, technical change and rebound effects, *Energies*, 7, 2850-2873.

Stern N., (2007), *The economics of climate change: The Stern review*, Cambridge University Press, Cambridge UK.

Tahvonen O. and C. Withagen, (1996), Optimality of irreversible pollution accumulation, *Journal of Economic Dynamics and Control*, vol. 20(9-10), 1775-1795.

Toman M.A. and C. Withagen, (2000), Accumulative pollution, clean technology and policy design, *Resource and Energy Economics*, 22, 367-384.

Van der Meijden G. and S. Smulders, (2018), Technological change during the energy transition, *Macroeconomic Dynamics*, 22(04), 805-836.

Van der Ploeg F. and C. Withagen, (2014), Growth, renewables and the optimal carbon tax, *International Economic Review*, 55(2), 283-311.

Weitzman M., (2010), What is the 'damages function' for global warming - and what differences might it makes, *Climate Change Economics*, 1(01), 57-69.

Withagen C., (1994), Pollution and exhaustibility of fossil fuels, *Resource and Energy Economics*, 16(3), 235-242.

Appendix

A.1 Building up the reduced system (5.19)

In (5.6) let us substitute $\zeta(1-\eta_z)g'(\eta_z)$ for μ_x (from (5.6) and (5.9)) and b' for u' (from (5.7)) to get:

$$b'(\eta_x)\eta_x - b(\eta_x) - \zeta\left[g(\eta_z) + (1-\eta_z)g'(\eta_z)\right] \;=\; \omega \;.$$

Differentiating results into:

$$b''(\eta_x)\eta_x d\eta_x - \zeta(1-\eta_z)g''(\eta_z)d\eta_z \;=\; d\omega \;. \tag{A.1.1}$$

Next differentiating (5.7) and substituting $(b''/c'')d\eta_x$ for dq_y (from (5.8)) we obtain:

$$\left[\frac{\alpha\bar{Z}}{\zeta(1-\eta_z)} + \frac{b''}{c''} - \frac{b''}{u''}\right] d\eta_x + \frac{\eta_x\alpha\bar{Z}}{\zeta(1-\eta_z)^2}d\eta_z \;=\; 0 \;, \tag{A.1.2}$$

hence, together with (A.1.1), (5.19) in matrix form.

A.2 Proof of the Proposition P.5

a. Abatement is required only if the ceiling constraint binds

Assume that pollution is abated during some time interval, $(\underline{t}_a, \bar{t}_a)$ but that $\sup\{Z(t), t > \underline{t}_a\} < \bar{Z}$. Then it is possible to save costs by reducing slightly the abatement while preserving the inequality, hence the non-optimality of the path.

b. Unicity of the abatement period

Let us assume that the pollution is abated during two separate intervals $\Delta' = (\underline{t}'_a, \bar{t}'_a)$ and $\Delta'' = (\underline{t}''_a, \bar{t}''_a)$, $0 \le \underline{t}'_a < \bar{t}'_a < \underline{t}''_a < \bar{t}''_a$:

$$\eta_z(t) \begin{cases} > 0 \quad , \quad t \in \Delta' \cup \Delta'' \\ \\ = 0 \quad , \quad t \notin \Delta' \cup \Delta'' \end{cases}$$

and

$$\lambda_Z(t) \begin{cases} = g'(\eta_z(t)) > g'(0^+) \quad , \quad t \in \Delta' \cup \Delta'' \\ \\ \qquad\qquad\quad \le g'(0^+) \quad , \quad t \notin \Delta' \cup \Delta'' \ . \end{cases} \tag{A.2.1}$$

b.1 Assume first that the ceiling constraint does not bind within the interval $[\bar{t}'_a, \underline{t}''_a]$. Then, by (3.14), the continuity of $\lambda_Z(t)$ and $\lambda_Z(\bar{t}'_a) > g'(0^+)$:

$$\lambda_Z(t) \ = \ \lambda_Z(\bar{t}'_a)e^{(\rho+\alpha)(t-\bar{t}'_a)} > g'(0^+) \ , \ t \in (\bar{t}'_a, \underline{t}''_a) \ ,$$

so that it would be optimal to abate within the interval, a contradiction.

b.2 Assume now that the ceiling constraint binds within an interval $(\underline{t}'_Z, \bar{t}'_Z)$ beginning before $\underline{t}''_a$ and consider the two cases $\bar{t}'_Z > \underline{t}''_a$ and $\bar{t}'_Z \le \underline{t}''_a$.

b.2.1 Case: $\bar{t}'_a > \underline{t}''_a$

In this case there would exist a first interval $(\underline{t}''_a - \delta, \underline{t}''_a)$, $\delta > 0$ and sufficiently small, at the ceiling without abatement followed by a second interval $(\underline{t}''_a, \underline{t}''_a + \delta)$ at the ceiling with abatement. During the first interval, $x(t) = \bar{x}(0)$ and during the second one, $x(t) = \bar{x}(\eta_z(t)) > \bar{x}(0)$, and is decreasing by the Proposition P.4-a (since $\eta_z(t)$ decreases then $\bar{x}(\eta_z(t))$ decreases during this second time interval). This is possible if and only if $x(t)$ jumps up at time $\underline{t}''_a$, but $x(t)$ must be time continuous.[36]

b.2.2 Case: $\bar{t}'_a \le \underline{t}''_a$

[36]From (3.6) and (3.7), $\eta_x(t)$ and $q_y(t)$ have to be time continuous along an optimal path since $p(t)$ is time continuous, together with $q(t)$. Thus $q_x(t) = q(t) - q_y(t)$ is time continuous and $x(t) = q_x(t)/\eta_x(t)$ is also time continuous.

Note that abatement within $(\underline{t}''_a, \bar{t}''_a)$ implies that there exists an interval $(\underline{t}''_a, \bar{t}''_Z)$, $\underline{t}''_a \leq \underline{t}''_Z < \bar{t}''_Z$, at the ceiling. If not it would not be optimal to abate during the interval $(\underline{t}''_a, \bar{t}''_a)$. Furthermore if $\underline{t}''_Z = \underline{t}''_a$, we are brought back to the preceding case in which there exists a phase at the ceiling, the first part of which before $\underline{t}''_a$ is without abatement and the second and following phase, after $\underline{t}''_a$, is a phase with abatement. Hence we have to consider only the case $\underline{t}''_a < \underline{t}''_Z$.

Thus let us assume that $\underline{t}''_a < \underline{t}''_Z$ and show that $x(\underline{t}''^{+}_a) > \bar{x}(0)$. If first $\underline{t}''_Z < \bar{t}''_a$, then there exists a phase $(\underline{t}''_a, \underline{t}''_Z)$ unconstrained with abatement followed by a phase $(\underline{t}''_Z, \underline{t}''_Z + \delta)$, $\delta > 0$ and $\underline{t}''_Z + \delta < \bar{t}''_a$, constrained with abatement. Within the whole interval $(\underline{t}''_a, \underline{t}''_Z + \delta)$, $x(t)$ is decreasing (by Propositions P.2-a for the first sub-interval $(\underline{t}''_a, \underline{t}''_Z)$ and by Proposition P.4-a for the second sub-interval $(\underline{t}''_Z, \underline{t}''_Z + \delta)$) and because $x(t) = \bar{x}(\eta_z(t)) > \bar{x}(0)$, $t \in (\underline{t}''_Z, \underline{t}''_Z + \delta)$, since $\eta_z(t) > 0$, then $x(\underline{t}''^{+}_a) > \bar{x}(0)$. The same kind of argument holds if $\underline{t}''_Z \geq \bar{t}''_a$. If $\underline{t}''_Z \geq \bar{t}''_a$, then $(\underline{t}''_Z, \underline{t}''_Z + \delta)$, $\delta > 0$ and sufficiently small, is a phase at the ceiling without abatement during which $x(t) = \bar{x}(0)$ preceded by either only one phase $(\underline{t}''_a, \bar{t}''_Z)$ during which $x(t)$ decreases, when $\bar{t}''_a = \underline{t}''_Z$, or by two phases $(\underline{t}''_a, \bar{t}''_Z)$ and $(\bar{t}''_a, \underline{t}''_Z)$ respectively during which $x(t)$ decreases, when $\underline{t}''_Z > \bar{t}''_a$.

Let us come back to what happens before $\underline{t}''_a$ taking care that $x(\underline{t}''^{+}_a) > \bar{x}(0)$. First, if $\bar{t}'_Z = \underline{t}''_a$, then $(\bar{t}'_Z - \delta, \bar{t}'_Z)$, $\delta > 0$ and sufficiently small, is a phase at the ceiling without abatement during which $x(t) = \bar{x}(0)$, so that $x(t)$ should jump up at time $t = \underline{t}''_a$. Second, if $\bar{t}'_a < \underline{t}''_a$, then there exist two phases $(\bar{t}'_Z - \delta, \bar{t}'_Z)$ and $(\bar{t}'_Z, \underline{t}''_a)$, the first one during which $x(t) = \bar{x}(0)$ and the second one during which $x(t)$ decreases and by continuity of $x(t)$, we get $x(\underline{t}''^{-}_a) < \bar{x}(0)$. Again $x(t)$ should jump up at time $t = \underline{t}''_a$.

c. Unicity of the ceiling period

Assume that there exist two successive disconnected periods at the ceiling, $[\underline{t}'_Z, \bar{t}'_Z]$ and $[\underline{t}''_Z, \bar{t}''_Z]$, $\underline{t}'_Z < \bar{t}'_Z < \underline{t}''_Z < \bar{t}''_Z$, with an unconstrained period in between: $Z(t) < \bar{Z}$, $t \in (\bar{t}'_Z, \underline{t}''_Z)$.

c.1 Assume first that there is no abatement during the intermediate period $(\bar{t}'_Z, \underline{t}''_Z)$. Then this period is an unconstrained period without abatement during which $x(t)$ decreases according to the Proposition P.2-a, hence also the emission flow $\zeta x(t)$. Note also that at $t = \bar{t}_Z^{'-}$, the abatement is nil. If not $\lambda_Z(t)$ would jump at time $t = \bar{t}'_Z$ from some value larger than $g'(0^+)$ down to some value at most equal to $g'(0^+)$. This implies that $x(\bar{t}'_Z) = \bar{x}(0)$. Thus the emission flow is strictly lower than $\alpha\bar{Z}/\zeta$ during the intermediate time interval so that $Z(t)$, starting from $\bar{Z}$ at time $\bar{t}'_Z$ is strictly lower than $\bar{Z}$ during the whole intermediate interval. We conclude that $Z(t)$ should be discontinuous at $\underline{t}''_Z$ jumping from some level lower than $\bar{Z}$ up to $\bar{Z}$, an impossibility.

c.2 Assume now that there is abatement during the intermediate period $(\bar{t}'_Z, \underline{t}''_Z)$ while taking care that the abatement period is a unique time interval $(\underline{t}_Z, \bar{t}_a)$, according to the above Proposition P.5-b. Let us show that the result is either an impossibility or a contradiction whatever the location of the interval $(\underline{t}_a, \bar{t}_a)$ with respect to the intervals $[\underline{t}'_Z, \bar{t}'_Z]$ and $[\underline{t}''_Z, \bar{t}''_Z]$. Let us consider the two cases $\bar{t}_a \leq \underline{t}''_Z$ and $\underline{t}''_Z < \bar{t}_a$.

c.2.1 Case : $\bar{t}_a \leq \underline{t}''_Z$

Consider first the case $\bar{t}'_Z \leq \underline{t}_a$, implying that $[\underline{t}''_Z, \bar{t}'_Z]$ is a period at the ceiling without abatement so that $x(\bar{t}'_Z) = \bar{x}(0) = \alpha\bar{Z}/\zeta$ sustaining the emission flow $\alpha\bar{Z}$. Then either $\underline{t}_a = \bar{t}'_Z$ or $\bar{t}'_Z < \underline{t}_a$. If $\underline{t}_a = \bar{t}'_Z$, then the interval $(\bar{t}'_Z, \underline{t}''_Z)$ is an unconstrained phase with abatement during which the emissions decrease so that $Z(t) < \bar{Z}$, $t \in (\bar{t}'_Z, \underline{t}''_Z)$, with $Z(\underline{t}_Z^{''-}) < \bar{Z}$ and at time $\underline{t}''_Z$, $Z(t)$ should jump up, an impossibility. If $\bar{t}'_Z < \underline{t}_a$, then there exist two successive intervals $(\bar{t}'_Z, \underline{t}_a)$ and $(\underline{t}_a, \underline{t}''_Z)$, the first one unconstrained without abatement and the second one unconstrained with abatement during which the emissions decrease and again $Z(\underline{t}_Z^{''-}) < \bar{Z}$.

In the case $\underline{t}_a < \bar{t}'_Z$, then $Z(\bar{t}'_Z) = \bar{Z}$ while $Z(\bar{t}_Z^{'+}) < \bar{Z}$. This is possible if and only if the emissions at time $\bar{t}_Z^{'+}$ are lower than $\alpha\bar{Z}$, that is $\zeta(1 - \eta_z(\bar{t}_Z^{'+}))x(\bar{t}_Z^{'+}) < \alpha\bar{Z}$. Next the interval $(\bar{t}'_Z, \underline{t}''_Z)$ is an interval unconstrained with abatement during which the emissions decrease by Proposition P.2-c, hence again $Z(\underline{t}_Z^{''-}) < \bar{Z}$ and $Z(t)$ should jump up at time $\underline{t}''_Z$.

d. Abatement begins before the arrival at the ceiling and ends

within the ceiling period: $0 \leq \underline{t}_a < \underline{t}_Z < \bar{t}_a < \bar{t}_Z$**.**

Let us first show that $\lambda_Z(\underline{t}_Z) > g'(0^+)$ if it is ever optimal to abate. Assume that $\lambda_Z(\underline{t}_Z) \leq g'(0^+)$, then by (3.17):

$$\lambda_Z(t) \quad = \quad \lambda_Z(\underline{t}_Z)e^{-(\rho+\alpha)(\underline{t}_a - t)} < g'(0^+) \ , \ t \in [0, \underline{t}_a] \ ,$$

and it is optimal to not abate during the pre-ceiling period. Next $\lambda_Z(t)$ decreases during the period at the ceiling according to Propositions P.3-c and P.4-c so that $\lambda_Z(t) < g'(0)$, $t \in (\underline{t}_Z, \bar{t}_Z)$ and it is not optimal to abate during this period. Last $\lambda_Z(t) = 0$, $t > \bar{t}_Z$, by (3.18) because the ceiling constraint is no more active after $\bar{t}_Z$ and again it is not optimal to abate after $\bar{t}_Z$. To sum up, it is never optimal to abate.

Now assume that $\lambda_Z(\underline{t}_Z) > g'(0^+)$. First define $\underline{t}_a$, $0 \leq \underline{t}_a < \underline{t}_Z$, and $\underline{t}_Z = \inf\{t : 0 \leq t < \underline{t}_Z \text{ and } \lambda_Z(t) > g(0^+)\}$. Then by (3.17), $\lambda_Z(t) > g'(0^+)$, $t \in (\underline{t}_a, \underline{t}_Z)$, and it is optimal to abate during the pre-ceiling phase $(\underline{t}_a, \underline{t}_Z)$ and at time $\underline{t}_a$ if $\underline{t}_a = 0$ and $\lambda_Z(0) > g'(0)$. Next note that $\lambda_Z(\bar{t}_Z) = 0$ since after $\bar{t}_Z$ the ceiling constraint never binds again. Thus $\lambda_Z(t)$ being time continuous, there exists some time $\bar{t}_a$, $\underline{t}_Z < \bar{t}_a < \bar{t}_Z$, defined as $\bar{t}_a = \sup\{t : \lambda_Z(t) > g'(0^+)\}$, such that, $\lambda_Z(t)$ being monotically decreasing on $(\underline{t}_Z, \bar{t}_Z)$ by the Propositions P.3-c and P.4-c, then $\lambda_Z(t) \gtreqless g'(0^+)$ according to $t \lesseqgtr \bar{t}_a$ and it is optimal to abate during the first ceiling sub-period $(\underline{t}_Z, \bar{t}_Z)$ and not during the second one, $[\bar{t}_a, \bar{t}_Z]$.

e. Existence of a post-ceiling period of coal exploitation: $\bar{t}_Z < t_X$

Because the useful energy price path $\{p(t) = u'(q(t) \ , \ t \geq 0\}$ is continuous, absent such a post-ceiling period $(\bar{t}_Z, t_X)$, we should have: $p(\bar{t}_Z^-) = p(\bar{t}_Z^+) = \tilde{p}$, where $\tilde{p} = u'(\tilde{q}_y)$ and $\tilde{q}_y$ solves $u'(q_y) = c'(q_y)$. Next because $p(\bar{t}_Z^-) = \tilde{p}$, then $q_y(\bar{t}_Z^-) = \tilde{q}_y$ and by the above point **d.** and the Propositions P.3-a and P.3-b, we get: $p(\bar{t}_Z^-) = u'(\eta_x(\bar{t}_Z^-)\alpha\bar{Z}/\zeta + \tilde{q}_y)$. Clearly $\eta_x(\bar{t}_Z^-)\alpha\bar{Z}/\zeta > 0$ and $p(\bar{t}_Z^-) < u'(\tilde{q}_y) = p(\bar{t}_Z^+)$, hence a contradiction since $p(t)$ is continuous.

A.3 Building up the reduced system (7.17)

With distinct capture and sequestration costs the problem (SSP) of the Section 5 must be rewritten as follows:[37]

$$\begin{aligned} \max_{x,\eta_x,\eta_z,q_y} \quad & u(\eta_x x + q_y) - \omega x - b(\eta_x)x - c(q_y) - [g(\eta_z) + \eta_z\theta]\,\zeta x \\ s.t. \quad & \bar{x}(\eta_z) - x \geq 0\ ,\ \ , \eta_z \geq 0 \text{ and } q_y \geq 0\ , \end{aligned}$$

where ω and θ are given.

The only f.o.c's which have to be modified are the ones relative to x and η_z. The system at stake is now:

$$\begin{aligned} \text{w.r.t. } x \quad &: \quad u'(\eta_x x + q_y)\eta_x = \omega + b(\eta_x) + \zeta\,[g(\eta_z) + \eta_z\theta] + \mu_x && \text{(A.3.1)} \\ \text{w.r.t. } \eta_x \quad &: \quad u'(\eta_x x + q_y) = b'(\eta_x) && \text{(A.3.2)} \\ \text{w.r.t. } q_y \quad &: \quad u'(\eta_x x + q_y) = c'(q_y) - \gamma_y && \text{(A.3.3)} \\ \text{w.r.t. } \eta_z \quad &: \quad [g'(\eta_z) + \theta]\,\zeta x = \mu_x \bar{x}'(\eta_z) + \gamma_{\eta z} && \text{(A.3.4)} \end{aligned}$$

where μ_x is the multiplier associated to the maximum extraction rate constraint $\bar{x}(\eta_z) - x \geq 0$.

Elimination of μ_x

Substituting $\bar{x}(\eta_z)$ for x in (A.3.4) with $\gamma_{\eta z} = 0$ since during the ceiling regime with abatement the maximum coal input constraint is active and $\eta_z > 0$, and taking care that $\bar{x}(\eta_z)/\bar{x}'(\eta_z) = 1 - \eta_z$ according to (5.1), we obtain:

$$\mu_x \;=\; [g'(\eta_z) + \theta]\,\zeta(1 - \eta_z) \qquad \text{(A.3.5)}$$

Next:

- in (A.3.1) substitute $b'(\eta_x)$ for $u'(\eta_x x + q_y)$ thank to (A.3.2), next substitute the r.h.s. of (A.3.5) for μ_x and simplify;

[37] Like in Section 5, we omit the non-active constraints $x > 0$ and $\eta_x > 0$ since the aim of the reformulation is to determine when it is optimal to abate while at the ceiling.

- in (A.3.2) substitute $\alpha\bar{Z}/(\zeta(1-\eta_z)) = \bar{x}(\eta_z)$ for x;
- assume that $q_y > 0$ hence $\gamma_y = 0$ in (A.3.3) and substitute $b'(\eta_x)$ for $u'(\eta_x x + q_y)$ again thank to (A.3.2),

to get the following system:

$$\begin{aligned} b'(\eta_x)\eta_x - b(\eta_x) - \zeta\left[g(\eta_z) + (1-\eta_z)g'(\eta_z)\right] &= \omega + \zeta\theta & \text{(A.3.6)} \\ u'\left(\eta_x \frac{\alpha\bar{Z}}{\zeta(1-\eta_z)} + q_y\right) - b'(\eta_x) &= 0 & \text{(A.3.7)} \\ b'(\eta_x) - c'(q_y) &= 0\ . & \text{(A.3.8)} \end{aligned}$$

Obtaining the reduced system (7.17)

Differentiate (A.3.6) and simplify to get:

$$b''(\eta_x)\eta_x d\eta_x - \zeta(1-\eta_z)g''(\eta_z)d\eta_z = d\omega + d\theta\ . \qquad \text{(A.3.9)}$$

Next differentiate (A.3.8) to obtain:

$$dq_y = \frac{b''(\eta_x)}{c''(q_y)} d\eta_x \qquad \text{(A.3.10)}$$

and differentiate (A.3.2), substitute the r.h.s of (A.3.10) for dq_y and divide by u'' to get:

$$\left[\frac{\alpha\bar{Z}}{\zeta(1-\eta_z)} + \frac{b''(\eta_x)}{c''(q_y)} - \frac{b''(\eta_x)}{u''(\eta_x\bar{x}(\eta_z)+q_y)}\right] d\eta_x + \frac{\eta_x\alpha\bar{Z}}{\zeta(1-\eta_z)^2} d\eta_z = 0\ . \qquad \text{(A.3.11)}$$

The system (7.17) is (A.3.9) and (A.3.11) in matrix form.

A.4 Dynamics during a phase at the ceiling with the abatement effort blockaded at its maximum level $\bar{\eta}_z$

With this specification of the costs functions laid down in Section 8:

$$\bar{\eta}_x - \eta_x \geq 0\ ,\ \bar{\eta}_z - \eta_z \geq 0 \text{ and } \bar{q}_y - q_y \geq 0\ .$$

Let us denote by $\bar{\gamma}_{\eta x}$, $\bar{\gamma}_{\eta z}$ and $\bar{\gamma}_y$ the corresponding multipliers and by $\underline{\gamma}_{\eta x}$, $\underline{\gamma}_{\eta z}$ and $\underline{\gamma}_y$ the multipliers associated to the non-negativity constraints.

The Lagrangian of the modified $(S.S.P)$ problem, $\mathcal{L}_{cm}$, is:

$$\begin{aligned}\mathcal{L}_{cm} &= u(\eta_x x + q_y) - \omega x - b(\eta_x)x - c(q_y) - g(\eta_z)\zeta x + \mu_x[\bar{x}(\eta_z) - x] \\ &\quad + \bar{\gamma}_x[\bar{\eta}_x - \eta_x] + \underline{\gamma}_x x + \bar{\gamma}_{\eta z}[\bar{\eta}_z - \eta_z] + \underline{\gamma}_{\eta z}\eta_z + \bar{\gamma}_y[\bar{\eta}_y - q_y] + \underline{\gamma}_y q_y \ .\end{aligned}$$

Assuming that only η_z is blockaded and that $0 < \eta_x < \bar{\eta}_x$ together with $0 < q_y < \bar{q}_y$, the f.o.c's read:

$$\begin{aligned}
\text{w.r.t.}x &: \ u'(\eta_x\bar{x}(\bar{\eta}_z) + q_y)\eta_x = \omega + b(\eta_x) + \zeta g(\bar{\eta}_z) + \mu_x && \text{(A.4.1)}\\
\text{w.r.t.}\eta_x &: \ u'(\eta_x\bar{x}(\bar{\eta}_z) + q_y)\bar{x}(\bar{\eta}_z) = b'(\eta_x)\bar{x}(\bar{\eta}_z) && \text{(A.4.2)}\\
\text{w.r.t.}q_y &: \ u'(\eta_x\bar{x}(\bar{\eta}_z) + q_y) = c'(q_y) && \text{(A.4.3)}\\
\text{w.r.t.}\eta_z &: \ g'(\bar{\eta}_z^-)\zeta\bar{x}(\bar{\eta}_x) = \mu_x\bar{x}'(\bar{\eta}_z) - \bar{\gamma}_{\eta z} \ . && \text{(A.4.4)}
\end{aligned}$$

Here we may not eliminate μ_x. What we obtain after the usual manipulations is a reduced system in $d\eta_x$, $d\mu_x$ and $d\omega$ instead of the reduced system (5.19) in $d\eta_x$, $d\eta_z$ and $d\omega$ since now η_z is blockaded:

$$\begin{bmatrix} b''\eta_x & -1 \\ \dfrac{\alpha\bar{Z}}{\zeta(1-\bar{\eta}_z)} + \dfrac{b''}{c''} - \dfrac{b''}{u''} & 0 \end{bmatrix} \begin{bmatrix} d\eta_x \\ d\mu_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\omega \ . \qquad \text{(A.4.5)}$$

Clearly:

$$\frac{d\eta_x}{d\omega} = 0 \quad \text{and} \quad \frac{d\mu_x}{d\omega} = -1 \ . \qquad \text{(A.4.6)}$$

Since η_x is constant then $\eta_x\bar{x}(\bar{\eta}_z)$ is constant and q_y solving (A.4.3) neither changes. Thus q_x, q_y, q and p are all constant.

With η_z blockaded at $\bar{\eta}_z$, the f.o.c relative to η_z in the initial $(S.P)$ problem, the equivalent of the above condition (A.4.4), would read:

$$\lambda_Z\zeta\bar{x}(\bar{\eta}_z) = g'(\bar{\eta}_z^-)\zeta\bar{x}(\bar{\eta}_z) + \bar{\gamma}_{\eta z} \ . \qquad \text{(A.4.7)}$$

In the above equation let us substitute $\mu_x \bar{x}'(\bar{\eta}_z)$ for $g'(\bar{\eta}_z^-)\zeta\bar{x}(\bar{\eta}_z) + \bar{\gamma}_{\eta z}$ (from (A.4.4)), we get:

$$\lambda_Z \zeta \bar{x}(\bar{\eta}_z) \quad = \quad \mu_x \bar{x}'(\bar{\eta}_z) \ ,$$

hence, together with (A.4.6):

$$\frac{d\lambda_Z}{d\omega} = \bar{x}'(\bar{\eta}_z)\frac{d\mu_x}{d\omega} = -\bar{x}'(\bar{\eta}_z) < 0 \ . \tag{A.4.8}$$

Thus, as asserted in Section 8, during such a phase, $\lambda_Z(t)$ is decreasing since $d\omega/dt > 0$.