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Jean Pierre Amigues - Michel Moreaux

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From Primary Resources to Useful Energy: The Pollution Ceiling Efficiency Paradox *

Jean-Pierre Amigues[†] and Michel Moreaux[‡]

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[†]Toulouse School of Economics(INRA, IDEI and LERNA), 21 allée de Brienne, 31000 Toulouse, France. E-mail:amigues@toulouse.inra.fr

[‡]Toulouse School of Economics (IDEI, IUF and LERNA), Manufacture des Tabacs, 21 allée de Brienne, 31000 Toulouse, France.

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Abstract

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We study an economy producing energy services from a polluting fossil fuel and a carbon free renewable resource under a constraint on the admissible atmospheric carbon concentration, equivalently under a constraint on the admissible temperature. The transformation rates of natural primary resources energy into useful energy are costly endogenous variables. Choosing higher efficiency rates requires to bring into operation more sophisticated energy transformation devices, that is more costly ones. We show that, independently of technical progress, along a perfect foresight equilibrium path which is Pareto optimal, the transformation rate of any exploited resource should increase throughout time, excepted within the period during which the carbon constraint is binding, a phenomenon we call the 'ceiling paradox'.

Keywords: energy efficiency; carbon pollution; non-renewable resources; renewable resources.

JEL classifications: Q00, Q32, Q43, Q54.

1 Introduction

It is a well documented fact that historically, the conversion rate of primary energy into useful energy has steadily increased. For example, the first steam power units operated in the British mining industry to pump water, in the early eighteenth century, the steam engine of Thomas Newcomen (1712), converted about 0.5% of its potential energy input (coal and/or wood) into useful work output. About sixty years later (1769)[1], the Watt engine was converting 3% of its potential energy input, six times the performance of its ancestor.[2] Both engines were built according to some loose thermodynamic principles.[3] However, a glance at the two engines design shows that the cooling system of the Watt engine was more sophisticated and more costly than the cooling system of the Newcomen machine. Improvement still went on during the nineteenth century and up to the middle of the twentieth century with more and more complex engines, especially by multiplying the number of cylinders and other devices for a better use of the steam produced in the boiler.

Although technical progress and growing scientific knowledge are certainly key explanations of the increasing efficiency of the energy uses in an historical perspective,[4] the objective of this paper is to show that it may result more simply from the mere working of competitive forces promoting efficiency efforts in a world of nonrenewable energy resource scarcity even without technical progress. Choosing a conversion rate of available energy into useful energy is an economic choice and retaining a more efficient process is also, generally, more costly. For example some part of the exhausted energy resulting from the burning of gas in a today engine can be exploited, via a turbo-charger, to improve the efficiency of combustion. But one has to bear the cost of the turbo-charger and its installation in the car. In hybrid cars, some part of the energy which would be otherwise dissipated, is used

[1]1769 is the date of the first patent obtained by Watt. The first engines were sold by Boulton and Watt seven years later, in 1776 according to Marsden (2002, p 102).

[2]Both rates from Kümmel (2011), p. 49 for the Newcomen engine and p. 50 for the Watt engine. A usual, one must be cautious about such estimations. For example, Brookes (2000, p 359) gives respectively 0.75% for the Newcomen engine and 4% for the Watt engine. However the scale of the increase factor is about the same, the Watt engine being 5.33 more efficient than the Newcomen machine.

[3]The first theoretical essay by Carnot (1824) was published only fifty years later.

[4]However note that technical progress is partly endogenous.

to save the gasoline consumption thank to electric devices. But clearly two engines, a classical gas one and an electric one, are more costly than only one.

In this paper, we take explicitly into account the fact that to obtain useful energy from a fossil resource, it is first necessary to transform the underground energy into what we call available crude energy, an operation undertaken by the extractive industry, and next to transform the available crude energy into useful energy, a task generally performed jointly by utilities and the final users themselves. We assume that the unitary extraction costs depend negatively upon the resource grade under exploitation. Moreover, the cost of transforming one unit of the extracted resource into useful energy increases with the energy conversion rate.[5]

Transforming extracted fossil fuel into useful energy generates also as a by-product polluting GHG emissions in the atmosphere with potential adverse consequences. These emissions are more or less proportional to the burnt fossil resource rather than to the useful energy output. Hence, improving the conversion rate may be seen as an indirect abatement device. The diesel engines emit less CO_2 per unit of gas than the gasoline engines because they are more efficient converters of potential energy, but they are also more costly to produce. However contrary to other abatement options, like carbon capture and sequestration installations, the conversion rate improvements simultaneously save the resource.

Several views of a climate change mitigation policy have been proposed in the climate economics literature. The most conventional one expresses the consequences of global warming as a combination of welfare losses (impacts on properties, health impacts) and productivity losses (e.g. agricultural yields losses). These losses are then aggregated as a 'damage' function, assumed to be some increasing function of the size of the atmospheric carbon stock (or of the average temperature rise).[6] The design of an optimal environmental reg-

[5]Also important empirically is the recovery rate of available crude energy from the underground one. This problem requires a specific analysis outside the scope of the present paper.

[6]Multiple theoretical and empirical studies have endorsed the 'damage function' approach. Main original contributions are Tahvonen (1991), Tahvonen and Kuuluvainen (1991), Farzin and Tahvonen (1996), Withagen (1994) Tahvonen and Withagen (1996), Toman and Withagen (2000). For more recent contributions see Golosov *et al.* (2014),

ulation scheme is then accommodated in a Pigouvian way. An optimal global carbon tax should identify with the marginal climate damage in annualized value equivalent. Since the damages evolve through time with the climate dynamics, the tax rate should also be adjusted to the climate damages trend. An alternative policy to the implementation of a carbon tax, perhaps less demanding with respect to the commitment abilities of the governments, is the creation of carbon emissions permits markets. The time adjustment of the regulation is in this case achieved through a periodic revision of the quantity of allowances issued by the regulator.

Instead of a climate damage function framework, we follow in the present paper another route pioneered by Chakravorty, Magné and Moreaux (2006). In their model, carbon accumulation in the atmosphere creates only negligible damages provided that the pollution carbon stock stays under some critical threshold. However, this threshold be crossed over, earth climate conditions would become catastrophic. The objective of the environmental regulation should then be to limit the atmospheric carbon concentration below the threshold level. Such a modeling framework echoes the current policy proposal of avoiding an average earth temperature rise above $+2^0C$ by the end of the current century.[7]

In our setting, the use of fossil fuels thus faces two kinds of constraints. The first one is the physical scarcity of the available fossil fuels reserves, the second one is the limited ability of the atmospheric compartment to store carbon emissions without triggering potentially damaging climate change. Raising the energy conversion rate of fossil fuels can alleviate these two constraints by saving the resource while mitigating carbon emissions.

An alternative to costly energy conversion efficiency efforts in fossil fuel exploitation is the development of carbon free renewable energy use. However, the transformation of clean energy primary sources into useful energy

Van der Ploeg and Withagen (2014), Hassler *et al.* (2012), Nordhaus (2008), Stern (2007).

[7]As pointed out by Weitzman (2010) and Mason and Wilmot (2015), damages are depending on the temperature rather than directly on the carbon stock. Thus the ceiling should be defined as a temperature ceiling, like the well-known $+2^0C$ ceiling. However as far as there exists a monotonic relationship between the temperature level and the atmospheric carbon stock, the qualitative properties of the optimal paths would be the same. The issue becomes more intricate when both the temperature level and the carbon stock size drive the temperature dynamics.

faces the same kind of physical and technical constraints than the exploitation of fossil fuels. We model this problem by assuming that the unitary production cost of useful energy from clean renewable energy increases with the transformation rate. We want to describe in this context the dynamics of the energy transition from fossil fuels toward clean energy.

Within this general framework, we show the following. When the demand function for useful energy is stationary, the equilibrium and/or optimal transformation rate of the fossil energy source broadly increases through time up to the end of its exploitation. The time and the grade at which exploitation ends are endogenously determined not only by the increasing costs of less accessible grades but also by the increasing transformation cost of crude energy into useful energy. When renewable energy is exploited in conjunction with fossil energy, its transformation rate also increases and it takes a larger share inside the energy mix until it replaces completely the use of fossil energy.

Rather surprisingly, when a constraint on the atmospheric pollution stock is added, then the transformation rate in the coal industry must stay constant when coal is the only resource which is exploited within the time period during which the constraint is active. Maybe even more surprising, when the renewable resource is also exploited, both conversion rates have to be kept constant. Restricting the use of coal does not open a larger market for the non-polluting renewable resource. The energy mix must stay unmodified. We call this feature of the optimally regulated perfect foresight equilibrium, equivalently of the Pareto optimal paths, the *ceiling paradox*.

The paper is organized as follows. The model of useful energy production from primary resources is laid down in Section 2. Section 3 describes the perfect foresight equilibrium of an economy without damages generated by the use of the polluting non-renewable resource. Section 4 characterizes the first best regulation policy of the pollution damages. Section 5 concludes.

2 A model of useful energy production from primary resources

We consider a stationary economy in which final energy services can be obtained through the exploitation of two primary sources. The first one is a nonrenewable polluting fossil fuel (say coal) while the second one is clean and renewable (say solar).

Let $X(t)$ be the available underground stock of coal at time t measured in energy units, X^0 be the initial endowment, $X^0 \equiv X(0)$, and $x(t)$ be the instantaneous extraction rate measured in the same units: $\dot{X}(t) = -x(t)$. For the renewable and carbon free primary energy source, we assume a constant available flow of this energy, y^n, also measured in energy units.

The use of coal generates carbon polluting emissions in the atmosphere. Assume an homogenous polluting content from coal burning to simplify.[8] Denoting by ζ the unitary pollution content of coal, the pollution emission flow at time t amounts to $\zeta x(t)$. Let $Z(t)$ be the pollution stock at time t and Z^0 be the historically given pollution stock, $Z^0 \equiv Z(0)$. The pollution stock is fed by the emission flow $\zeta x(t)$ and is self-regenerating at some proportional rate $\alpha > 0$, assumed to be constant to simplify.[9] Hence the dynamics of $Z(t)$ is driven by: $\dot{Z}(t) = \zeta x(t) - \alpha Z(t)$.

The energy industry transforms the two primary energy sources into useful energy supplied to the final users, either consumers or firms. We assume that the industry is composed of identical competitive firms with equal access to the same set of processing technologies. Thus we do not assign specialization to the firms in converting primary energies, either coal or solar, into useful energy.

[8] The issue of heterogenous polluting resources is thoroughly examined in Chakravorty, Moreaux, Tidball (2008).

[9] More general formulations of the self-regeneration process are explored in Forster (1975), Farzin (1996), Tahvonen and Salo (1996), Tahvonen and Withagen (1996) and Toman and Withagen (2000) in which the proportional rate α depends upon Z. Some self-regenerating processes give rise to non convex dynamic programs in which the necessary first order conditions are not sufficient to characterize the optimal paths. Such regeneration processes would induce the same type of difficulty in the present context.

Concerning their access to the primary energy sources, two industry structures may be considered. The first one is an integrated structure, where the firms own the coal mines and the land areas used for renewable energy generation (solar, wind or biomass). The second one is a disintegrated structure, where the industry must purchase coal from the extractive industry and hire land to produce renewable energy. We assume an integrated structure for the production of renewable energy to escape the problem of favorable sites access for wind or solar energy generation. For coal, we retain a disintegrated structure and we have thus to describe the features of the coal mining industry.

The coal extractive industry is composed of competitive firms facing an identical unit cost schedule depending on the grade under exploitation. Denote by $a(X)$ this unit cost, hence a total extraction cost $a(X)x$ at the mining industry level. $a(X)$ follows the following usual assumptions in mining economics:[10,11]

Assumption A. 1 $a : (0, X^0] \to \mathbb{R}_+$ *is twice continuously differentiable on* $(0, X^0)$*, strictly decreasing,* $a'(X) < 0$*, strictly convex,* $a''(X) > 0$*, with* $a(0^+) = +\infty$*.*

The last property in A.1 together with the assumption A.3 imply that some part of the initially available coal endowment, X^0, will be kept underground.

Transforming primary energy from any source into final energy services implies a loss depending on the processing technologies and the type of energy source. First, consider coal energy. Let us denote by η_x the unitary transformation rate of extracted coal into useful energy, $0 < \eta_x < 1$. [12] The

[10] See for example Heal (1976), Hanson (1980). The underlying assumption is that the coal deposits have different extraction costs and that they are exploited by increasing order of extraction costs as a result of the minimization of discounted extraction costs under positive discounting.

[11] Let $f(x)$ defined on $X \subseteq \mathbb{R}$ and $\bar{X}$ the closure of X. Then for any $x_0 \in \bar{X}$ we denote respectively by $f(x_0^+)$ and $f(x_0^-)$ the limits $\lim_{x \downarrow x_0} f(x)$ and $\lim_{x \uparrow x_0} f(x)$, when such limits exist.

[12] To simplify, we assume that the upper bound of the transformation rate η_x is equal to one, although it is actually strictly lower. The same remark applies to the conversion rate of solar energy, η_y, introduced below. See Section 5 for a sketch of the more complex

useful coal energy consumption rate at time t, denoted by $q_x(t)$, amounts to $\eta_x(t)x(t)$.

Getting more useful energy than less from a given flow x of extracted coal, requires more efficient, hence more sophisticated processing devices, that is more costly ones. Let $b(\eta_x)$ be the unitary conversion cost of extracted coal energy (per unit of processed coal input) into ready-to-use energy services as a function of η_x, the chosen efficiency rate. Then the total transformation cost of x units of crude energy from extracted coal into $q_x = \eta_x x$ units of useful energy services amounts to $b(\eta_x)x$.

The unitary production cost of useful coal energy amounts to $b(\eta_x)/\eta_x$, equal to the marginal produnction cost, that is a total cost equal to $(b(\eta_x)/\eta_x)q_x$. We assume that this unit cost is an increasing function of the transformation rate. This implies that $b(\eta_x)$, the unit conversion cost of coal energy should also be an increasing function of η_x.[13] The unit conversion cost function $b(.)$ satisfies:

Assumption A. 2

- $b : [0, 1) \to \mathbb{R}_+$ *is twice continuously differentiable on* $(0, 1)$, *strictly increasing,* $b'(\eta_x) > 0$, *strictly convex,* $b''(\eta_x) > 0$, *with* $b(0^+) = 0$, $b'(0^+) > 0$, $b(1^-) = +\infty$ *and* $b'(1^-) = +\infty$.
- *The unit production cost of useful coal energy (and so the marginal production cost) is a strictly increasing function of* η_x*:* $b'(\eta_x) > b(\eta_x)/\eta_x$ *and* $\lim_{\eta_x \downarrow 0} b(\eta_x)/\eta_x > 0$.

Burning coal to obtain, for example, electricity requires other inputs, at least equipment, hence a strictly positive marginal cost of useful coal energy

optimal paths which could result from technical constraints imposing upper bounds below 1 on the conversion efficiency rates.

[13]Differentiating the unit production cost yields:

$$\frac{d}{d\eta_x}\frac{b(\eta_x)}{\eta_x} = \frac{1}{\eta_x}\left[b'(\eta_x) - \frac{b(\eta_x)}{\eta_x}\right] .$$

It is immediate that $b'(\eta_x) > 0$ is a necessary condition for $(d/d\eta_x)[b(\eta_x)/\eta_x] > 0$.

at 0^+: $\lim_{\eta_x \downarrow 0} b(\eta_x)/\eta_x > 0$. The same should apply to the derivative of the function $b(.)$ at $\eta_x = 0^+$. The assumptions $b(1^-) = +\infty$ and $b'(1^-) = +\infty$ mean that it is not physically possible to transform totally the potential energy of crude coal into useful energy.

Turn to solar energy generation from the natural flow y^n. Like for coal, transforming this primary resource into ready-to-use energy services implies a loss. Let us denote by η_y, $0 < \eta_y < 1$, the conversion rate of primary solar energy into final energy, so that the current consumption of useful solar energy, denoted by q_y, amounts to $\eta_y y^n$. Let $c(\eta_y)$ be the unitary processing cost of primary solar energy into useful energy, thus a total processing cost of useful solar energy equal to $c(\eta_y)y^n$. The average production cost of useful solar energy amounts to $c(\eta_y)/\eta_y$, equal to the marginal cost. As for coal, higher solar conversion rates are more costly and the $c(.)$ function satisfies:

Assumption A. 3

- $c : [0, 1) \to \mathbb{R}_+$ *is twice continuously differentiable over* $(0, 1)$*, strictly increasing,* $c'(\eta_y) > 0$*, strictly convex,* $c''(\eta_y) > 0$*, with* $c(0^+) = 0$, $c'(0^+) > 0$, $c(1^-) = +\infty$ *and* $c'(1^-) = +\infty$.
- *The unit production cost of useful solar energy (and so the marginal cost) is a strictly increasing function of* η_y*:* $c'(\eta_y) > c(\eta_y)/\eta_y$ *and* $\lim_{\eta_y \downarrow 0} c(\eta_y)/\eta_y > 0$.

The rationale for $\lim_{\eta_y \downarrow 0} c(\eta_y)/\eta_y > 0$, $c'(0^+) > 0$, $c(1^-) = +\infty$, $c'(1^-) = +\infty$ is the same than the rationale for the similar assumptions on $b(.)$ in A.2.

3 Perfect foresight competitive equilibrium

Assume that all markets are competitive, including the capital market, and that the interest rate, denoted by r, is a strictly positive constant. The purpose of this section is two-fold. First we identify a set of short run equilibrium properties under an arbitrary public regulation scheme of the pollution resulting from coal burning. These properties will facilitate considerably the characterization of the optimal policy done in Section 4. Second, we describe the perfect foresight equilibrium of the energy sector when burning coal generates only negligible damages.

3.1 Demand and energy supply plans

We first consider the two ends of the energy sector, the extractive industry and the final users of useful energy, and next the in-between transformation industry.

Extracted coal supply

Let $p_x(t)$ be the price of extracted coal at time t, $x^s(t)$ the quantity supplied by the extractive industry and $x^d(t)$ the quantity demanded as an input by the transformation industry.

Consider first the extraction sector. Given a price path $\{p_x(t); t \geq 0\}$, the industry has to design a coal supply plan $\{x^s(t); t \geq 0\}$ maximizing its cumulated discounted profit under the coal resource availability constraint, that is solve the problem:

$$\begin{aligned} \max_{x^s(t)} \quad & \int_0^\infty \left[p_x(t)x^s(t) - a(X(t))x^s(t) \right] e^{-rt} dt \\ s.t. \quad & \dot{X}(t) = -x^s(t) \quad , \quad X(0) = X^0 \ \text{ given} \\ & x^s(t) \geq 0 \quad , \quad X(t) \geq 0 \ . \end{aligned}$$

Denote by $\mu_X(t)$ the current level of the mining rent. Then profit maximization by the coal extraction industry requires that:

$$p_x(t) \ = \ a(X(t)) + \mu_X(t) \ , \tag{3.1}$$

and when $\mu_X(t)$ is time differentiable:[14]

$$\dot{\mu}_X(t) \ = \ r\mu_X(t) + a'(X(t))x^s(t) \ . \tag{3.2}$$

[14]It is later shown at the end of the section that $\mu_X(t)$ is differentiable over the whole time interval during which coal is exploited.

The linearity of the coal extraction cost implies that the supply plan of the extractive industry is indeterminate at the equilibrium on the coal market. For a given price p_x of extracted coal and a unitary full marginal cost of extraction at grade X, $a(X) + \mu_X$, either x^s is nil if $p_x < a(X) + \mu_X$ or infinite in the reverse case. Thus the equilibrium on the extracted coal market at time t requires that $p_x(t) = a(X(t)) + \mu_X(t)$, that is (3.1), the usual zero profit equilibrium condition under constant unitary costs, and the extractive industry is ready to supply at this price any amount x^s determined elsewhere provided that no benefit could be earned by supplying earlier of later the coal of the grade at stake.

Equation (3.2) expresses the Hotelling rule when the extraction costs depend upon the grade which is mined. Here $\mu_X(t)$ is the mining rent of the grade $X(t)$ exploited at time t. Under constant marginal costs, when the extraction cost is the same for all the grades, the rent should grow at a proportional rate equal to the interest rate under competitive capital market conditions, that is the rent must increase by $r\mu_X(t)dt$ between t and $t + dt$. In the present context, the extraction cost is larger at $t + dt$ than at t by an amount approximatively equal to $-a'(X(t))x(t)dt$.[15] Thus the mining rent of the grade $X(t + dt)$ must be equal to $\mu_X(t + dt) = \mu_X(t) + r\mu_X(t)dt + a'(X(t))x(t)dt$, hence (3.2), in order that the extraction of $x(t)$ be not postponed to $t + dt$ and the extraction of $x(t + dt)$ be not switched earlier at t.

Useful energy demand

At the other end of the energy sectors are the final users of useful energy. At this end-users stage, we assume perfect substitutability between useful energy produced from any primary source. We assume in addition that useful energy is not storable, so that any amount of transformed energy which is not immediately consumed by the end-users is definitively lost. Denote by q, $q = q_x + q_y$, the aggregate instantaneous rate of useful energy consumed by the end-users and let $u(q)$ denote the gross surplus thus generated. The gross surplus function $u(.)$ satisfies the following standard assumption:

Assumption A. 4 $u : (0, \infty) \to \mathbb{R}_+$ *is twice continuously differentiable,*

[15] Remember that $a'(X)$ is negative.

strictly increasing, $u' > 0$, strictly concave, $u'' < 0$, and satisfies the first Inada condition: $u'(0^+) = +\infty$.

We shall denote by $p^d(q)$ the inverse demand function of useful energy, $p^d(q) \equiv u'(q)$, and by p the price of useful energy. The inverse of $p^d(q)$, denoted by $q^d(p)$, is the standard direct demand function.

Demand of extracted coal and supply of useful energy

The transformation industry, which lies in-between the extractive industry and the final users, takes as given the useful energy price, $p(t)$, together with the extracted coal price, $p_x(t)$, and determines the supplies of useful energies, $q_x^s(t)$ and $q_y^s(t)$, and the demand of extracted coal $x^d(t)$ which, together with the choice of $\eta_x(t)$ and $\eta_y(t)$, maximize its profits.

Although we characterize in the present section the perfect foresight competitive equilibrium when pollution damages are negligible, we assume here that the transformation sector could be subjected to monetary penalties for its polluting emissions. Hence we will not have to repeat the same arguments in the next section when pollution will be assumed to be actually damaging. Let $T(\zeta x^d(t), t)$ be the burden, or monetary transfer, imposed to the energy sector at time t. As far as more pollution is worsening the welfare, this transfer should be some increasing function of the emissions level resulting from coal burning, $\partial T / \partial \zeta x > 0$, and would have to be adjusted through time to accommodate the dynamics of the atmospheric carbon stock. We shall make precise the transfer scheme in section 4 when introducing the environmental regulation objective.

Since $q_x^s = \eta_x x^d$ and $q_y^s = \eta_y y$, the profit maximization problem of the transformation industry may be laid down in terms of η_x, η_y, y and x^d. Thus the industry solves at any time t:[16,17]

$$\max_{x^d,\eta_x,\eta_y,y} \quad p\left[\eta_x x^d + \eta_y y\right] - b(\eta_x)x^d - c(\eta_y)y - p_x x^d - T(\zeta x^d, t)$$

$$s.t. \quad x^d \geq 0\ ,\ \eta_x \geq 0 \text{ and } \eta_y \geq 0 \tag{3.3}$$

$$y^n - y \geq 0\ . \tag{3.4}$$

Let γ_x, $\gamma_{\eta x}$ and $\gamma_{\eta y}$ be the multipliers associated to the non-negativity constraints (3.3) over the choice variables x^d, η_x and η_y respectively. Let $\bar{\gamma}_y$ be the multiplier associated to the solar flow constraint (3.4). The first order conditions are:

$$\text{w.r.t. } x^d \ :\ p\eta_x = b(\eta_x) + p_x + \frac{\partial T}{\partial x^d} - \gamma_x \tag{3.5}$$

$$\text{w.r.t. } \eta_x \ :\ px^d = b'(\eta_x)x^d - \gamma_{\eta x} \tag{3.6}$$

$$\text{w.r.t. } \eta_y \ :\ py = c'(\eta_y)y - \gamma_{\eta y} \tag{3.7}$$

$$\text{w.r.t. } y \ :\ p\eta_y = c(\eta_y) + \bar{\gamma}_y\ , \tag{3.8}$$

together with the usual complementary slackness conditions.

The specification of the cost structure implies that if the transfer scheme is a linear function of the emission rate, ζx^d, then the supply of useful coal energy is indeterminate. Assume that $T(\zeta x^d, t) = T_0 + T(t)\zeta x^d$. Denote by $p_x^T = p_x + \zeta T(t)$ the full price of coal, marginal transfer included. For any given transformation rate η_x, the unitary production cost of useful coal energy is constant and equal to $\left(p_x^T + b(\eta_x)\right)/\eta_x$ (per unit of useful energy). Thus either $p(t) < \left(p_x^T(t) + b(\eta_x)\right)/\eta_x$ and the supply is nil or the reverse holds and the supply is infinite. Hence at the equilibrium, the profit must be nil, the meaning of (3.5), as usual under constant average costs. Then the industry is ready to supply any quantity q_x^s determined elsewhere. This feature translates to the demand for extracted coal input, x^d. Given any efficiency rate η_x, $x^d = q_x^s/\eta_x$ is indeterminate since q_x^s itself is indeterminate.[18]

The condition (3.6) states that the choice of the coal conversion efficiency rate, η_x, depends only on p, the price of useful energy. Let $\eta_x^e(p)$ be the profit

[16] We neglect the constraints $\eta_x \leq 1$ and $\eta_y \leq 1$, since they cannot bind at the equilibrium.

[17] We omit the time index when no confusion is possible.

[18] Note however that for transfer schemes increasing at an increasing rate with the emission flow ζx, $\partial^2 T/\partial(\zeta x)^2 > 0$, the supply of useful coal energy and the demand of extracted coal would be unambiguously determined as functions of p and p_x^T.

maximizing choice function:[19]

$$\eta_x^e(p) \begin{cases} = 0 \text{ for } p \leq b'(0^+) \\ \\ > 0 \text{ for } b'(0^+) < p \end{cases} \quad \text{and} \quad \frac{d\eta_x^e}{dp} = \begin{cases} 0 \text{ for } p < b'(0^+) \\ \\ 1/b''(\eta_x^e(p)) > 0 \\ \\ \text{for } b'(0^+) < p \,. \end{cases} \tag{3.9}$$

Thus we may define what we call a quasi-supply function of useful coal energy that we denote by $\hat{q}_x^s(p, x^d) \equiv \eta_x^e(p) x^d$:[20]

$$\hat{q}_x^s(p, x^d) \begin{cases} = 0 \quad \text{for } p \leq b'(0^+) \\ \quad\quad \text{or } x^d = 0 \\ \\ > 0 \quad \text{for } b'(0^+) < p \\ \quad\quad \text{and } x^d > 0 \end{cases} \quad \text{and} \quad \frac{\partial \hat{q}_x^s}{\partial p} = \begin{cases} 0 \text{ for } p < b'(0^+) \\ \quad \text{or } x^d = 0 \\ \\ \dfrac{x^d}{b''(\eta_x^e(p))} > 0 \\ \\ \quad \text{for } b'(0^+) < p \\ \quad \text{and } x^d > 0 \,. \end{cases} \tag{3.10}$$

Positive solar energy production implies that $p = c'(\eta_y)$ through (3.7). On the other hand, the assumption A.3 implies that $c'(\eta_y) > c(\eta_y)/\eta_y$. Thus (3.8) implies that: $p\eta_y = c'(\eta_y)\eta_y = c(\eta_y) + \bar{\gamma}_y > c(\eta_y)$. Hence $\bar{\gamma}_y > 0$, implying that $y = y^n$. What would be the quasi-supply function of useful solar energy resulting from the profit maximization condition (3.7), is *de facto* a standard supply function. As for coal, the profit maximizing rate of solar energy into useful energy depends only on p, the useful energy price. Let $\eta_y^e(p)$ be the choice function resulting from (3.7). Clearly:

$$\eta_y^e(p) \begin{cases} = 0 \text{ for } p \leq c'(0^+) \\ \\ > 0 \text{ for } c'(0^+) < p \end{cases} \quad \text{and} \quad \frac{d\eta_y^e}{dp} = \begin{cases} 0 \text{ for } p < c'(0^+) \\ \\ 1/c''(\eta_y^e(p)) > 0 \\ \\ \text{for } c'(0^+) < p \,. \end{cases} \tag{3.11}$$

The supply function of useful solar energy that we denote by $q_y^s(p)$, $q_y^s(p) \equiv$

[19]The function $\eta_x^e(p)$ is not differentiable at $p = b'(0^+)$ as far as $b''(0^+) > 0$. Furthermore $\lim_{p\uparrow\infty} \eta_x^e(p) = 1$.

[20]The function $\hat{q}_x^s(p, x^d)$ is not differentiable at $p = b'(0^+)$ and $x^d > 0$. Furthermore $\lim_{p\uparrow\infty} \hat{q}_x^s(p, x^d) = x^d$.

$\eta^e_y(p)y^n$, satisfies:

$$q^s_y(p) \begin{cases} = 0 \text{ for } p \le c'(0^+) \\ > 0 \text{ for } c'(0^+) < p \end{cases} \quad \text{and} \quad \frac{dq^s_y}{dp} = \begin{cases} 0 \text{ for } p < c'(0^+) \\ y^n / c''(\eta^e_y(p)) > 0 \\ \quad \text{for } c'(0^+) < p \ . \end{cases} \tag{3.12}$$

Last we define the aggregate quasi-supply function of useful energy, denoted by $\hat{q}^s(p, x^d)$, as the sum of the useful coal energy quasi-supply and useful solar energy supply functions: $\hat{q}^s(p, x^d) \equiv \hat{q}^s_x(p, x^d) + q^s_y(p)$, nil for $p \le \min\{b'(0^+), c'(0^+)\}$ and next increasing up to $x^d + y^n$ for $p \to \infty$:

$$\frac{\partial \hat{q}^s}{\partial p} \begin{cases} = 0 \quad , \quad p < \min\{b'(0^+), c'(0^+)\} \\ > 0 \quad , \quad p \in \mathcal{D}(b'(0^+), c'(0^+), x^d) \ , \end{cases} \tag{3.13}$$

where $\mathcal{D}$ is the domain of the (p, x^d) space in which $\partial \hat{q}^s / \partial p$ is well defined.[21]

3.2 Short run equilibrium relationships

We show now that at any time t, the short run equilibrium condition on the useful energy market allows defining the useful energy outputs from the two primary sources $q_x(t)$ and $q_y(t)$, and the extracted coal demand of the transformation industry, $x^d(t)$, as functions of the useful energy price $p(t)$. Furthermore, since $\eta_x(t) = \eta^e_x(p(t))$ and $\eta_y(t) = \eta^e_y(p(t))$, then all the variables, excepted the supply of extracted coal, may be seen as functions of $p(t)$. Next, using the profit maximization condition of the transformation industry with respect to the demand of extracted coal, we show that $p(t)$ may itself

[21]

$$\begin{aligned} &\mathcal{D}(b'(0^+), c'(0^+), x^d) = \\ &\{p : p > c'(0^+)\} \text{ if } x^d = 0 \ , \\ &\{p : p > b'(0^+) \text{ and } p \ne c'(0^+)\} \text{ if } x^d > 0 \text{ and } b'(0^+) < c'(0^+) \ , \\ &\{p : p > c'(0^+) \text{ and } p \ne b'(0^+)\} \text{ if } x^d > 0 \text{ and } c'(0^+) < b'(0^+) \ . \end{aligned}$$

For $p = \min\{b'(0^+), c'(0^+)\}$ and $p = \max\{b'(0^+), c'(0^+)\}$ the derivative $\partial \hat{q}^s / \partial p$ is not defined when $x^d > 0$. When $x^d = 0$ the derivative is defined at $p = c'(0^+)$.

be expressed as a function of the full price of extracted coal at the same date, $p_x^T(t) = p_x(t) + \partial T(\zeta x^d, t)/\partial x^d$. Since $p_x(t)$ depends of the coal grade under exploitation at time t and of the corresponding mining rent $\mu_X(t)$, we must wait until the characterization of the complete equilibrium dynamics, included the coal market equilibrium, before getting a complete picture of what happens at time t.

Consider first the useful energy market equilibrium. The market clearing equilibrium condition implies that:

$$q^d(p(t)) \;=\; \hat{q}^s\left(p(t), x^d(t)\right) \;. \tag{3.14}$$

Let us denote by $x^{de}(p(t))$ the solution of (3.14). Differentiating, we get from (3.13):

$$\frac{dx^{de}}{dp} \;=\; \frac{dq^d/dp - \partial\hat{q}^s/\partial p}{\partial\hat{q}^s/\partial x^d} < 0 \;, \tag{3.15}$$

provided that $p \in \mathcal{D}(b'(0^+), c'(0^+), x^d)$ and $x^d > 0$ in order that the derivatives of the right hand side be defined and that $\partial\hat{q}^s/\partial x^d \neq 0$.[22]

Next the profit maximization condition with respect to x^d, eq.(3.5), determines a relationship between $p(t)$ and $p_x^T(t)$ as:

$$p(t)\eta_x^e(p(t)) \;=\; p_x^T(t) + b\left(\eta_x^e(p(t))\right) \;. \tag{3.16}$$

[22] Note that, since $\hat{q}^s(p, x^d) = \hat{q}_x^s(p, x^d) + q_y^s(p)$, $\eta_x = \eta_x^e(p)$ and $\hat{q}_x^s = \eta_x^e(p)x^d$, then $x^{de}(p(t))$ may be equivalently defined as:

$$x^{de}(p(t)) \;=\; \frac{q^d(p(t)) - q_y(p(t))}{\eta_x^e(p(t))} \;,$$

provided that $q^d(p(t)) \geq q_y^s(p(t))$, inequality which is satisfied at the equilibrium on the useful energy market. Differentiating, we obtain:

$$\frac{dx^{de}}{dp} \;=\; \frac{1}{(\eta_x^e)^2}\left\{\left(\frac{dq^d}{dp} - \frac{dq_y^s}{dp}\right)\eta_x^e - \left(q^d - q_y^s\right)\frac{d\eta_x^e}{dp}\right\} < 0 \;.$$

Let $p^e\left(p_x^T(t)\right)$ be the solution of (3.16): [23,24]

$$p^e(0^+) = b'(0^+) \quad \text{and} \quad \frac{dp^e}{dp_x^T} = \frac{1}{\eta_x^e(p^e(p_x^T))} > 0 \quad \text{for } p_x^T > 0 \ . \quad (3.17)$$

Using this relation we may express all the model variables, the supply of extracted coal excepted, as functions of $p_x^T(t)$, given that the useful energy market is in equilibrium. Especially useful for the determination of the dynamic equilibrium of the extracted coal market is the expression of the extracted coal demand as a function of p_x^T, we denote by $x^{de}\left(p_x^T(t)\right)$ by a slight abuse of notation. It is immediate from (3.15) and (3.17) that the demand of extracted coal by the transformation industry is a decreasing function of its full price:

$$\frac{dx^{de}(p_x^T)}{dp_x^T} = \frac{dx^{de}}{dp} \cdot \frac{dp^e}{dp_x^T} < 0 \ . \quad (3.18)$$

Note that $x^{de}(p_x^T)$ is not an usual input demand function because it takes into account the equilibrium condition on the useful energy market.

Before turning to the dynamic equilibrium on the extracted coal market, note that a higher full price of coal induces higher efficiency rates of both coal and solar when solar is simultaneously exploited. Denoting by $\eta_x^e\left(p_x^T(t)\right)$ and $\eta_y^e\left(p_x^T(t)\right)$ the efficiency rates as functions of the full price of extracted coal (again by a slight abuse of notation), we get from (3.9), (3.11) and (3.17):

$$\frac{d\eta_x^e}{dp_x^T} = \frac{d\eta_x^e}{dp} \cdot \frac{dp^e}{dp_x^T} > 0 \quad \text{and} \quad \frac{d\eta_y^e}{dp_x^T} = \frac{d\eta_y^e}{dp} \cdot \frac{dp^e}{dp_x^T} > 0 \ . \quad (3.19)$$

When the price of the extracted coal increases, the transformation industry decreases its demand for coal while increasing its transformation efficiency rate. However, because the useful energy consumption decreases, the

[23] For $p_x^T = 0$ we get $p\eta_x^e(p) = b(\eta_x^e(p))$ which is satisfied by any $p \le b'(0^+)$ since then $\eta_x^e(p) = 0$.

[24] Differentiating (3.16), we get:

$$\left[\eta_x^e(p) + (p - b'(\eta_x^e(p)))\frac{d\eta_x^e}{dp}\right] dp = dp_x^T \ .$$

By (3.6): $p - b' = 0$ if $\eta_x > 0$, hence $\eta_x^e(p)dp = dp_x^T$.

increase of the transformation rate does not compensate for the decrease of the extracted coal input rate. The equilibrium of the useful energy market requires that the supply of useful coal energy decreases. Thus, the decrease of the useful coal energy production is even more severe when solar energy is also exploited. The increase of p_x^T induces, *via* the increase of p and hence of η_y, a larger supply of useful solar energy.

3.3 Perfect foresight equilibrium paths absent environmental damages

In the preceding sub-section we have shown that the problem of characterizing the paths of all the key features of the energy sectors may be reduced to the problem of characterizing the equilibrium path of the extracted coal market. Let us now describe the main features of the equilibrium path on this market.

Perfect foresight equilibrium of the extracted coal market

Absent environmental damages, the full price of extracted coal for the transformation industry is equal to the selling price of the extractive industry, $T(\zeta x^d, t) = 0$, $x^d \geq 0$ and $t \geq 0$, hence $p_x^T(t) = p_x(t)$, $t \geq 0$.

Next, from the supply side of the extracted coal market, we know that: $p_x(t) = a\,(X(t)) + \mu_X(t)$ (c.f. (3.1)). Time differentiating and substituting for $\dot{\mu}_X(t)$ its expression (3.2) yields:

$$\dot{p}_x(t) \;=\; -a'(X(t))x(t) + r\mu_X(t) + a'(X(t))x(t) = r\mu_X(t) > 0 \;. \tag{3.20}$$

Thus the price of extracted coal increases up to the time, we denote by t_x, at which ends coal exploitation. Let us now determine the highest price thus attained.

Once coal exploitation is closed, useful energy is supplied only by solar energy at the price $\tilde{p}$ balancing supply and demand: $q^d(\tilde{p}) = q_y^s(\tilde{p})$. We denote by $\tilde{q}$ the corresponding consumption and/or production rate, $\tilde{q} =$

$q^d(\tilde{p}) = q^s_y(\tilde{p})$ and by $\tilde{\eta}_y = \eta^e_y(\tilde{p})$, equivalently $\tilde{\eta}_y = \tilde{q}/y^n$, the solar energy transformation rate. Under the assumptions A.3 and A.4, $\tilde{p} > c'(0^+)$ and is unique.

At the same time t_x^-, the rent of the underground coal grade which is exploited must be nil: $\mu_X(t_x^-) = 0$. On the other hand, since the price path of useful energy must be time continuous, the price of extracted coal is determined *via* the function $p^e(p_x)$ as the solution of $\tilde{p} = p^e\left(p_x(t_x^-)\right)$. Let us denote by $\tilde{p}_x$ this solution. Then, by the equation (3.2) with $\mu_X = 0$, the last grade which is exploited is this grade $\tilde{X}$ for which $\tilde{p}_x = a(\tilde{X})$.

We conclude that the perfect foresight equilibrium of the extracted coal market may be defined as the path $\{(p_x(t), X(t))\}_{t=0}^{t=t_x}$, together with the coal exploitation closing time t_x, solution of the following system of equations:[25]

$$\begin{aligned} \dot{p}_x(t) &= r\left(p_x(t) - a\left(X(t)\right)\right) && (3.21)\\ \dot{X}(t) &= -x^{de}\left(p^e(p_x(t))\right) , && (3.22) \end{aligned}$$

with:

$$X(0) = X^0 \ , \ X(t_x) = \tilde{X} \ \text{ and } \ p_x(t_x) = \tilde{p}_x = a(\tilde{X}) \ . \qquad (3.23)$$

Equilibrium path properties

Some properties of the equilibrium path of the extracted coal market and the induced paths of the other model variables are worth to be pointed out.

First the amount of underground coal which is extracted up to the time t_x at which the extraction is closed, $X^0 - \tilde{X}$, equivalently the amount of coal kept underground forever, $\tilde{X}$, does not depend upon the price path of the extracted coal $\{p_x(t)\}_{t=0}^{t=t_x}$, but only upon the conditions prevailing in the useful energy market when solar is the only supplier. Second, from this amount of cumulated extraction, $X^0 - \tilde{X}$, only that part we denote by Q_x is converted into useful energy during the coal exploitation period, $Q_x = \int_0^{t_x} \eta^e_x\left(p(t)\right) x^{de}\left(p(t)\right) dt$, and the average conversion rate of coal energy is *in fine* equal to $Q_x/(X^0 - \tilde{X})$. Thus, not only some fraction of the available

[25] Equation (3.21) results first from (3.20): $\dot{p}_x = r\mu_X$, and second from (3.1): $\mu_X = p_x - a(X)$, that is nothing but the Hotelling rule in the present context.

coal stock, X^0, is left underground, but also from the extracted fraction, another fraction is also left unused.

Next, since $p(t) = p^e(p_x(t))$ and $dp^e/dp_x > 0$ (c.f. (3.17)), then the useful energy price $p(t)$ increases up to $\tilde{p} > c'(0^+)$ and the exploitation of solar energy must begin at some time t_y before the end of coal exploitation: $t_y = 0$ if $p(0) \geq c'(0^+)$ and $t_y \in (0, t_x)$ if $p(0) < c'(0^+)$. When $t_y > 0$, then the transformation rate of coal $\eta_x(t)$ increases during the time interval $[0, t_y]$ and both transformation rates $\eta_x(t)$ and $\eta_y(t)$ increase during the interval (t_y, t_x). Note that once both resources are exploited $b'(\eta_x(t)) = p(t) = c'(\eta_y(t))$. Time differentiating: $b''\dot{\eta}_x = c''\dot{\eta}_y$ yields together with $b' = c'$:

$$\frac{\dot{\eta}_x/\eta_x}{\dot{\eta}_y/\eta_y} = \frac{c''(\eta_y)\eta_y/c'(\eta_y)}{b''(\eta_x)\eta_x/b'(\eta_x)}, \tag{3.24}$$

where $b''\eta_x/b'$ and $c''\eta_y/c'$ are the elasticities of the marginal transformation costs in respectively the coal and solar energy transformation processes. Hence, the energy conversion rate of coal energy increases at a higher relative growth rate than the energy conversion rate of solar energy when the elasticity of the solar marginal cost function is higher than the elasticity of the coal marginal cost function, and the reverse happens in the opposite case.

The last point to examine is the time differentiability of the different paths, especially the price path of useful energy. Differentiability is a potential problem at $t = t_y$ when $t_y > 0$, that is the time at which solar energy becomes competitive when it is initially too costly to exploit, and at $t = t_x$ that is the time at which ends the coal exploitation when solar energy becomes the only provider of useful energy. The study is done in Appendix A.1 and the results presented in the below Proposition 1.

Proposition P. 1 *Along a perfect foresight equilibrium of the energy sector, absent any environmental damages:*

a. *The cumulative quantity of extracted coal depends only of the cost function in the solar transformation industry (for a given gross surplus function of useful energy).*

b. *During the period of coal extraction* $[0, t_x)$*:*

b.1. The price of useful energy $p(t)$ *increases and is time differentiable with* $\lim_{t \uparrow t_x} \dot{p}(t) = 0$*. The production rate* $q(t)$ *decreases and is also time differentiable with* $\lim_{t \uparrow t_x} \dot{q}(t) = 0$*.*

b.2. The production of useful coal energy $q_x(t)$ *decreases, the coal extraction rate* $x(t)$ *decreases while its transformation rate,* $\eta_x(t)$*, increases but not sufficiently to compensate for the decrease of the extracted coal production.*

The price of extracted coal, $p_x(t)$*, increases up to* $\tilde{p}_x$ *and is time differentiable with* $\lim_{t \uparrow t_x} \dot{p}_x(t) = 0$*. The paths of* $q_x(t)$ *and* $x(t)$ *are both time differentiable excepted at* $t = t_y$ *when* $t_y > 0$ *and* $\lim_{t \uparrow t_x} \dot{q}_x(t) = \lim_{t \uparrow t_x} \dot{x}(t) = 0$*.*

b.3. Solar energy begins to be exploited before the end of coal exploitation. When it is exploited the production of useful solar energy $q_y(t)$ *increases but not sufficiently to compensate for the decrease of useful coal energy. The conversion rate,* $\eta_y(t)$*, increases. The highest proportional increase of conversion rates* η_i*,* $i = x, y$*, occurs in the transformation industry where the elasticity of the marginal unitary transformation rate cost, either* b' *or* c'*, is the largest. Last* $\lim_{t \uparrow t_x} \dot{\eta}_y(t) = 0$*.*

c. Once the transition to a pure renewable energy economy is over, the price of useful energy is constant, hence also the production of useful solar energy and its conversion rate.

4 Climate regulation

Denote by $\bar{Z}$ the critical atmospheric carbon stock, or 'carbon ceiling', triggering catastrophic climate damages, for example the GHG stock triggering a temperature rise above $+2^o$. For the model to make sense, we have to assume that $Z^0 < \bar{Z}$. Initially the global economy is not constrained by the ceiling. The objective of the regulator is to maximize the sum of discounted net surpluses while maintaining the carbon pollution stock below the ceiling, that is implement the constraint $\bar{Z} - Z(t) \geq 0$, $t \geq 0$.

Observe that when the economy is constrained by the ceiling, the use of crude coal energy should not be higher than what is allowed by natural dilution. Then $\bar{x} \equiv \alpha \bar{Z}/\zeta$ is the upper bound on the extracted coal input of the coal transformation industry corresponding to the pollution stock threshold $\bar{Z}$. One policy option for the regulator could be to set a global emission norm $\alpha \bar{Z}$, equivalently a maximum extraction rate $\bar{x}$, when the carbon ceiling constraint binds. But this cannot be optimal in a perfect foresight context.[26] An optimal regulation of the climate problem is not only a matter of carbon concentration stabilization in the atmosphere but also a timing problem. If an active stabilization policy cannot be dispensed with, the problem is to determine the time at which the ceiling constraint should start to bind and for how long. This requires a more sophisticated carbon regulation scheme than just partially or totally banning emissions when it is too late in some sense.

The regulator has to design a dynamic linear tax scheme applied to coal emissions which decentralizes the first best optimum. Let $T(t)$ be the unitary carbon tax rate and $\zeta T(t)x(t)$ be the instantaneous carbon tax revenue at time t. If along the equilibrium path without environmental regulation described in the subsection 3.3 the carbon ceiling never binds, there is of course no need for a carbon tax. To give content to the problem we therefore assume that $Z(t)$ would override $\bar{Z}$ in the no-regulation, perfect foresight scenario.

The problem is now to determine the best slowing down of coal consumption with respect to the non regulated rate, at least initially. First, must the curbing be so strong that the constraint $\bar{Z} - Z(t) \geq 0$ be never effective? We show that it is never the case. More precisely, assuming that, without a tax scheme, the ceiling constraint would be violated, implies that under the optimal tax scheme, the ceiling constraint necessarily binds along the optimal path. Curbing the curb is necessary but not too much.

Thus the problem reduces to determine the time at which the constraint begins to bind and the length of the constrained period, more precisely to determine the coal consumption path leading to $\bar{Z}$ and the duration of the period during which the burned coal rate is equal to $\bar{x}$, and last the coal consumption path once the ceiling constraint may be forgotten. What makes attractive this arbitrage problem is that burning coal may result into more or less useful energy *via* the choice of the transformation rate η_x, a costly

[26]See Corollary 1 of the Proposition 2 for a proof.

device which enters into the determination of the net surplus.

4.1 The social planner problem

In the absence of income effects, the social planner can use the interest rate as a discounting device of the indirect utilities of the end users in a partial equilibrium context. The social planner solves the following problem ($S.P.$):[27]

$$\begin{aligned}
\max_{x,\eta_x,\eta_y} \quad & \int_0^\infty \{u(\eta_x(t)x(t) + \eta_y(t)y^n) - a(X(t))x(t) \\
& -b(\eta_x(t))x(t) - c(\eta_y(t))y^n\}\, e^{-rt}dt \\
s.t. \quad & \dot{X}(t) = -x(t)\ ,\ X(0) = X^0 \ \text{ given} && (4.1)\\
& \dot{Z}(t) = \zeta x(t) - \alpha Z(t)\ ,\ Z(0) = Z^0 < \bar{Z} \ \text{ given} && (4.2)\\
& \text{and } \ \bar{Z} - Z(t) \geq 0 && (4.3)\\
& x(t) \geq 0\ ,\ \eta_x(t) \geq 0 \ \text{ and } \eta_y(t) \geq 0\ .
\end{aligned}$$

Let λ_X and $-\lambda_Z$ be the costate variables associated to the relations (4.1) and (4.2) describing the dynamics of X and Z respectively,[28] ν be the Lagrange multiplier associated to the non-negativity constraint in (4.3). The first order conditions are:

$$\begin{aligned}
u'(\eta_x x + \eta_y y^n)\eta_x &= a(X) + b(\eta_x) + \lambda_X + \zeta\lambda_Z - \gamma_x && (4.4)\\
u'(\eta_x x + \eta_y y^n)x &= b'(\eta_x)x - \gamma_{\eta x} && (4.5)\\
u'(\eta_x x + \eta_y y^n)y^n &= c'(\eta_y)y^n - \gamma_{\eta y}\ , && (4.6)
\end{aligned}$$

together with the usual complementary slackness conditions.

Let $\mu_X^E(t)$ denote the equilibrium level of the mining rent at time t under the transfer schedule $\{T(t), t \geq 0\}$ and $\lambda_X^*(t)$ together with $\lambda_Z^*(t)$ denote respectively the levels of the costate variables at time t along an optimal path. The inspection of (4.4) reveals that any linear unit transfer schedule $\{T(t), t \geq 0\}$ satisfying $\mu_X^E(t) + T(t) = \lambda_X^*(t) + \zeta\lambda_Z^*(t)$ at any time $t \geq 0$ implements the social optimum. Pick $T(t) = \zeta\lambda_Z^*(t)$, $\forall t \geq 0$, as the optimal

[27]We omit the constraints $\eta_x \leq 1$ and $\eta_y \leq 1$ which are never active under A.2 and A.3. Also under A.1 and A.3, the constraint $X(t) \geq 0$ never binds.

[28]Using $-\lambda_Z$ as the co-state variable of Z allows to interpret λ_Z as the shadow marginal cost of Z.

carbon tax rate, then $\mu_X^E(t) = \lambda_X^*(t)$, the mining rent simply identifies with the optimal level of the coal resource scarcity rent under such a tax schedule.

The dynamics of the costate variables when differentiable, must satisfy:

$$\dot{\lambda}_X = r\lambda_X + a'(X)x \tag{4.7}$$

$$\begin{aligned} \dot{\lambda}_Z &= (r+\alpha)\lambda_Z - \nu \ , \\ & \nu \geq 0 \ , \bar{Z} - Z \geq 0 \text{ and } \ \nu \left[\bar{Z} - Z\right] = 0 \end{aligned} \tag{4.8}$$

Last the transversality condition at infinity is:

$$\lim_{t\uparrow\infty} e^{-rt} \left[\lambda_X(t)X(t) + \lambda_Z(t)Z(t)\right] = 0 \ . \tag{4.9}$$

Identifying μ_X and λ_X, (4.7) shows that the coal scarcity rent obeys the same law of motion than the mining rent along the equilibrium path.

As observed before, the carbon regulation makes sense only if the ceiling constraint actually binds. This means that the use of the coal input is initially so large that $Z(t)$ increases over time until it reaches the ceiling level $\bar{Z}$ at a time we denote by $\underline{t}_Z$. Initially $\nu = 0$ since $Z^0 < \bar{Z}$. Thus we get from (4.8):

$$\lambda_Z(t) = \lambda_{Z0}e^{(r+\alpha)t} \ , \ t \leq \underline{t}_Z \ \text{ where } \lambda_{Z0} \equiv \lambda_Z(0) \ . \tag{4.10}$$

Once the ceiling constraint binds, the coal consumption rate must stay constant at the level $\bar{x} = \alpha\bar{Z}/\zeta > 0$. On the other hand, at the end of the coal exploitation period, extraction should end smoothly, $x(t_x^-) = 0$. Since the optimal coal extraction path must be time continuous, we conclude that the ceiling constraint cannot bind until the end of coal exploitation. Thus the last phase of coal exploitation is an unconstrained one (see Appendix A.2 for a formal proof).

Let $\bar{t}_Z < t_x$ be the time at which ends the phase at the ceiling, then once the ceiling constraint is no more active and forever, the shadow marginal cost of the pollution stock is nil:

$$\lambda_Z(t) = 0 \ , \ t \geq \bar{t}_Z \ . \tag{4.11}$$

Denote by $\omega^*(t) \equiv a(X^*(t)) + \lambda_X^*(t) + \zeta\lambda_Z^*(t)$, the optimal level of the full shadow marginal cost of the coal input for the transformation industry evaluated at time t along an optimal path $\{(X^*(t), \lambda_X^*(t), \lambda_Z^*(t)), t \geq 0\}$. If the regulator sets the carbon tax rate $T(t) = \zeta\lambda_Z^*(t)$ at any time t, it implements the optimal climate policy. Hence $p_x^T(t) = \omega^*(t)$ is the coal price level, tax included, which decentralizes the first best optimum. Let $x^*(t)$ and $Z^*(t)$ denote the optimal levels of the coal extraction rate and the pollution stock at any time t. From (4.7), (4.10) and (4.11) we conclude:

Proposition P. 2 *Along the optimal path, during any period of unconstrained coal extraction, the full marginal cost of the coal input of the transformation industry, $\omega^*(t)$, and thus the full price of extracted coal, tax included, implementing the first best optimum, $p_x^T(t)$, are increasing:*

$$\begin{aligned} \forall t \ &: \ x^*(t) > 0 \ \text{and} \ Z^*(t) < \bar{Z} \\ \Longrightarrow \ &\quad \dot{\omega}^*(t) = r\lambda_X^*(t) + \zeta(r+\alpha)\lambda_Z^*(t) = \dot{p}_x^T(t) \ > 0 \ . \end{aligned} \tag{4.12}$$

Instead of a carbon tax, the regulator could set a norm on polluting emissions $\zeta x(t) \leq \alpha\bar{Z}(t)$ at any time t when $Z(t) = \bar{Z}$, while letting free the choice of $\zeta x(t)$ by the energy industry when $Z(t) < \bar{Z}$. Define a perfect foresight equilibrium with an emission norm when at the ceiling as a pair of useful energy price and extracted coal price paths together with a time interval during which the emissions cannot be larger than $\alpha\bar{Z}$, such that, taking as given these prices paths, the time interval and the norm, the extractive industry and the energy transformation industry both deliver profit-maximizing outputs justifying the price paths and satisfying the emission norm during the specified interval. Then an implication of Proposition 2 is that such a perfect foresight equilibrium cannot be optimal.

Corollary 1 *Assume that the pollution stock regulation is set as an emission norm $\alpha\bar{Z}$ when the ceiling constraint binds, then the corresponding perfect foresight equilibrium is not optimal.*

A formal proof is given in Appendix A.3. The intuition behind the result is that, in such an equilibrium, the transformation industry has not to take into account the constraint before the time at which it starts to be active. The transformation industry takes as given the day-to-day price of the extracted coal and the day-to-day price of useful energy and, given those prices, chooses the best (profit-maximizing) transformation rates together with the quantity of useful energy the market is able to absorb at the corresponding given price. This applies in particular up to the time at which the constraint begins to bind. But, along an optimal path, the emissions have to be taxed before hitting the ceiling constraint. This is the missing information that the transformation industry cannot take into account in a perfect foresight competitive equilibrium. Only putting a cap on emissions when attaining the ceiling is a too simple policy instrument inducing a too late mitigation of the emission flow. Although the ceiling constraint would not be violated, the intertemporal profile of the useful energy consumption restrictions would not be optimal. Caps on emissions (equivalently, carbon taxation) should be introduced right from the start up to the time at which the ceiling constraint ceases to be active.

4.1.1 Optimal dynamics: Periods of unconstrained coal extraction

During any unconstrained period, Proposition 2 states that p_x^T, the price of coal, tax included, rises along an optimally regulated equilibrium. We can thus infer from the Proposition 1 that the useful energy price also increases, while the useful energy consumption rate declines together with coal use. A time increasing useful energy price trend spurs more efficiency efforts from the transformation industry, η_x increases through time, and if solar energy is produced in combination with coal energy, η_y also increases. Last the production of useful energy from coal, q_x, constantly decreases with the decline of the coal consumption rate, x, and solar energy takes progressively a larger share of the energy mix when the two sources are simultaneously exploited.

4.1.2 Periods of constrained coal extraction at the pollution stock ceiling

When the ceiling constraint binds, the coal extraction rate must be constant, $x(t) = \bar{x}$. We show that, during such a phase, the price of useful energy is constant and thus the optimal conversion rate of coal η_x is constant, and also the efficiency rate of solar energy, η_y, when both solar and coal are simultaneously exploited.

Since $x(t) = \bar{x}$, the supply function of useful energy is now well defined $\hat{q}^s(p, \bar{x}) = \hat{q}^s_x(p, \bar{x}) + q^s_y(p)$. The useful energy price at the ceiling is the price $\bar{p}$ at which the total supply balances the demand: $q^d(\bar{p}) = \hat{q}^s(\bar{p}, \bar{x})$. We thus conclude that the useful energy price should be constant during any constrained phase of coal extraction. Let $\bar{q}$ denote the equilibrium useful energy consumption rate, $\bar{q} = q^d(\bar{p})$. Since the useful energy price is constant, the optimal conversion rates from any primary energy source should also be constant. Let $\bar{\eta}_x = \eta^e_x(\bar{p})$ denote the constant level of the coal energy conversion rate during a constrained phase. Similarly denote by $\bar{\eta}_y$ the solar energy conversion rate, $\bar{\eta}_y = \eta^e_y(\bar{p})$, if solar energy is produced during the constrained phase. Last, let $\bar{q}_x = \bar{\eta}_x \bar{x}$, and $\bar{q}_y = \bar{\eta}_y y^n$ denote the also constant useful energy consumption rates from the two sources when at the ceiling.

Different cases may appear according to only coal or both coal and solar feed the useful energy needs. In the case illustrated in Figure 1, only coal is exploited when the cap constraint $\bar{Z} - Z \geq 0$ is active. This is possible if first $b'(0^+) < c'(0^+)$ and second if the useful energy demand is not too strong so that the price $\bar{p}$ at which $q^d(\bar{p}) = \hat{q}^s(\bar{p}, \bar{x})$ is lower than $c'(0^+)$. The demand curve, $q^d(p)$, intersects the total supply curve in this part of the curve for which the coal energy supply is equal to the total energy supply, useful solar energy being not competitive at the price $\bar{p}$, thus: $q^d(\bar{p}) = \hat{q}^s_x(\bar{p}, \bar{x})$. The efficiency rate of the coal energy industry is given by $\bar{\eta}_x = \eta_x(\bar{p})$ and $\bar{q} = \bar{q}_x = \bar{\eta}_x \bar{x}$ while $\bar{q}_y = 0$.

Figure 1 about here

In the case illustrated in Figure 2, both coal and solar energies are exploited when the pollution cap constraint is active, while $b'(0^+) < c'(0^+)$ like in the preceding case. Now $\bar{p} > c'(0^+)$. The reason is that the useful energy demand is strong and cannot be fed by the coal industry alone without triggering the competition of solar energy. In Figure 2, absent the solar alternative, the price which would balance the useful energy demand and supply while using only coal would be the price $\bar{p}_x$ at which $\hat{q}^s_x(p, \bar{x}) = q^d(p)$, higher than $c'(0^+)$. Clearly the optimality condition (4.6), here $\bar{p}_x = c'(0^+) - \gamma_{\eta y}$ with $\gamma_{\eta y} \geq 0$, could not be satisfied.

Figure 2 about here

When $c'(0^+) < b'(0^+)$ the solar energy is always exploited, even when the coal industry sector is active. If coal is exploited then, by (4.5), $p = b'(\eta_x)$ with $\eta_x > 0$, so that (4.6): $p = c'(\eta_y)$ may be satisfied only by some $\eta_y > 0$. When the ceiling constraint is active and the coal extraction regime must thus stay at $\bar{x}$, the dispatching between coal useful energy and solar useful energy is illustrated in the Figure 3.

Figure 3 about here

The most striking fact of the ceiling phase is that because the price of useful energy is constant, the conversion rate of the coal industry, η_x, is constant, and that, when the solar energy is exploited, the conversion rate of the solar industry, η_y, is also constant. We now show that the constancy of the useful energy price also implies that the shadow marginal cost of the pollution stock, equivalently the optimal carbon tax, should decline during the ceiling phase.

Since the equilibrium price of useful energy must be constant during the ceiling phase, the full cost of the extracted coal input, tax included, defined by $\bar{p} = p^e(p_x^T)$, is also constant. Denote by $\bar{p}_x^T$ this constant level. Along an optimal path, $p_x^T = \omega^*$ implies that the shadow full marginal cost of the coal input for the coal transformation industry when at the ceiling must also be constant, a constant we denote by $\bar{\omega}^*$. During the ceiling period:

$$\bar{\omega}^* \quad = \quad a(X^*(t)) + \lambda_X^*(t) + \zeta\lambda_Z^*(t) \ .$$

Time differentiating yields:

$$0 \quad = \quad -a'(X^*(t))\bar{x} + \dot{\lambda}_X^*(t) + \zeta\dot{\lambda}_Z^*(t) \ ,$$

where $-a'(X^*(t))\bar{x} + \dot{\lambda}_X^*(t) = r\lambda_X^*(t) > 0$ according to (4.7). Hence:

$$\dot{\lambda}_Z^*(t) \quad = \quad -\frac{r}{\zeta}\lambda_X^*(t) < 0 \ . \tag{4.13}$$

The phase at the ceiling is a phase of decreasing shadow marginal cost of the pollution stock and at the end of the ceiling phase, this shadow cost must be nil.[29] Maintaining a constant rate of coal exploitation, $\bar{x}$, in order to satisfy the carbon cap target, requires a constant coal price, tax included, $\bar{p}_x^T$, all along the ceiling phase. Since the coal market price, $p_x(t)$, continues to grow during the ceiling phase because of the permanent increase of the marginal extraction cost, the optimal regulation policy of the atmospheric carbon stock requires to decrease progressively the optimal carbon tax rate, $T(t) = \zeta\lambda_Z^*(t)$, until the constraint ceases to bind, and forever, the carbon tax rate being nil after $\bar{t}_Z$, the end of the ceiling phase.

We conclude as follows:[30]

Proposition P. 3 *Assume that along the optimal path, there exists a period during which the pollution stock cap $\bar{Z}$ constrains the use of coal, then during such a period:*

a. *The useful energy production q_x and the transformation rate η_x of the coal industry are both constant, together with its extracted coal input level $\bar{x}$.*

b. *When both coal and solar energy are simultaneously exploited, then the production of useful solar energy q_y and its transformation rate η_y are also both constant, so that, whatever the case, $q_y = 0$ or $q_y > 0$, the total consumption of useful energy, q, is constant.*

c. *The optimal carbon tax rate charged for using the coal input in the coal transformation industry, $T = \zeta\lambda_Z^*$, decreases and exactly balances the increase of the extracted coal price p_x, so that the full cost of the extracted coal input, tax included, p_x^T, stays constant within the period.*

[29] Since the constraint $\bar{Z} - Z \geq 0$ is tight, when at the ceiling $\nu(t)$ is positive. From (4.8) and the value of $\dot{\lambda}_Z^*(t)$ given by (4.13), the optimal level of ν, we denote by ν^*, is given by:

$$\nu^*(t) \quad = \quad (r+\alpha)\lambda_Z^*(t) + r\lambda_X^*(t)/\zeta \ > \ 0 \ .$$

[30] At this stage, one may wonder if the model results would be robust to more general costs structures than our linear cost assumptions. We show in Appendix A.4 that the whole qualitative results presented in the Propositions 1 and 3 remain valid when considering cost functions of the form $B(q_x, \eta_x)$, assumed increasing and convex in both q_x and η_x.

4.2 Optimal paths

Assume that (X^0, Z^0) lies in the relevant zone of the plane (X, Z) for the constraint $\bar{Z} - Z \geq 0$ to be violated along the unconstrained equilibrium path described in subsection 3.3.[31] Taking this constraint into account, there exist two types of optimal paths according to the solar energy is exploited or not during the phase at the ceiling. We focus only on the first type of path because exploiting the solar source when at the ceiling is empirically the most probable case.

When solar energy is exploited during the phase at the ceiling and in addition $p(0) < c'(0^+)$, so that initially only coal is exploited, then the optimal path is a five phases path. The price paths of useful energy, $p(t)$, and of extracted coal, $p_x(t)$, are illustrated in Figure 4. The useful energy consumption paths from the two sources are pictured in Figure 5, on the top panel for coal energy and on the bottom panel for the solar one.

Figure 4 about here

Figure 5 about here

Before the ceiling phase, the useful energy price path is defined as $p(t) = [p_x^T(t) + b(\eta_x(t))]/\eta_x(t)$, $t < \underline{t}_Z$. Such a path corresponds to the trajectory (1) in Figure 4. At the ceiling, $p(t) = \bar{p}$. After the ceiling phase, the price path is given by $p(t) = [p_x(t) + b(\eta_x(t))]/\eta_x(t)$, corresponding to the price trajectory (2) in the Figure 4, since then $T(t) = 0$.

Phase 1: Initial pre-ceiling phase of only coal exploitation

The first phase $[0, t_y)$ is a phase of only coal exploitation during which the price of useful energy increases, the consumption of coal energy decreases and

[31]The phase portrait of the stock variables dynamics in the state space (X, Z) for the unconstrained equilibrium is presented in Appendix A.5. It identifies the non empty set of initial endowments (X^0, Z^0) for which the constraint $\bar{Z} - Z \geq 0$ would be violated.

its extraction rate decreases while its conversion rate increases. The phase ends at time t_y when $p(t_y) = c'(0^+)$ and solar energy becomes competitive. The pollution stock increases because $x(t) > \bar{x}$ and $Z(t) < \bar{Z}$, $\dot{Z}(t) > 0$ and at the end of the phase the pollution stock ceiling is not yet reached: $Z(t_y) < \bar{Z}$.[32]

Phase 2: Phase of simultaneous exploitation of coal and solar before the ceiling

The second phase $(t_y, \bar{t}_Z)$ is a phase of simultaneous exploitation of both resources. Again the price of useful energy increases and its consumption decreases. As shown in paragraph 4.1.1 (Proposition 2) the consumption of useful solar energy increases while the consumption of useful coal energy decreases, the optimal conversion rate of the coal industry increases and the coal extraction rate decreases, and the optimal conversion rate of the solar industry also increases.[33]

During the phase, $x(t) > \bar{x}$ and since $Z(t_y) < \bar{Z}$ at its beginning, then the pollution stock increases like in the preceding phase. The phase ends at time $\underline{t}_Z$ when the pollution stock hits the cap $\bar{Z}$ and the extraction rate has decreased down to $\bar{x}$.

Phase 3: Phase of constrained coal extraction

During this phase $[\underline{t}_Z, \bar{t}_Z]$ all the command variables controlled by the regulator through the carbon tax schedule, $T(t)$, that is: x, η_x and η_y, are constant. The phase ends at time $\bar{t}_Z$ when $T(t) = 0$.

[32]It may happen that the initial price $p(0)$ could be larger than $c'(0^+)$ in which case this first phase disappears, hence a four phases path. This would be the case for initial endowments X^0 and Z^0 equal to their levels at any time t^0 between t_y and $\underline{t}_Z$ in the scenario illustrated in Figures 4 and 5. Since the solution of the social planner problem is dynamically consistent, then the optimal paths starting from $X(t^0)$ and $Z(t^0)$ instead of X^0 and Z^0 would be the tails of the illustrated paths starting at t_0.

[33]Note that at t_y, when switching from Phase 1 to Phase 2, $p(t)$ and $p_x(t)$ are time differentiable since $\lambda_X(t)$, $\dot{\lambda}_Z(t)$, $u''(q(t))$ and $\eta_X(t)$ are all continuous time functions. This is also the case for $q(t)$ and $\eta_x(t)$. However, the same arguments developed in the unconstrained equilibrium case (c.f. Proposition 1 and Appendix A.1), show that $q_x(t)$, $q_y(t)$, $\eta_y(t)$ and $x(t)$ are continuous but not time differentiable functions at t_y, as illustrated in Figure 5.

Phase 4: Post-ceiling phase of simultaneous exploitation of coal and solar energies

During this phase $(\bar{t}_Z, t_x)$ the price of useful energy increases again, q, q_x and x decrease while q_y, η_x and η_y increase. The phase ends at the time t_x when the price of useful energy equals $\tilde{p}$ and the production of useful solar energy amounts to $\tilde{q}_y = \tilde{\eta}_y y^n$. Since $\lim_{t \uparrow t_x} q(t) = \lim_{t \uparrow t_x} q_y(t) = \tilde{q}_y$, then $\lim_{t \uparrow t_x} q_x(t) = \lim_{t \uparrow t_x} [q(t) - q_y(t)] = 0$. However $\lim_{t \uparrow t_x} \eta_x(t) = \tilde{\eta}_x > 0$, hence $\lim_{t \uparrow t_x} x(t) = 0$. Thus, the grades $X < \tilde{X} = X(t_x)$ are too costly to be exploited for whatever efficiency rate η_x.[34]

Phase 5: Pure solar economy

The last phase $[t_x, \infty)$ is the phase of permanent pure solar economy: $q(t) = \tilde{q}_y$, $p(t) = \tilde{p}$ and $\eta_y(t) = \tilde{\eta}_y$.

For the sake of completeness, Appendix A.6 presents an algorithmic argument able to determine the optimal scenario of the regulated economy.

5 Concluding remarks

One main conclusion of the study is that the progressive depletion of fossil fuels and/or the implementation of climate change mitigation policies should drive up over time the energy conversion rates, not only for fossil fuels but also for carbon free renewable sources. Our work has also raised the rather paradoxical conclusion that the economy should stop improving the energy conversion rates when being constrained by a global atmospheric carbon stock stabilization objective. It is thus when the climate constraint binds that the improvement efforts are postponed. However the effects of the constraint are not restricted to the phase at the ceiling but are spread over the whole optimal path. Next, tightening the constraint results into higher transformation rates both when at the ceiling and before and after the ceiling phase. Several

[34]Since the phase is an unconstrained one, the results of Proposition 1 hold, showing that $p(t)$, $p_x(t)$, $q_x(t)$ and $x(t)$ are continuous and time differentiable at t_x.

remarks and potential extensions of our work are worth pointing out in that respect.

In our stylized framework, the energy transformation industry is broadly defined as a set of useful energy production activities from two main kinds of crude energy sources, we summarized as 'coal' and 'solar'. Actual industries transform energy from several types of energy sources, fossil or not, mobilizing a large array of techniques, more or less dependent on the characteristics of the energy sources. The energy consumption itself also relies on a variety of techniques to generate useful energy services in transportation, heating, cooling of lightning, for example. Thus any efficiency increase within the transformation chain of energies is included within increases of η_x, η_y, or both in our setting. For example, although efficiency improvements are not yet possible in electricity generation from coal, improvements in the lightning system are captured by increases of η_x and η_y. This is suggesting that in a more detailed model, improvements of the transformation rates, η_x and η_y would be sometimes correlated and sometimes not.

For the sake of simplicity, we have assumed that potentially any level lower than one of the energy conversion rate could be attained with the existing panel of energy transformation techniques. A more realistic approach could be to introduce upper technical limits on conversion rates. Assume for example that the possible conversion rates of coal and solar energy are confined below upper bounds $\bar{\bar{\eta}}_x$ and $\bar{\bar{\eta}}_y$ respectively, with $\bar{\bar{\eta}}_x < 1$ and $\bar{\bar{\eta}}_y < 1$. An immediate consequence of such an assumption is that the rise of the conversion efficiency rates predicted by our model may be technically constrained. In particular, the optimality condition on the choice of conversion rates when both sources are simultaneously used, $b'(\eta_x) = c'(\eta_y)$, may fail to hold over some range of conversion rates.

This multiplies the number of possible kinds of optimal paths according to the type of phase in which the maximum efficiency rate of either coal or solar transformation is attained. For example assume that $b'(\bar{\bar{\eta}}_x) < c'(\bar{\bar{\eta}}_y)$. Then the five phases optimal path described in the previous subsection 4.2 may become a seven phases path. In this new possible optimal scenario, the initial phases below and at the ceiling remain the same as before. But under our cost assumption, the post-ceiling phase before the complete transition toward solar energy may be composed of three successive sub-phases. The first one is an unconstrained phase as previously described until the coal energy conversion rate reaches its upper bound $\bar{\bar{\eta}}_x$. During the second sub-phase, the coal energy conversion rate stays at the level $\bar{\bar{\eta}}_x$ while the solar energy conversion rate continues to rise until the upper limit $\bar{\bar{\eta}}_y$ is attained. During the third sub-phase, the two energy sectors face their respective conversion efficiency constraints. The production of solar energy remains constant at a level $\bar{\bar{q}}_y = \bar{\bar{\eta}}_y y^n$, while the production of coal energy, $\bar{\bar{q}}_x = \bar{\bar{\eta}}_x x$, falls down because of fossil fuels depletion. Note that in such a scenario, the solar energy conversion constraint also limits the useful energy consumption possibilities after the complete transition toward renewable energy.

Other simplifications have been made when modeling the mining sector. For example, petroleum exploitation involves frequently oil recovery processes in order to extend the life duration of the field. This may be accommodated by assuming an average exploitation cost function of the form $a(X, \eta_m)$, where η_m denotes the recovery rate from the resource deposit. It should be expected that η_m increases over time for similar reasons that make rise the energy conversion rates. In contrast with the conversion rates, which stop increasing during the ceiling phase, η_m should continuously rise because of the increasing scarcity of fossil fuels. Exploration and development of new resources is another way to alleviate the scarcity constraint. Assuming that the exploration cost is also a convex function of the exploration effort leads to similar cost structures.[35]

Our analysis has shown that the currently observed increasing long run trend of energy efficiency may be the result of the mere working of market forces, the exhaustion of fossil fuels, but also carbon pollution mitigation policies, creating incentives to raise the energy conversion performance of the transformation industry. The literature has frequently stressed the role of technical progress to explain this trend. A quite natural extension of the present model would be to introduce exogenous technical progress, by making the transformation costs functions explicitly time dependent, with $\partial b(\eta_x, t)/\partial t \leq 0$ and $\partial c(\eta_y, t)/\partial t \leq 0$. Endogenous technical change settings could also be considered, raising the issue of the optimal direction of research efforts either toward cost reductions in fossil fuels energy conversion or in solar energy conversion.

[35] See Gaudet and Lasserre (1988) for a study of the consequences of exploration over the management of a non-renewable resource, both in a competitive and a monopoly context.

No rebound effect arises in the present model although the efficiency rates η_x and η_y steadily increase so long as the transition toward a totally clean economy is not brought to an end (excepted during the phase at the ceiling).[36] The reason is that the improvements of these efficiency rates do not reduce the full marginal cost of useful energy even without pollution damages as shown in Sub-section 3.3. The marginal transformation costs increase as long as coal is exploited and also the marginal cost of extracted coal. Thus when facing a stationary useful energy demand, the consumption of useful energy necessarily decreases while the share of solar energy increases. Hence the production of useful coal energy decreases and since the efficiency rate η_x increases, the production of extracted coal decreases.

Apart from technical progress considerations, three routes toward a 'greener' useful energy production economy are currently considered in the climate debate. The first one is the improvement of energy conversion performances, the second one the substitution of 'dirty' primary energy sources by 'clean' renewable energies, the third one is the abatement of polluting emissions and their sequestration inside underground reservoirs. The present work has described the optimal mix between the two first options. Introducing the third one inside the model would permit to draw an almost complete view of the energy production 'greening' problem.

[36] That energy efficiency improvements could trigger a bound in energy consumption was first pointed out by Jevons (1865) and is also known as the Jevons paradox. The effect was recently rediscovered by Brookes (1978) and Khazzoom (1980). See Gavankhar and Geyer (2010) and Sorrel (2014) for recent surveys.

References

Brookes L. G., (1978), The energy price fallacy and the role of nuclear energy in the U.K., *Energy Policy*, 6, 94-106.

Brookes L. G., (2000), Energy efficiency fallacies revisited, *Energy Policy*, 28, 355-366.

Carnot S., (1824), *Réflexions sur la puissance motrice du feu sur les machines propres à développer cette puissance*, Bachelier, Paris.

Chakravorty U., Magné B., and M. Moreaux, (2006), A Hotelling model with a ceiling on the stock of pollution, *Journal of Economic Dynamics and Control*, 30, 2875-2904.

Chakravorty U., Moreaux M., and M. Tidball, (2008). Ordering the extraction of polluting nonrenewable resources, *American Economic Review*, 98, 1128-1144.

Farzin Y. H., (1996), Optimal pricing of environmental and natural resource use with stock externalities, *Journal of Public Economics*, vol. 62(1-2), 31-57.

Farzin Y. H. and O. Tahvonen, (1996), Global carbon cycle and the optimal time path of a carbon tax, *Oxford Economic Papers*, 48(4), 515-536.

Forster B. A., (1975), Optimal pollution control with a non constant exponential rate of decay, *Journal of Environmental Economics and Management*, Volume 2, Issue 1, 1-6.

Gaudet G. and P. Lasserre, (1988), On comparing monopoly and competition in exhaustible resource exploitation, *Journal of Environmental Economics and Management*, 15(4), 412-418.

Gavankar S. and R. Geyer, (2010), The rebound effect: State of the debate and implications for energy efficiency research, *mimeo*, Bren School of Environmental Science and Management, University of California, Santa Barbara.

Golosov M., Hassler J., Krusell P. and A. Tsyvinski, (2014), Optimal tax on fossil fuel in general equilibrium, *Econometrica*, 82(1), 41-88.

Grafton R. Q., Kompass T. and N. G. Long, (2012), Substitution between biofuel and fosill fuels: Is there a green paradox? *Journal of Environmental Economics and Management*, 64(3), 328-341.

Hanson, D., (1980), Increasing extraction costs and resource prices: Some further results, *Bell Journal of Economics*, 11(1), 335-345.

Hassler J., Krusell P. and C. Olovsson, (2012), Energy-saving technical change, WP NBER 18456.

Heal, G., (1976), The relationship between price and extraction cost for a resource with a backstop technology, *Bell Journal of Economics*, 7(2), 371-378.

Jevons W. S., (1865), *The coal question; An inquiry concerning the progress of the Nation, and the probable exhaustion of our coal mines*, Macmillan and Co, London.

Khazzoom J. R., (1980), Economic implications for mandated efficiency in standards for households appliances, *The Energy Journal*, 1(4), 21-40.

Kümmel R., (2011), *The second law of economics*, Springer, Heidelberg.

Marsden B., (2002), *Watt's perfect engine. Steam at the age of invention*, Icon Books, Duxford.

Mason C. F. and N. Wilmot, (2015), Modeling damages in climate policy models: Temperature-based or Carbon-based?, CESifo WP 5287.

Nordhaus W., (2008), *A question of balance: Weighting the options on global warming policies*, Yale University Press, New Haven, CT.

Sorrell S., (2014), Energy substitution, technical change and rebound effects, *Energies*, 7, 2850-2873.

Smulders S., Tsur Y. and A. Zemel, (2012), Announcing climate policy: Can a green paradox arise without scarcity, *Journal of Environmental Economics and Management*, 64(3), 364-376.

Stern N., (2007), *The economics of climate change: The Stern review*, Cambridge University Press, Cambridge UK.

Tahvonen O. and J. Kuuluvainen, (1991), Optimal growth with renewable resources and pollution, *European Economic Review*, 35(2-3), 650-661.

Tahvonen O. and S. Salo, (1996), Non convexities in optimal pollution accumulation, *Journal of Environmental Economics and Management*, 31 (2), 160-177.

Tahvonen O. and C. Withagen, (1996), Optimality of irreversible pollution accumulation, *Journal of Economic Dynamics and Control*, vol. 20(9-10), 1775-1795.

Toman M.A. and C. Withagen, (2000), Accumulative pollution, clean technology and policy design, *Resource and Energy Economics*, 22, 367-384.

Van der Ploeg F. and C. Withagen, (2014), Growth, renewables and the optimal carbon tax, *International Economic Review*, 55(2), 283-311.

Weitzman M., (2010), What is the 'damages function' for global warming - and what differences might it makes, *Climate Change Economics*, 1(01), 57-69.

Withagen C., (1994), Pollution and exhaustibility of fossil fuels, *Resource and Energy Economics*, 16(3), 235-242.

Appendix

A.1 Differentiability of the useful energy price trajectory

Consider first what happens when the solar energy becomes competitive at a time $t_y > 0$. At any time before the end of coal extraction, we have from (3.5):

$$p(t)\eta_x(t) \quad = \quad p_x(t) + b(\eta_x(t)) \quad , \; t < t_x \; .$$

Time differentiating at times $t \neq t_y$ at which $\dot{p}(t)$ is well defined, we get since $p - b' = 0$ by (3.6):

$$\dot{p}(t) \quad = \quad \dot{p}_x(t)/\eta_x(t) \quad , \; t \neq t_y \;\text{ and } t < t_x \; .$$

Because $\eta_x(t) = \eta_x^e(p(t))$ and $p(t)$ is continuous, then $\eta_x(t_y)$ is well defined and either both $p(t)$ and $p_x(t)$ are time differentiable at t_y, or the both derivatives jump and their proportional jumps are equal, $\dot{p}(t_y^-)/\dot{p}(t_y^+) = \dot{p}_x(t_y^-)/\dot{p}_x(t_y^+)$. But since $\mu_X(t)$ must be continuous, then from (3.20), $\dot{p}_x(t)$ is well defined at $t = t_y$, hence also $\dot{p}(t)$.

Although both $\dot{p}(t)$ and $\dot{p}_x(t)$ are well defined, $q_x(t)$, $q(t)$ and $x^d(t)$ are not time differentiable at t_y. Because $\dot{p}(t_y)$ is well defined then $\dot{\eta}_x(t_y)$ is also well defined: $\dot{\eta}_x(t_y) = \dot{p}(t_y)/b''(\eta_x(t_x))$ by (3.6). Also, time differentiating (3.7), we get:

$$\dot{\eta}_y(t_y^+) \equiv \lim_{t \downarrow t_y} \frac{\dot{p}(t)}{c''(\eta_y(t))} = \frac{\dot{p}(t_y)}{c''(0^+)} > 0 \; .$$

Hence:

$$\dot{q}_x(t_y^+) \quad = \quad \dot{q}(t_y) - \dot{q}_y(t_y^+) < \dot{q}(t_y) = \dot{q}_x(t_y^-) \; .$$

Thus $q_x(t)$ is not differentiable at $t = t_y$. Since $q_x(t) = \eta_x(t)x^d(t)$ for $t \neq t_y$, time differentiating, we obtain:

$$\dot{x}^d(t_y^-) \quad = \quad \frac{\dot{q}_x(t_y^-) - \dot{\eta}_x(t_y)x^d(t_y)}{\eta_x(t_y)} \quad > \quad \frac{\dot{q}_x(t_y^+) - \dot{\eta}_x(t_y)x^d(t_y)}{\eta_x(t_y)} = \dot{x}^d(t_y^+) \; .$$

However $x^d(t)$ is continuous at t_y because both $q_x(t)$ and $\eta_x(t)$ are continuous. Hence from (3.2): $\dot{\mu}_X(t_y)$ is also well defined.

Consider now the time t_x at which ends coal exploitation. Since $p(t)$ is continuous and tends to $\tilde{p}$ at t_x, then $q_y(t) = \eta_y(t)y^n$ tends to $\tilde{q} = \eta_y(\tilde{p})y^n$ hence $q_x(t) = q(t) - q_y(t)$ tends to 0. Also since $p(t)$ is increasing, then $\dot{p}(t_x) = 0 = p(t_x^+)$ and $\dot{p}_x(t_x^-) = 0$, and $\dot{q}(t_x^-) = \dot{q}_x(t_x^-) = \dot{q}_y(t_x^-) = 0$.

A.2 Proof that the last phase of coal exploitation is an unconstrained one

Assume to the contrary that this last period is a constrained one. Then $x(t) = \bar{x}$ during the phase. Let $\bar{p}$ be the useful energy price clearing the useful energy market: $q^d(\bar{p}) = \hat{q}^s(\bar{p}, \bar{x})$. Since the useful energy price path is continuous, that implies that $\bar{p} = \tilde{p}$, where $\tilde{p}$ is the useful energy price which allows the solar industry to supply the whole market, hence we should have $q(t) = \tilde{q} = q_y(t)$ that is $x(t) = 0$ during the period, a contradiction.

Now assume that $x(t_x^-) > 0$ hence $q(t_x^-) > 0$, then we should have $q_y^s(t_x^-) < \tilde{q}$ since $p(t)$ is continuous at t_x and $q_y^s(t_x^-) = \tilde{q}$. Thus the function $q_y^s(p)$ should be discontinuous at $p = \tilde{p}$. But we have shown in appendix A.1 that it is differentiable. ∎

A.3 Proof of the corollary 1 of Proposition 2

We show that assuming that a perfect foresight equilibrium with an emission norm $\alpha\bar{Z}$ once at the ceiling is an optimal path implies a contradiction.

In the perfect foresight competitive equilibrium, the coal extraction sector solves the problem laid down in sub-section 3.1. Profit maximizing conditions are:

$$p_x(t) = a(X(t)) + \mu_X(t) \qquad \text{(A.3.1)}$$

$$\dot{\mu}_X(t) = r\mu_X(t) + a'(X(t))\, x^s(t) \qquad \text{(A.3.2)}$$

The first relation (A.3.1) is just (3.1) while (A.3.2) is (3.2). Remember that the extracted coal supply level is indeterminate, the extractive industry being ready to produce any quantity demanded by the transformation industry provided that (A.3.1) and (A.3.2) be satisfied. Thus, at any time t the transformation industry would have to satisfy the emission norm, $x^s(t) = \bar{x}$ on the perfect foresight equilibrium path.

Consider a time interval (t_1, t_2) preceding the arrival at the ceiling along the equilibrium path. If this path is optimal then $0 \leq t_1 < t_2 < \underline{t}_Z$, where $\underline{t}_Z$ is the optimal arrival time at the ceiling. The alleged optimality of the equilibrium path also implies that the following should hold within the time interval:

$$p_x(t) = p_x^T(t) = \omega^*(t) = a(X^*(t)) + \lambda_X^*(t) + \zeta\lambda_Z^*(t)\ , \ t \in (t_1, t_2)\ . \qquad \text{(A.3.3)}$$

The first equality comes from the fact that there is no taxation along the equilibrium path, hence the price of coal, tax included, for the transformation sector is the price of extracted coal for the extraction industry. The other equalities are just the optimality conditions having to hold along the equilibrium path. In order that (A.3.1) and (A.3.3) be simultaneously satisfied, we must have:

$$\mu_X(t) = \lambda_X^*(t) + \zeta\lambda_Z^*(t)\ . \qquad \text{(A.3.4)}$$

Time differentiating and using (4.12) of Proposition 2, in virtue of the alleged optimality of the equilibrium path, yields:

$$\dot{\mu}_X(t) = \rho\lambda_X^*(t) + \zeta(\rho+\alpha)\lambda_Z^*(t) = \rho\mu_X(t) + \zeta\alpha\lambda_Z^*(t) > \rho\mu_X(t) = r\mu_X(t)\ .$$

The third equality comes from (A.3.4) while the last one results from the assumption that the social planner discounts welfare at the interest rate $(\rho = r)$. On the other hand, according to (A.3.2):

$$\dot{\mu}_X(t) = r\mu_X(t) + a'(X^*(t))\, x^*(t) < r\mu_X(t)\ ,$$

hence a contradiction. ■

A.4 More general cost structures

We focus here on the possibility of decreasing returns in coal energy generation. The same issue in solar energy production has little interest in the present model, since the supply of this energy has been assumed to be fixed at the level y^n. Instead of a coal energy transformation cost function $b(\eta_x)x$, take the more general form $B(q_x, \eta_x)$. Assume that $B(q_x, \eta_x)$ is increasing and convex in both q_x and η_x:

$$\begin{aligned}
\frac{\partial B(q_x,\eta_x)}{\partial q_x} > 0 \quad &; \quad \frac{\partial^2 B(q_x,\eta_x)}{\partial q_x^2} > 0 \\
\frac{\partial B(q_x,\eta_x)}{\partial \eta_x} > 0 \quad &; \quad \frac{\partial^2 B(q_x,\eta_x)}{\partial \eta_x^2} > 0 \\
\frac{\partial^2 B(q_x,\eta_x)}{\partial q_x^2}\frac{\partial^2 B(q_x,\eta_x)}{\partial \eta_x^2} &- \left[\frac{\partial^2 B(q_x,\eta_x)}{\partial q_x \partial \eta_x}\right]^2 \geq 0 \\
\frac{\partial^2 B(q_x,\eta_x)}{\partial q_x \partial \eta_x} &\geq \left(\frac{1}{q_x}\right)\frac{\partial B(q_x,\eta_x)}{\partial \eta_x} > 0 \; . \qquad \text{(A.4.1)}
\end{aligned}$$

This assumption states that the industry faces decreasing returns to scale in coal transformation. The marginal cost of rising the conversion rate increases with the operating scale of coal energy processing, reflecting the increasing difficulties to deploy more efficient techniques when the coal energy transformation scale is larger. Moreover, we assume that an increase of transformation efficiency has a larger positive effect over the marginal cost $\partial B/\partial q_x$ than on the average cost B/q_x so that $(\partial/\partial\eta_x)\,\partial B/\partial q_x \geq (\partial/\partial\eta_x)\,(B/q_x)$.

Since $x = q_x/\eta_x$, the profit maximization criterion of the transformation industry may be written as:

$$\max_{\eta_x,\eta_y,q_x} \quad p(q_x + \eta_y y^n) - p_x\frac{q_x}{\eta_x} - B(q_x,\eta_x) - c(\eta_y)y^n - T(\zeta x, t) \; .$$

Since nothing is changed in the optimization problem with respect to solar energy, we can infer that the profit maximizing efficiency rate of solar energy is still given by (3.11) while the solar energy supply function is still defined by (3.12).

The market clearing condition on the useful energy market writes:

$$q^d(p) \;\; = \;\; q_x + q_y^s(p) \; .$$

This defines implicitly q_x as a function of p. Let $q^e_x(p)$ denote this function. Differentiating yields:

$$\frac{dq^e_x(p)}{dp} = q^{d'}(p) - q^{s'}_y(p) < 0 \; , \tag{A.4.2}$$

since $q^{d'}(p) < 0$ and $q^{s'}_y(p) > 0$. Denote by $p^e(q_x)$ the inverse function. Let $p^T_x = p_x + \zeta \partial T/\partial x$. The profit maximization condition w.r.t. η_x writes:

$$\frac{\eta_x^2}{q_x}\frac{\partial B}{\partial \eta_x} = p^T_x \; . \tag{A.4.3}$$

On the other hand, the profit maximization condition w.r.t. q_x writes:

$$\left(p^e(q_x) - \frac{\partial B}{\partial q_x}\right)\eta_x = p^T_x \; . \tag{A.4.4}$$

To simplify notations, let $p' \equiv dp^e(q_x)/dq_x < 0$. Differentiating (A.4.3)-(A.4.4) yields in matrix form:

$$\begin{bmatrix} \eta_x \dfrac{\partial^2 B}{\partial q_x \partial \eta_x} - \dfrac{p^T_x}{\eta_x} & 2\dfrac{\partial B}{\partial \eta_x} + \eta_x \dfrac{\partial^2 B}{\partial \eta_x^2} \\ \left[p' - \dfrac{\partial^2 B}{\partial q_x^2}\right]\eta_x & \dfrac{p^T_x}{\eta_x} - \eta_x \dfrac{\partial^2 B}{\partial q_x \partial \eta_x} \end{bmatrix} \begin{bmatrix} dq_x \\ d\eta_x \end{bmatrix} = \begin{bmatrix} \dfrac{q_x}{\eta_x} \\ 1 \end{bmatrix} dp^T_x$$

Let D be the determinant of this system.

$$D = -\left[\frac{p^T_x}{\eta_x} - \eta_x \frac{\partial^2 B}{\partial q_x \partial \eta_x}\right]^2 - \left(p' - \frac{\partial^2 B}{\partial q_x^2}\right)\left[2\frac{\partial B}{\partial \eta_x} + \eta_x \frac{\partial^2 B}{\partial \eta_x^2}\right]\eta_x \; .$$

Rearranging and taking (A.4.1) and (A.4.3) into account yields:

$$\begin{aligned} D &= -p'\left[2\frac{\partial B}{\partial \eta_x} + \eta_x \frac{\partial^2 B}{\partial \eta_x^2}\right]\eta_x \\ &\quad + \eta_x^2\left[\frac{\partial^2 B(q_x,\eta_x)}{\partial q_x^2}\frac{\partial^2 B(q_x,\eta_x)}{\partial \eta_x^2} - \left(\frac{\partial^2 B(q_x,\eta_x)}{\partial q_x \partial \eta_x}\right)^2\right] \\ &\quad + 2\frac{\partial^2 B}{\partial q_x^2}\frac{\partial B}{\partial \eta_x}\eta_x + p^T_x\left[2\frac{\partial^2 B}{\partial q_x \partial \eta_x} - \left(\frac{1}{q_x}\right)\frac{\partial B}{\partial \eta_x}\right] \\ &> 0 \; . \end{aligned}$$

Next, applying the Cramer rule:

$$\begin{aligned}
\frac{dq_x}{dp_x^T} &= \frac{1}{D}\left\{q_x\left(\left(\frac{1}{q_x}\right)\frac{\partial B}{\partial \eta_x} - \frac{\partial^2 B}{\partial q_x}\partial\eta_x\right) - \left(2\frac{\partial B}{\partial \eta_x} + \eta_x\frac{\partial^2 B}{\partial \eta_x^2}\right)\right\} < 0 \\
\frac{d\eta_x}{dp_x^T} &= \frac{1}{D}\left\{\eta_x\frac{\partial^2 B}{\partial q_x \partial \eta_x} - \left(\frac{1}{q_x}\right)\frac{\partial B}{\partial \eta_x} - q_x\left(p' - \frac{\partial^2 B}{\partial q_x^2}\right)\eta_x\right\} > 0 \ .
\end{aligned}$$

Since $x = q_x/\eta_x$, we conclude that:

$$\frac{dx}{dp_x^T} = \frac{1}{\eta_x^2}\left[\frac{dq_x}{dp_x^T}\eta_x - q_x\frac{d\eta_x}{dp_x^T}\right] < 0 \ .$$

Last $q_x^e(p) = q_x$ defines implicitly a relationship between p and p_x^T such that:

$$\frac{dp}{dp_x^T} = \frac{dq_x/dp_x^T}{dq_x^e(p)/dp} > 0 \ .$$

The coal conversion rate is increased by an higher coal input price level not only because of the induced increase of the useful energy price, p, but also because of the reduction of the scale of coal processing induced by the fall of x, the rate of coal use in the transformation industry. Concerning solar energy generation, the increase of the useful energy price induces a parallel increase of $\eta_y = \eta_y^e(p)$.

Considering the equilibrium dynamics, with or without carbon regulation, we thus conclude that the whole qualitative features of the variables dynamics during any unconstrained phase, summarized in the Proposition 1, are conserved under a convex transformation cost structure. It is also immediate that when the economy is constrained by the carbon pollution ceiling, the coal use rate being fixed to the level $\bar{x}$, the same applies to η_x and η_y, hence the qualitative results summarized in the Proposition 3 remain valid.

A.5 Geometry of the phase plane (X, Z)

We here describe the dynamics of the state variables X and Z in the unconstrained equilibrium case. Without an environmental constraint, $T(\zeta x, t) = 0$

and $p_x^T(t) = p_x(t)$. The equilibrium trajectories of X and p_x are solutions of the following autonomous dynamic system:

$$\begin{aligned}\dot{X}(t) &= -x^e(p_x(t)) \\ \dot{p}_x(t) &= r\left(p_x(t) - a(X(t))\right) .\end{aligned}$$

With the particular solution at t_x, $X(t_x) = \tilde{X}$ and $p_x(t_x) = a(\tilde{X})$, this system defines the unconstrained equilibrium trajectory $\{(X(t), p_x(t)), 0 \leq t \leq t_x\}$. Since $p_x(t) - a(X(t)) = \mu_X(t) > 0$, $t \in [0, t_x)$, we conclude that $\dot{p}_x(t) > 0$ while $\dot{X}(t) < 0$ within the time interval $[0, t_x)$.

Denote by $p_x = p_x^E(X)$ the relationship implicitly defined between p_x and X along the equilibrium trajectory. Note that $dp_x^E(X)/dX = \dot{p}_x/\dot{X} < 0$, $p_x^E(X)$ is a decreasing function of X. The dynamics of $Z(t)$ along the unconstrained trajectory is given by:

$$\dot{Z}(t) = \zeta x^e(p_x^E(X(t))) - \alpha Z(t) .$$

The locus $\dot{Z} = 0$ is defined by $Z^Z(X) \equiv \zeta x^e(p_x^E(X))/\alpha$. Since $x^e(p_x)$ is a decreasing function of p_x and $p_x^E(X)$ is a decreasing function of X:

$$\frac{dZ^Z(X)}{dX} = \frac{\zeta}{\alpha}\frac{dx^e(p_x)}{dp_x} \cdot \frac{dp_x^E(X)}{dX} > 0 .$$

Fixing X and increasing slightly Z above the curve $Z^Z(X)$, $\dot{Z}$ becomes negative. Thus, in the phase plane (X, Z), $Z(t)$ decreases through time above the curve $Z^Z(X)$, while it increases below the curve.

The phase diagram in the (X, Z) plane is pictured in the Figure 6. Depending on X^0 and Z^0, different types of unconstrained equilibrium trajectories emerge. All trajectories should converge toward the $X = \tilde{X}$ vertical in finite time. Starting from an initial pair (X^0, Z^0) located above the $\dot{Z} = 0$ locus, both $X(t)$ and $Z(t)$ decrease through time. The trajectory illustrated in the Figure 6 starts from (X^0, Z^0) located below the $\dot{Z} = 0$ locus. Such trajectories correspond to two phases paths. During a first time phase, $Z(t)$ increases while $X(t)$ decreases as the trajectory approaches the $\dot{Z} = 0$ locus. During the second time phase, the trajectory having crossed the $\dot{Z} = 0$ border at a point we denote by (X_m, Z_m), $Z(t)$ decreases down to $\tilde{Z}$ while $X(t)$ decreases down to $\tilde{X}$.

Figure 6 about here

Note that because of the extraction cost structure, $x(t) \to 0$ implies that before the closure of coal exploitation, the carbon pollution stock must begin to decrease for any vector of initial conditions (X^0, Z^0). This feature is absent from models with constant marginal extraction costs, positive carbon pollution accumulation being possible until the complete transition toward solar energy.

The equilibrium trajectory is unique in the phase plane and trajectories solution of the differential system cannot cross themselves in the phase plane. This implies that to any initial pair (X^0, Z^0) located below the $\dot{Z} = 0$ locus is associated a unique turning point (X_m, Z_m) lying along the curve $Z^Z(X)$ and $Z_m = Z^Z(X_m)$. Denote by $X_m(X^0, Z^0)$ and $Z_m(X^0, Z^0)$ the relationships between (X_m, Z_m) and (X^0, Z^0).

Since trajectories cannot cross in the phase plane, $\partial X_m/\partial X^0 > 0$ and $\partial X_m/\partial Z^0 > 0$. Since $Z^Z(X)$ is an increasing function of X, this implies that $\partial Z_m(X^0, Z^0)/\partial X^0 > 0$ and $\partial Z_m(X^0, Z^0)/\partial Z^0 > 0$.

Now consider a given level of Z_m denoted by $\bar{Z}_m$. Then $\bar{Z}_m = Z_m(X^0, Z^0)$ defines implicitly a relationship between X^0 and Z^0. Denote by $\bar{Z}^0_m(X^0)$ this relationship. Note that $d\bar{Z}^0_m(X^0)/dX^0 = -(\partial Z_m/\partial X^0)/(\partial Z_m/\partial Z^0) < 0$. Fix X^0 and increase slightly Z^0 above the curve $\bar{Z}^0_m(X^0)$ then Z_m should be increased. This implies that to larger levels of Z_m correspond higher iso-$\bar{Z}_m$ curves $\bar{Z}^0_m(X^0)$.

The previous characterization allows separating the set of initial endowment pairs (X^0, Z^0) such that the unconstrained equilibrium would never meet the atmospheric carbon cap from the set of initial pairs that should violate the constraint. Consider the particular $\bar{Z}_m$ level $\bar{Z}$ and by a slight abuse of notation, let $\bar{Z}^0(X^0)$ denote the corresponding critical curve $\bar{Z}^0_m(X^0)$ when $Z_m = \bar{Z}$ in the (X, Z) plane. Then, it can be concluded from the previous discussion that for initial endowments vectors (X^0, Z^0) located below the critical curve $\bar{Z}^0(X^0)$, the unconstrained equilibrium trajectory never attains the $\bar{Z}$ level. Such vectors lie in the Zone I on Figure 6. On the other hand, for trajectories initiated in Zone II, like the one starting from $(X^0, Z^{0'})$ in Figure 6, the ceiling constraint should be violated along the equilibrium unconstrained path. The zone II thus defines the set of initial endowments vectors (X^0, Z^0) corresponding to an active carbon constraint situation.

A.6 Determination of the optimal policy

Consider the five phases optimal path described in sub-section 4.2. Denote by $T^p \equiv t_x - \bar{t}_Z$, the time duration of the first post-ceiling phase 4, by $T^c \equiv \bar{t}_Z - \underline{t}_Z$, the time length of the ceiling phase 3 and by $T^a \equiv \underline{t}_Z$ the time length of the pre-ceiling phase, the union of the phases 1 and 2. The optimal path of the regulated economy may be determined through the following backward induction algorithm.

During the phase 4, the optimal trajectories of $X(t)$ and p_x^T are solution of the following autonomous differential system:

$$\begin{aligned} \dot{X}(t) &= -x^e(p_x^T(t)) \\ \dot{p}_x^T(t) &= r\left(p_x^T(t) - a(X(t))\right) . \end{aligned} \tag{A.6.1}$$

With the particular solution $(\tilde{X}, a(\tilde{X}))$ at time t_x, the end of the coal exploitation phase, the system (A.6.1) defines a unique trajectory $\{(X(t), p_x^T(t)), \bar{t}_Z \leq t \leq t_x\}$. Let $p_x^{T*}(X)$ denote the implicit relation so defined in the phase plane (X, p_x^T). Since $p_x^T(t) - a(X(t)) = \lambda_X(t) \geq 0$ and > 0 if $t < t_x$, we conclude that $\dot{p}_x^T(t) > 0$ over the open time interval $(\bar{t}_Z, t_x)$. In the phase plane, the derivative of $p_x^{T*}(X)$ is given by $\dot{p}_x^T/\dot{X} < 0$. Hence the graph of the optimal trajectory in the phase plane is represented by a decreasing function of X. Last, remember that since the economy is no more constrained by the carbon ceiling during the phase 4, $\lambda_Z = 0$ entails $p_x^T(t) = p_x(t)$.

At the beginning of the phase 4, that is at time $\bar{t}_Z$, the condition $x^e(p_x^T) = \bar{x}$ defines a unique level of p_x^T that we denote by $\bar{p}_x^T$. Remember that $\lambda_Z(\bar{t}_Z)$ being nil, $\bar{p}_x^T = p_x^T(\bar{t}_Z) = p_x(\bar{t}_Z)$. Then $p_x^{T*}(X) = \bar{p}_x^T$ defines the coal grade $\bar{X}_Z = X(\bar{t}_Z)$ at $\bar{t}_Z$. Furthermore, the relation $\bar{p}_x^T = a(\bar{X}_Z) + \lambda_X(\bar{t}_Z)$ defines $\bar{\lambda}_X = \lambda_X(\bar{t}_Z)$. Last, the time needed to move the system from the vector $(\bar{X}_Z, \bar{p}_x^T)$ toward its final position at t_x, $(\tilde{X}, a(\tilde{X}))$, is given, the differential system being time autonomous. Thus T^p is determined.

During the ceiling phase 3, $p_x^T(t)$ is maintained at the constant level $\bar{p}_x^T$. The dynamics of $(X(t), p_x(t))$ are defined by the following autonomous time differential system with the particular solution at $\bar{t}_Z$, $(\bar{X}_Z, \bar{p}_x^T)$:

$$\begin{aligned}\dot{X}(t) &= -\bar{x} \\ \dot{p}_x(t) &= r\left[p_x(t) - a(X(t))\right] .\end{aligned}$$

Let $p_x^*(X)$ denote the implicit relationship between p_x and X corresponding to the graph of the optimal trajectory in the phase plane (X, p_x). Since $\dot{p}_x = r\lambda_X > 0$ during the ceiling phase, $p_x^*(X)$ is a decreasing function of X.

A given time length of the ceiling phase, T^c, defines the vector $(\underline{X}_Z(T^c), \underline{p}_x(T^c))$ where $\underline{X}_Z(T^c) = X(\underline{t}_Z)$ and $\underline{p}_x(T^c) = p_x(\underline{t}_Z)$. On the one hand: $\underline{X}_Z(T^c) = \bar{X}_Z + T^c\bar{x}$. On the other hand, applying the change of variables: $\varphi(\theta) = \theta + \underline{t}_Z$:

$$\underline{p}_x(T^c) = \bar{p}_x^T e^{-rT^c} + r\int_0^{T^c} a(\bar{X}_Z + \bar{x}(T^c - \theta))e^{-r\theta}d\theta .$$

$\underline{X}_Z(T^c)$ is an increasing function of T^c while:

$$\begin{aligned}\frac{d\underline{p}_x(T^c)}{dT^c} &= -r\left(\bar{p}_x^T - a(\bar{X}_Z)\right)e^{-rT^c} \\ &\quad + r\int_0^{T^c} a'(X(T^c - \theta))\bar{x}e^{-r\theta}d\theta < 0 ,\end{aligned}$$

$a'(X)$ being negative under A.1.

The condition $\bar{p}_x^T = \underline{p}_x(T^c) + \zeta\lambda_Z(\underline{t}_Z)$ then defines $\underline{\lambda}_Z(T^c) = \lambda_Z(\underline{t}_Z)$, an increasing function of T^c, $\underline{p}_x(T^c)$ being itself a decreasing function of T^c. For a given T^a, the time length of the pre-ceiling phase covering the phases 1 and 2, this defines $\lambda_Z(t)$ before the carbon constraint begins to bind.

$$\lambda_Z(t) = \underline{\lambda}_Z(T^c)e^{-(r+\alpha)(T^a - t)} \equiv \lambda_Z(t; T^a, T^c) .$$

It is immediate that $\lambda_Z(t)$ is an increasing function of T^c and a decreasing function of T^a.

Next, $X(t)$ and $p_x^T(t)$ are solutions of the following non autonomous differential system before $\underline{t}_Z$:

$$\begin{aligned}\dot{X}(t) &= -x^e(p_x^T(t)) \\ \dot{p}_x^T(t) &= r\left[p_x^T(t) - a(X(t))\right] + \zeta\alpha\lambda_Z(t; T^a, T^c) .\end{aligned}$$

With the particular solution $(\underline{X}_Z(T^c), \bar{p}_x^T)$, this system defines a unique trajectory of X and p_x^T for any given pair (T^a, T^c). Let $\{(X(t; T^a, T^c), p_x^T(t; T^a, T^c)), 0 \le t \le T^a\}$ be the corresponding trajectory. Then the initial condition: $X^0 = X(0, T^a, T^c)$ defines an implicit relation between T^a and T^c, a relation we denote $T^c(T^a)$. This relation defines in turn $p_x^T(t, T^a) \equiv p_x^T(t; T^a, T^c(T^a))$ and thus $x(t, T^a) \equiv x^e(p_x^T(t; T^a))$. Last, the ceiling attainment condition:

$$\bar{Z} e^{\alpha T^a} = Z^0 + \zeta \int_0^{T^a} x(t, T^a) e^{\alpha t} dt ,$$

determines T^a. Thus $\underline{t}_Z = T^a$ is determined, together with $\bar{t}_Z = T^c(T^a) + T^a$ and $t_x = T^p + T^c(T^a) + T^a$. On the other hand: $\lambda_Z(0) = \underline{\lambda}_Z(T^c(T^a)) e^{-(r+\alpha)T^a}$ is determined together with $\lambda_X(0) = p_x^T(0, T^a) - a(X^0) - \zeta \lambda_Z(0)$. Last $X(\underline{t}_Z) = \underline{X}_Z(T^c(T^a))$ and t_y is the unique solution of $c'(0^+) = p^e(p_x^T(t; T^a))$.

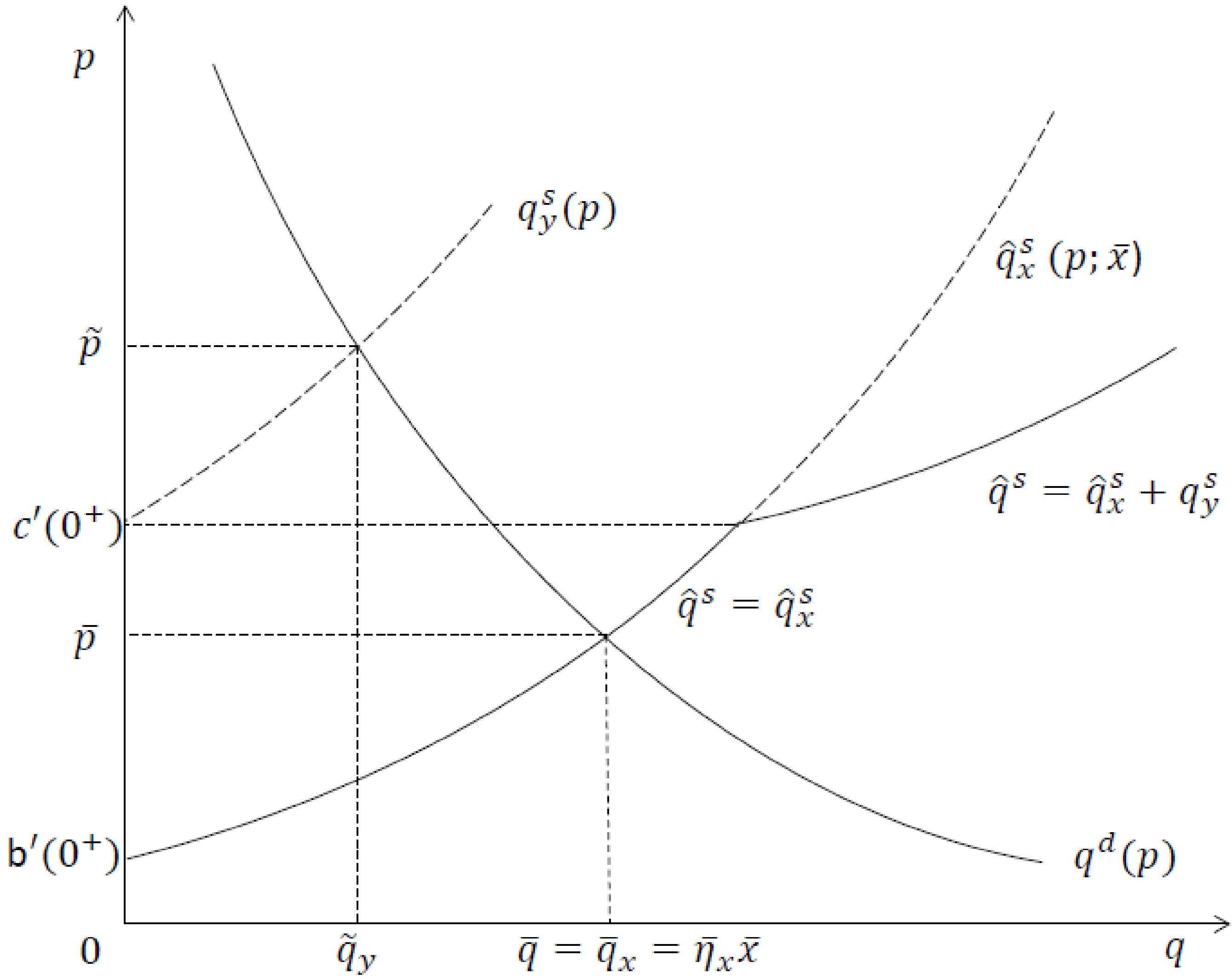

Figure 1: **Useful Energy Consumption when at the Ceiling and only Coal is Exploited.**

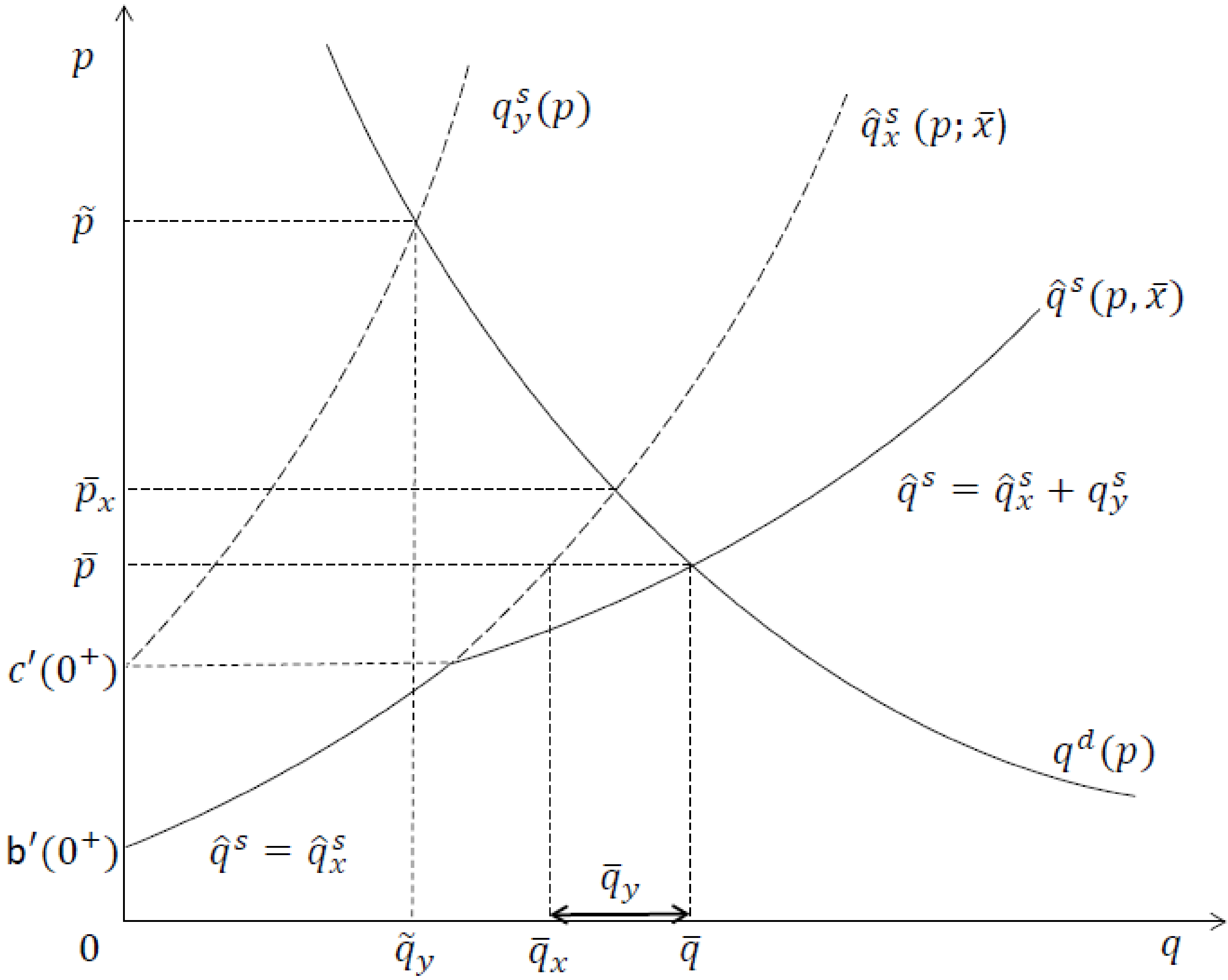

Figure 2: **Useful Energy Consumption at the Ceiling when both Coal and Solar are exploited and** $b'(0^+) < c'(0^+)$**.**

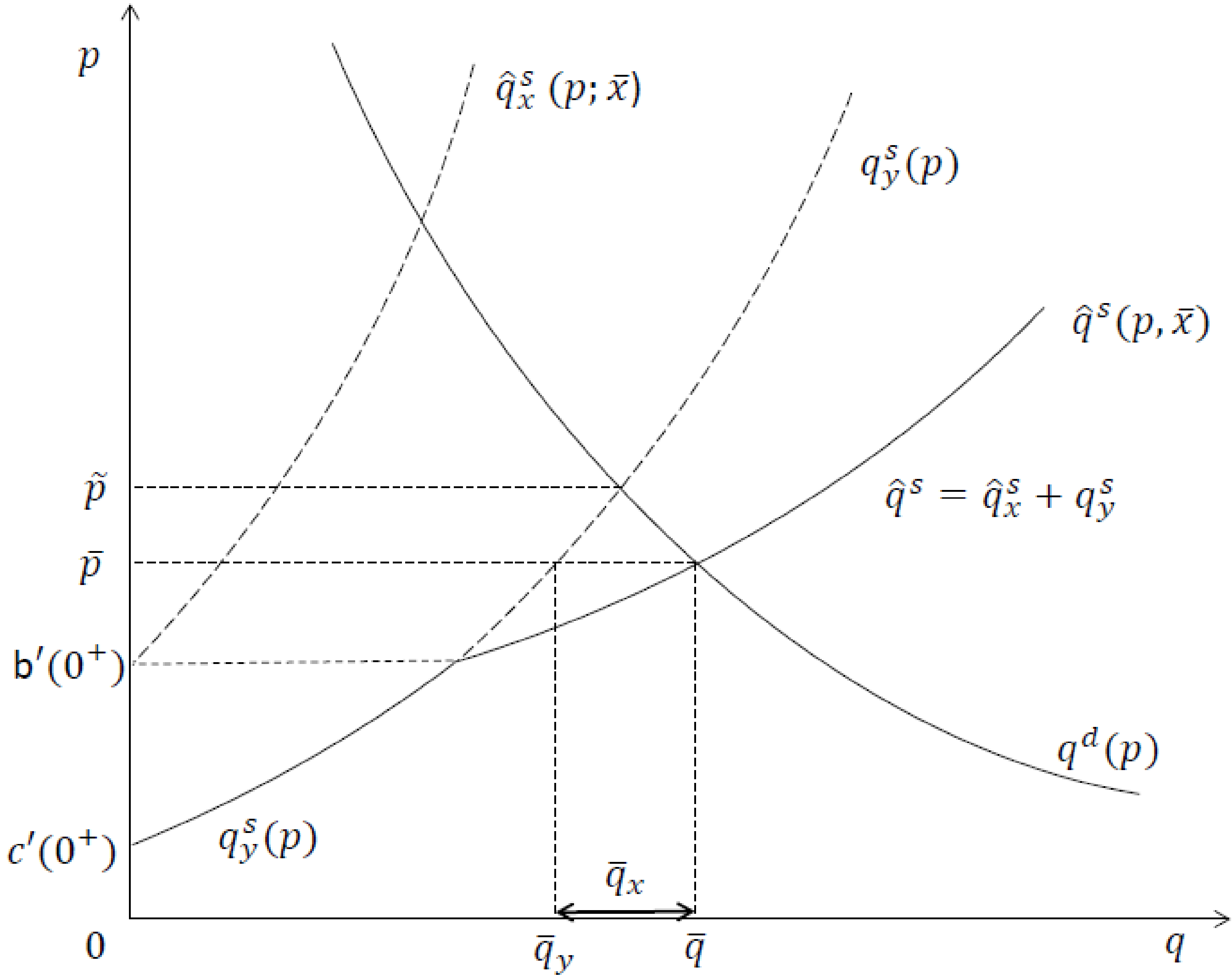

Figure 3: **Useful Energy Consumption when at the Ceiling and both Coal and Solar are exploited when** $c'(0^+) < b'(0^+)$**.**

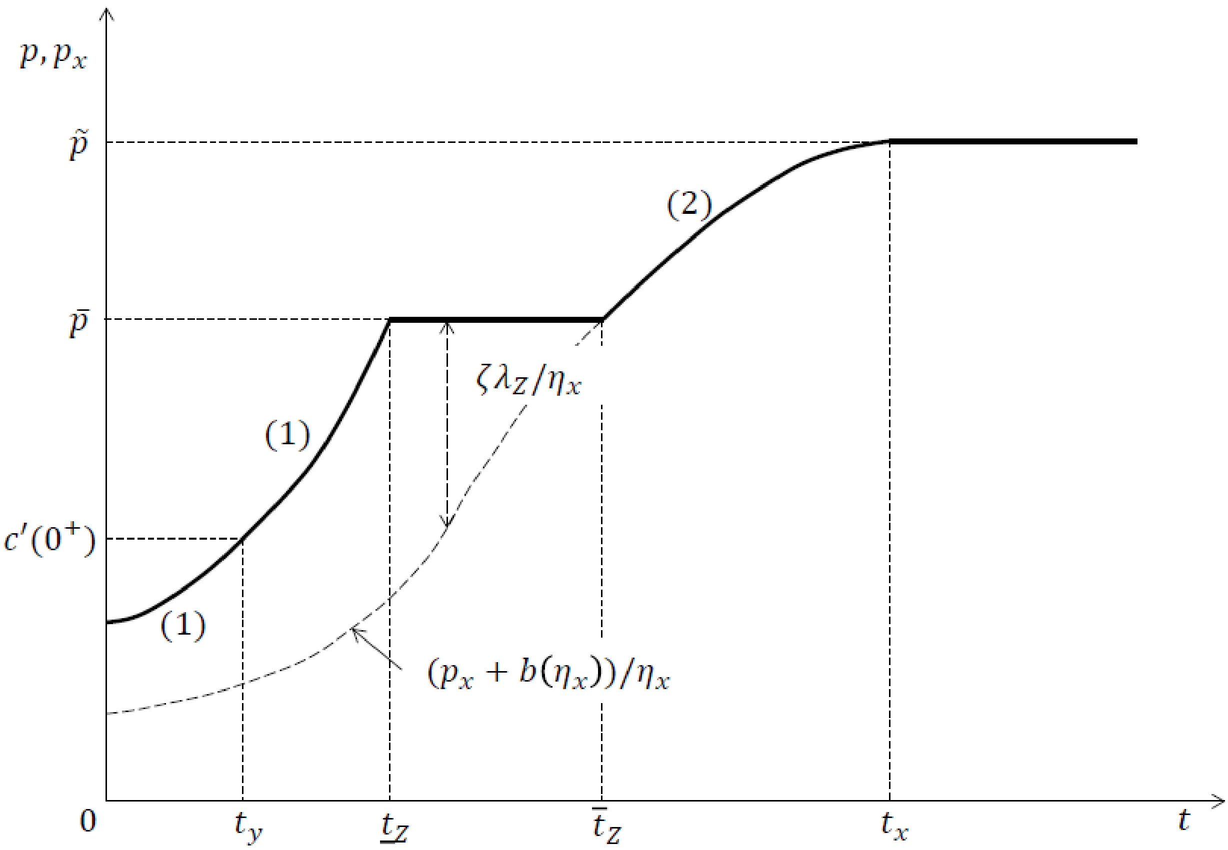

Figure 4: **Optimal Path of the Useful Energy Price.**

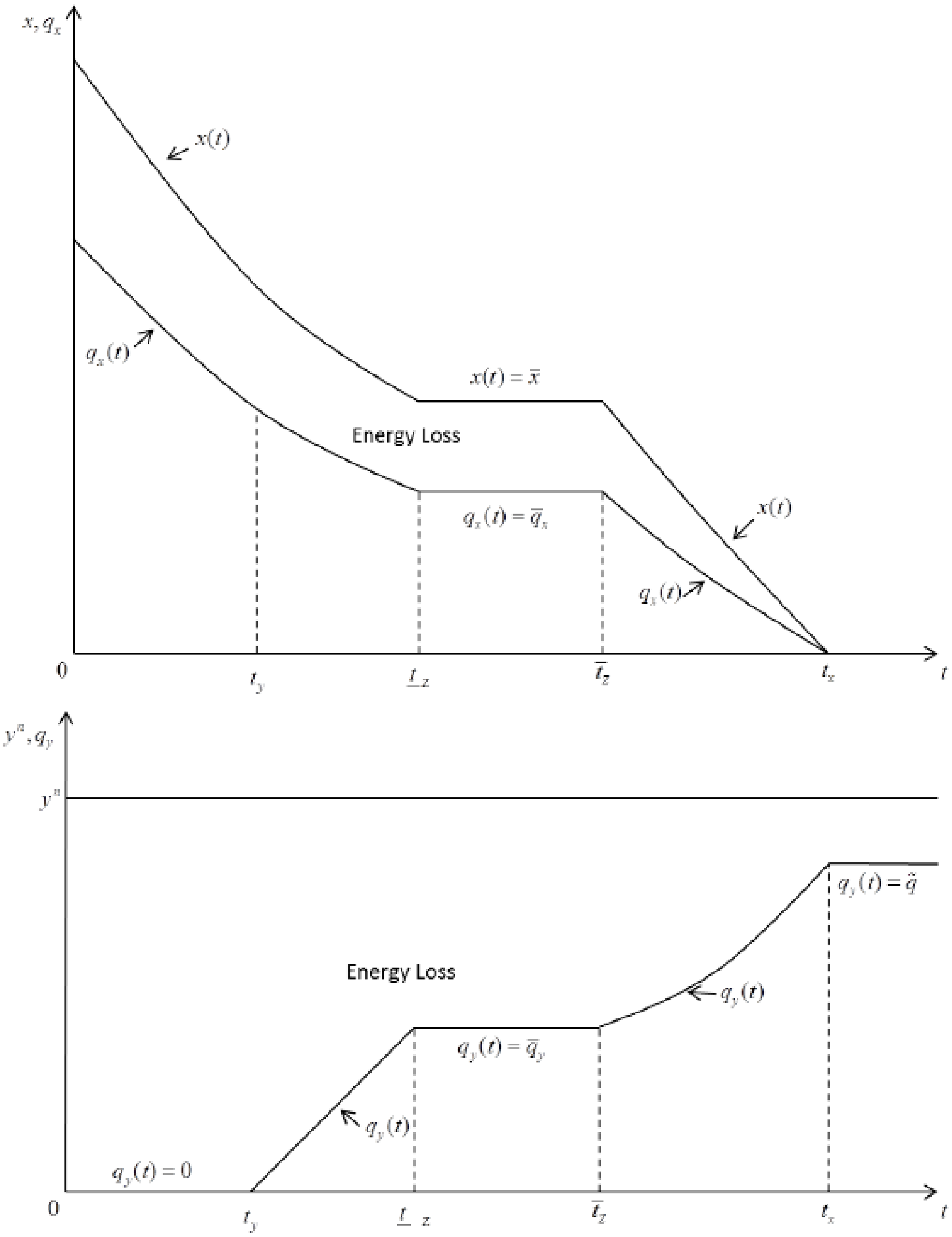

Figure 5: **Useful Energy Consumption Rates: Coal (top panel) and Solar (bottom panel).**

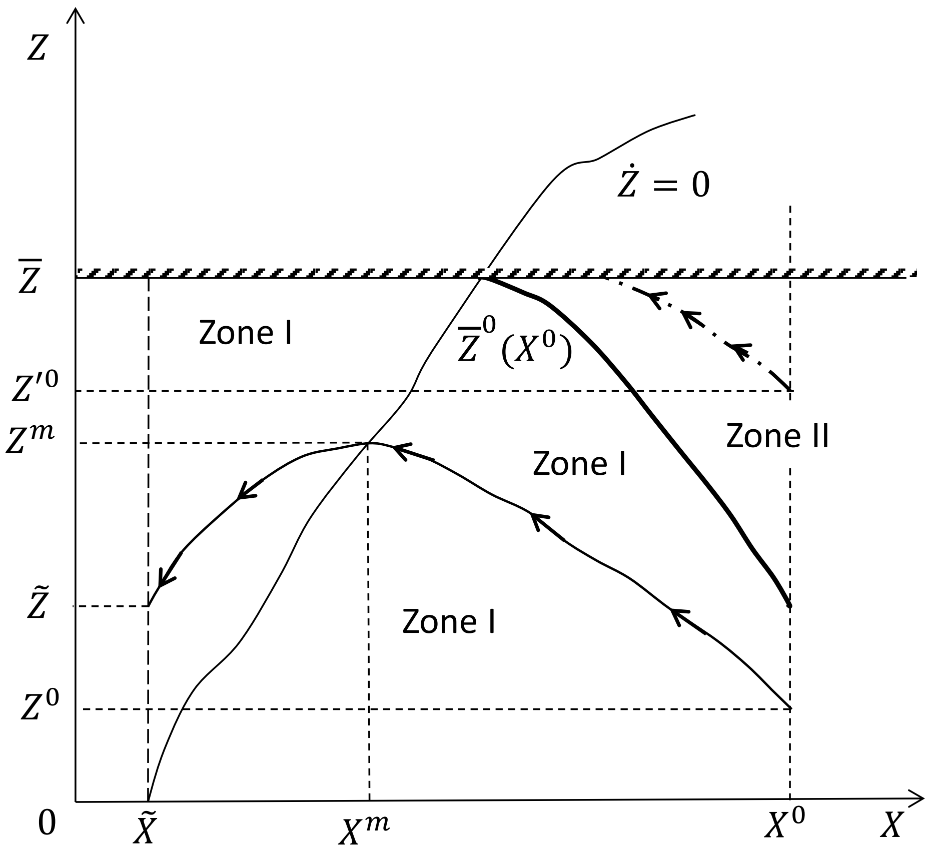

Figure 6: **Geometry of the plane** (X, Z)