Spacetime Discounted Value of Network Connectivity

Abstract

In order to unveil the value of network connectivity, discounted both in space and time, we formalize the construction of networks as an optimal control dynamic graph-theoretic problem. The network is based on a set of leaders and followers linked through edges. The node dynamics, built upon the consensus protocol, form a time evolutive Mahalanobis distance weighted by the opportunity costs. The results show that the network equilibrium depends on the influence of leader nodes, while the network connectivity depends on the cohesiveness among followers. Through numerical simulations, we find that – past a threshold level of opportunity costs – the values of shadow prices become stationary. Likewise, the model outputs show that, at a fixed level of foregone gains, agents value the safeguard of connections less in time than in space.

Keywords: Bioeconomics, Graph Theory, Optimal Control, Connectivity Value, Spacetime discounting

1 Introduction

Connectivity refers to any assemblage, interaction or linkage between human and non-human agents (Nicholls et al., 2016). Its expression can take multiple forms, of which economic (Stromquist, 2002; Wenz and Levermann, 2016), social (Miritello, 2013), environmental (Noss, 1987; Crooks and Sanjayan, 2006; Moritz et al., 2013; Dragicevic et al., 2017), technological (Webb, 2007) and organizational (Tillquist, 2002; Unhelkar, 2009).

The performance of complex systems and networks, such as socio-ecological systems (Ostrom, 2010; Gonzalès and Parrott, 2012), relies on their robustness and resilience (Dragicevic, 2016; Lu et al., 2016), which depend on the ability to maintain the topological structures through the vertex connectivity (Frank and Frisch, 1970). For example, forces such as global environmental change and globalization have pushed connectivity to such a level (Young et al. 2006; Brondizio et al., 2009), that ignoring the impacts of connectivity on the governance of interactions is perilous (Clark, 2000).

More specifically, the ecological fragmentation of territories and habitats leads to the current biodiversity decline. The supply of ecological networks, based on green corridors, has thus been engaged by the conservationists. These networks are meant to facilitate the adaptation of vulnerable species to local or broader environmental disturbances (Williams *et al.*, 2005). And yet, in order to increase the efficiency of the ecological network design (Wunder, 2005; Naidoo and Adamowicz, 2006), conservation issues must be assessed by taking into account the economic variables, simply because greater environmental benefits are obtained when the reserve site selection incorporates budget constraints (Conrad *et al.*, 2012). In forest ecosystems, for example, the establishment of corridors induces a series of opportunity costs in wood production by immobilizing the standing timber (Dragicevic et al., 2017).[1]

In the literature on social networks, the use of which can be seen as a source of innovation (Powell et al., 1996), a learning mechanism (Cassi and Zirulia, 2008), or as a medium of information diffusion (Aral and Walker, 2012; Yu and Kak, 2014), the role of connectivity in the economy is built on the idea that connections among agents do matter, especially when it comes to the process of knowledge creation and know-how trading (Cassi and Zirulia, 2008). The last authors argue that the use of networks is an economic choice based on cost-benefit comparisons. In their model, agents can choose to exchange knowledge through the social network or learn individually instead. The two mechanisms being mutually exclusive, the value obtained from individual learning represents the opportunity cost of using the network.

Although the literature on network connectivity has now grown abundant, little attention has been paid to the value of connectivity. For instance, in viral marketing, the non-economic network value is determined by two factors, that is, the intrinsic value of a user and the connectivity value of the network (Naik and Yu, 2015). The first measures

[1] The opportunity cost applies to two mutually exclusive options and refers to a benefit that an agent could have received, but gave up, to choose either option.

the level of a user as an influencer, while the second denotes how the user is connected in the network with respect to the level of connectivity of the neighborhood. Dragicevic et al. (2017) considered the construction of ecological networks in forest environments as the optimal control dynamic graph-theoretic problem. Through shadow prices, they managed to provide an economic value to the network connectivity, which was found to be greater in partially connected networks than in fully connected networks. They also showed that the value was of aggregated nature, which was then distributed among its constituents. Nevertheless, their theoretical framework did not consider any form of discounting.

The concept of discounting and present value are based on a behavior called time preference, which suggests that people prefer to realize benefits (costs) sooner (later) rather than later (sooner) (Kahn, 2005). Nonetheless, discounting is a controversial concept in light of the complexity related to the long-term perspective (Müller, 2013). Examples within reach are the risks of failure of preventing the global warming yielded by the emissions of greenhouse gases (Stern, 2007; Nordhaus, 2007) or that of biodiversity loss (Gowdy et al., 2010). In like manner, research on the valuation of environmental externalities shows that decision makers tend to discount not only over time but also across space (Jaunzeme, 2016). In that context, the preferences over space, just as the preferences over time, can be implemented through a discount rate (Perrings and Hannon, 2001). Therefore, the spacetime discounting can be envisaged. Recognizing this, Baum and Easterling (2010) showed, by emphasizing on inaccurate evaluations of adaptation projects to climate change, that proper discounting should include space as well as time.

We consider the construction of networks as an optimal control dynamic graph-theoretic problem, discounted both in space and time, which allows us to unveil the current economic value of network connectivity. The network at stake is composed of leader nodes, provided with an attraction function, and of follower nodes connected by edges. The node dynamics, built upon the consensus protocol, form a time evolutive Mahalanobis distance weighted by the opportunity costs. We formalize the existence of network equilibrium and provide conditions upon which depends the maintenance of network connectivity. The results show that the former depends on the influence of leader nodes, while the latter depends on the cohesiveness among followers. Through numerical simulations, we find that – past a threshold level of opportunity costs – the values of shadow prices become stationary. Likewise, the model outputs show that, at a fixed level of foregone gains, securing connectivity is much more sensitive to spatial discounting than to the temporal one, implying than agents value the safeguard of connections less in time than in space.

After this starting section, we present the graph-theoretic framework and its dynamic controllability in Section 2. Section 3 is devoted to illustrating simulation examples. Conclusive remarks are given in Section 4.

2 Model

Consider an ecological or a social network, based on the Euclidean metric, of dimension $\mathbb{R}^N$. The link formation depends on the distance between the constituents of the network which represent the areas of ecological significance selected by the resource manager or the social agents involved in information trading.

Both living species and information can move in all directions, such that the network is represented by an undirected graph $\Gamma = \{V, E\}$, which consists of vertices $V = \{1, ..., N\}$ indexed by the node members, where i and j represent two neighboring nodes, and of the set of edges $E = \{(i, j) \in V \times V\}$, which represent the inter-node interactions. The grid of ecological corridors or that of information flows stands for these interactions. Fig. 1 illustrates the network framework with a graph-theoretic mapping.

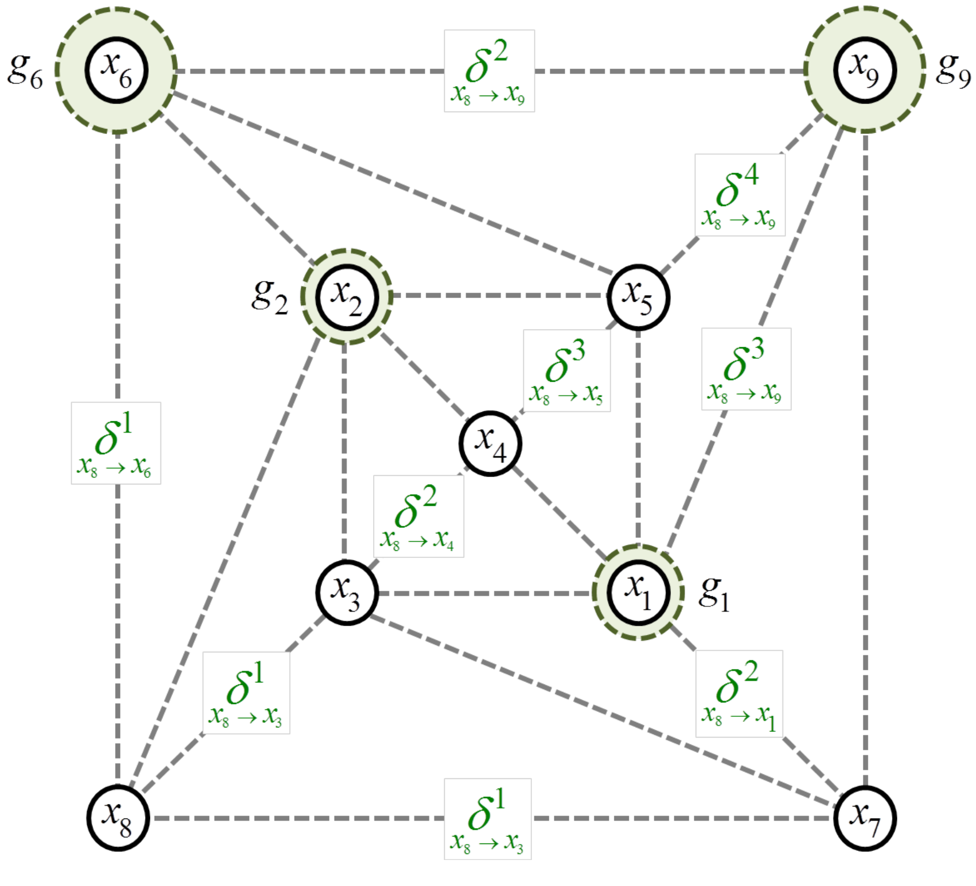

Figure 1: The network is composed of four leader nodes $N_l = \{x_1, x_2, x_6, x_9\}$ and five follower nodes $N_f = \{x_3, x_4, x_5, x_7, x_8\}$. A set of twenty edges forms the network $E = \{(1,3),(1,4),(1,5),(1,7),(1,9)\} \cup \{(2,3),(2,4),(2,5),(2,6),(2,8)\} \cup \{(3,4),(3,7),(3,8)\} \cup \{(4,5)\} \cup \{(5,6),(5,9)\} \cup \{(6,8),(6,9)\} \cup \{(7,8),(7,9)\}$. The spatial discount factor δ^l, where $\sum_{l=1}^{4} \delta^l$ in the longest path, weighs up the distance between two nodes.

Consider A to be an adjacency matrix[2] with nonnegative elements a_{ij} and assume that $(i, j) \in E$, implying that two nodes are connected if and only if $a_{ij} > 0$. The set of edges

[2]The adjacency matrix of an undirected graph is symmetric.

E and graph Γ vary in finite time for $t \in [0, T]$.

Assume the existence of a convex hull of vertices Ω to be an N-simplex, with the Euclidean norm in $\mathbb{R}^N$. Let $x_i(e_i, t) \in \mathbb{R}^N$, where $i = 1, ..., N$, denote the state of node i, characterized by the environmental feature or the informational content e_i at time t.[3] This state consists of two features: (1) a scalar value of the node, which permits computing its distance from the neighboring nodes; and (2) the nature of the node like the type of ecological area or the level of influence that a social agent can exert on other agents.

The set of all possible states of the dynamic system is the configuration space. It is spanned by the stack vector of all the control inputs $x = \left[x_1^T, ..., x_N^T\right]^T$, which denotes the global state vector. The state of each node evolves according to the dynamics which maps control inputs to states through

$$\dot{x} = u_i \tag{1}$$

where u_i denotes the control input of node i. The latter is selected such that the network evolution is constrained to invariant reachable sets. It drives the network from any initial condition to some arbitrary point in finite time and implies $\dot{x} = 0$ at steady state.

2.1 Distance

Let Λ be the set of nodes, such that the nodes connected to node $i \in \Lambda$ are referred as to subset Λ_i. For $\forall i, j \in \Lambda$, $d_{ij} = |x_i(e_i) - x_j(e_j)|$, and for $\Lambda_i = \{j \in \Lambda : 0 < d_{ij} \leq z\}$, d_{ij} and z respectively stand for the Euclidean distance between ecological spots and that between social agents, and the respective species transportation capacity as well as the information transmission capacity. To measure utility between two connected nodes, we take into account the capacities in flows, for networks built on unrealistic distances are ineffective. Nodes thus obtain and provide utility u_{ij} from and to other nodes, which can be defined as follows.

Definition 1 *For* $\forall i, j \in \Lambda$, $d_{ij} = |x_i(e_i) - x_j(e_j)|$, *and for* $\Lambda_i = \{j \in \Lambda : 0 < d_{ij} \leq z\}$,

$$u_{ij} \begin{cases} > 0 & \textit{if} \;\; a_{ij} > 0 \\ = 0 & \textit{otherwise} \end{cases}$$

By that, an improperly connected network implies the lack of value creation. Although the existence of ecological areas or that of informative agents could in itself be valued through utility, and their connections defined as sources of positive externalities, we assume that only connections provide utility, knowing that the nodes predate the network construction.

[3] In what follows, the explicit indication of time, such as in $x_i(e_i, t)$, is not specified unless necessary.

All nodes connected to node i form its utility set $U_i(\Lambda)$. In our case, the network utility set reflects the overall adequate connectivity in the network.

Definition 2 *For $i, j \in \Lambda$, the network utility set of node i is the union of utilities issued from the network connectivity or* $U_i(\Lambda) = \bigcup_{j \in \Lambda} u_{ij}$.

The network utility can thus be interpreted as the connectivity of the set of relevant nodes separated by distances which satisfy the capacities of flows.

Furthermore, the Euclidean distance becomes irrelevant if the hosting environment is unsuitable, that is, a sufficiently close but radically different habitat cannot provide any utility to the migratory species. The same comment applies to the strategic complementarity of information each agent holds (Chamley, 2010). For that reason, let us introduce the Mahalanobis distance. The latter measures the dissimilarity between the vectors and accounts for the variance of each variable and the covariance between the variables. It has been often used by biologists to take into account the intercorrelation of attributes (Relethford, 2016). As such, it has been introduced in habitat selection studies for the estimation of environmental suitability, where smaller distances correspond to areas that are more likely to be occupied by the species (Calenge *et al.*, 2008). The Mahalanobis distance can equally be used to measure the dissimilarity in informative content (Niu et al., 2014).

Following Shaw *et al.* (2011), consider an Euclidean distance metric parameterized by a positive semidefinite matrix $\Pi = L^T L \equiv S^{-1}$, where $\Sigma \in \mathbb{R}^{N \times N}$ and $L \in \mathbb{R}^{N \times N}$.[4] The latter reflects the feature similarity between the nodes, with L a positive semidefinite Laplacian matrix (Godsil and Royle, 2001). It is considered to be network structure preserving if the weighted graph $\Gamma(V, E, \Sigma)$ yields $A(\Gamma)$, with A the adjacency matrix.

According to the foregoing, the Mahalanobis distance $m(C)_{ij}$ between nodes i and j corresponds to

$$m(C)_{ij} = \left[\left(x_i(e_i) - \sum_{l=0}^{L} \delta^l x_j(e_j) \right)^T S^{-1} \left(x_i(e_i) - \sum_{l=0}^{L} \delta^l x_j(e_j) \right) \right]^{\frac{1}{2}} C \tag{2}$$

where $\sum_{l=0}^{L} \delta^l$ denotes the composite factor of the spatial discounting dependent on the sequence of vertices the distances of which are being measured. When the metric is the identity matrix or $\Pi \equiv S^{-1} = I$, $m(C)_{ij}$ falls back to the standard Euclidean distance between i and j. In order to ensure flowing through the network, the nodes with similar environmental features or endowed with informational complementarity shall be linked, which is verified by $S^{-1} \geq 0$. In different words, the Mahalanobis distance metric

[4] The matrix Π is equivalent to the inverse of the covariance matrix S^{-1}. If two vertices are unconnected they are conditionally independent in the graph (Bell *et al.*, 2000).

guarantees that the connection only occurs when two nodes display either compatible environmental characteristics or informational additivity.

In addition, as the resource manager implements the ecological grid, it immobilizes the exploitation of the natural resource. Let scalar C be this economic opportunity cost, from the sacrificed harvesting, computable at the market value. On the side of the social network interpretation, the opportunity cost of information trading corresponds to the abandon of individual learning, which can also be evaluated, through investment costs, in monetary terms.

The resource manager or the network administrator is able to identify the subset of leader nodes Λ^l, which either correspond to bioreserves, in case of ecological networks, or to the influential agents, in case of social networks, and the subset of follower nodes Λ^f, which represent patches or noninfluential individuals, such that

$$\Lambda^l \cup \Lambda^f = \Lambda \text{ and } \Lambda^l \cap \Lambda^f = \varnothing \tag{3}$$

The number of nodes in each subset is respectively given by $\left|\Lambda^l\right| = N_l$ and $\left|\Lambda^f\right| = N_f$. Provided the absence of a complete graph, that is, not all leader nodes are connected to all follower nodes, let $\Lambda_i^l \subset \Lambda^l$ be the subset of leader nodes connected to follower node i with $|\Lambda_i^l| = N_{li}$. Likewise, let $\Lambda_j^f \subset \Lambda^f$ be the subset of follower nodes connected to leader node j with $|\Lambda_j^f| = N_{fj}$.

2.2 Dynamics

Following Gustavi *et al.* (2010), the follower node dynamics is given by the Laplacian-based control strategy (consensus) differential equation, meaning that the state of a follower evolves according to the states of the nodes to which it is connected. In detail, the rate of change of a node's state is governed by the sum of states of the neighboring nodes. This property provides evidence for the cascade effect in natural environments, whereby a change is proportional to the node's number of interactions (Bascompte and Stouffer, 2009). In the social network analysis, the spread of information also takes the form of a cascade behavior (Myers et al., 2012).

The dynamics for a follower node can be written as

$$\dot{x_i} = -Nx_i(e_i) + \frac{\delta^{L+1} - 1}{\delta - 1}\left[Nx_j(e_j) + \sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right] \tag{4}$$

where $x_i(e_i)$ is the state of the follower node, $x_j(e_j)$ is the leader node state, and $\sum_{k\in\Lambda_i^l} x_k(e_k)$ and $\sum_{k\in\Lambda_j^l} x_k(e_k)$ respectively denote the sum of states of leader nodes connected to follower node i and the sum of states of leader nodes connected to leader node j. As can be noticed, the expression within parentheses is weighted by the spatial discount factor with respect to node i. It corresponds to the graph diameter (West, 2000), be it the largest number of nodes which must be traversed in order to travel from one node to another. Its annulment yields the follower node equilibrium state.

Lemma 1 *The consensus equilibrium under follower dynamics is equal to* $x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) = \frac{\delta^{L+1}-1}{N(\delta-1)} \left[\sum_{k \in \Lambda_i^l} x_k(e_k) - \sum_{k \in \Lambda_j^l} x_k(e_k) \right]$.

The proof is provided in the appendix.

The leader dynamics is not only based upon the consensus equation, the state of which evolves according to the neighboring nodes, but also upon the attraction function which depends on the size of j delimited by its frontier g_j. At any given state, the magnitude of attraction is a continuous scalar function $f\left(|g_j - x_j(e_j)|\right) \geq 0$. Its continuity is achieved by requiring $f(0) = 0$ and $\lim_{x_j(e_j) \to g_j} f\left(|g_j - x_j(e_j)|\right) = 0$. The frontier enables to give weight to the leader node, that is, the value associated with the bioreserve size or the ability to influence other trading agents on the network. The rationale is that the greater the gap between $x_j(e_j)$ and g_j encapsulated in f, the greater the attraction.

For an arbitrary leader node, we thus have

$$\begin{aligned} \dot{x_j} &= -N x_j(e_j) + f\left(|g_j - x_j(e_j)|\right) \\ &+ \frac{\delta^{L+1}-1}{\delta-1} \left[N x_i(e_i) + \sum_{k \in \Lambda_j^f} x_k(e_k) - \sum_{k \in \Lambda_i^f} x_k(e_k) \right] \end{aligned} \quad (5)$$

where $x_i(e_i)$ represents the follower node state, $x_j(e_j)$ represents the leader node state, $\sum_{k \in \Lambda_j^f} x_k(e_k)$ and $\sum_{k \in \Lambda_i^f} x_k(e_k)$ respectively stand for the sum of states of followers connected to leader j and the sum of states of followers connected to follower i, and where $f\left(|g_j - x_j(e_j)|\right) \geq 0$ is the attraction function toward the rest of the network.

Once again, the expression within parentheses is weighted by the spatial discount factor with respect to the node at stake. Its annulment yields the leader node equilibrium state.

Lemma 2 *The consensus equilibrium under leader dynamics is equal to* $x_j(e_j) - \frac{\delta^{L+1}-1}{\delta-1} x_i(e_i) = \frac{\delta^{L+1}-1}{N(\delta-1)} \left[\sum_{k \in \Lambda_j^f} x_k(e_k) - \sum_{k \in \Lambda_i^f} x_k(e_k) \right] + \frac{1}{N} f\left(|g_j - x_j(e_j)|\right)$.

The proof is provided in the appendix.

Equalizing the two equilibria yields the following.

Theorem 1 *When the consensus problem is well-defined in the initial state, such that* $(x_i(e_i, 0), x_j(e_j, 0)) \in \Omega(0)$*, where* $\Omega(0)$ *is an invariant set, the network equilibrium, which is discounted at the level of the graph diameter, with respect to the aggregated state*

$\sum_{k\in\Lambda_j^f\cup\Lambda_j^l} x_k(e_k)$ *of nodes connected to leader node* j*, the attraction function* $f\left(|g_j - x_j(e_j)|\right)$*, and to the aggregated state* $\sum_{k\in\Lambda_i^f\cup\Lambda_i^l} x_k(e_k)$ *of nodes connected to follower node* i*, corresponds to*

$$
\begin{aligned}
f\left(|g_j - x_j(e_j)|\right) &= \frac{\delta^{L+1}-1}{\delta-1}\left[N\left(x_i(e_i) - x_j(e_j)\right)\right] \qquad (6)\\
&- \frac{\delta^{L+1}-1}{\delta-1}\left[\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k) + \sum_{k\in\Lambda_i^f} x_k(e_k) - \sum_{k\in\Lambda_j^f} x_k(e_k)\right]
\end{aligned}
$$

The proof is provided in the appendix.

Remark 1 *In a network, the distance between the states of leader and follower nodes is equal to the overall difference between the states of nodes connected to follower nodes and the states of nodes connected to leader nodes supplemented by the attraction function discounted in space.*

The result shows that, at equilibrium, the success of flowing depends on the size of the reception zone to which the species migrate, or the magnitude of influence of a social agent involved in information trading, which has to be sufficiently large to initiate the exchange with respect to the graph diameter or in the longest path.

Let us now derive general conditions for the subgraphs to remain connected.

2.3 Connectivity

The connectivity relation $\dot{m}(C)_{ij}$ defines the preservation of the network connectedness. In other words, the network does not disconnect in time. Under the assumptions on differentiability and boundedness of dynamics, the connectivity between two initially nodes remains valid in time if the time derivative of the Mahalanobis distance between them is nonpositive. Thus, the condition for nodes i and j to evolve connected is $\dot{m}(C)_{ij} \leq 0$. When the latter is true, it proves that the convex hull Ω containing the nodes is invariant and, therefore, that the network is Lyapunov-stable (Dragicevic and Sinclair-Desgagné, 2013). The time derivative $\dot{m}(C)_{ij}$ may not be defined when $m(C)_{ij} = 0$, so the squared distance derivative shall be considered instead (Gustavi *et al.*, 2010). It depends on dynamics of nodes i and j and equals

$$
\dot{m}(C)_{ij}^2 = 2\left(x_i(e_i) - \sum_{l=0}^{L}\delta^l x_j(e_j)\right)^T S^{-1}\left(\dot{x}_i - \dot{x}_j\right) C^2 \qquad (7)
$$

2.3.1 Connectivity between arbitrary followers

For arbitrary nodes $i, j \in \Lambda^f$ and $k \in \Lambda^f \cup \Lambda^l$, the connectivity is defined by

$$\begin{aligned}
\dot{m}(C)_{ij}^2 &= 2C^2 \left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) \right)^T S^{-1} \left[\frac{2-\delta(\delta^L+1)}{\delta-1} N \left(x_i(e_i) - x_j(e_j) \right) \right] \\
&+ 4C^2 \left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) \right)^T S^{-1} \left[\frac{\delta^{L+1}-1}{\delta-1} \left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k) \right) \right]
\end{aligned} \quad (8)$$

The condition for nodes to evolve connected is $\dot{m}(C)_{ij} \leq 0$. This results in having $\sum_{k\in\Lambda_i^l} x_k(e_k) > \sum_{k\in\Lambda_j^l} x_k(e_k)$. The inequality is thus verified when the follower nodes are densely connected between them.

Corollary 1 *Necessary and sufficient condition for arbitrary followers to evolve connected in a graph is that the leader nodes be more sparsely connected between them than to the follower nodes.*

The proof is provided in the appendix.

2.3.2 Connectivity between arbitrary leaders and followers

For arbitrary nodes $i \in \Lambda^f$, $j \in \Lambda^l$ and $k \in \Lambda^f \cup \Lambda^l$ and for $f\left(|g_{i,j} - x_j(e_j)|\right) \neq 0$, the connectivity is defined by

$$\begin{aligned}
\dot{m}(C)_{ij}^2 &= 2C^2 \left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) \right)^T S^{-1} \left[\frac{2-\delta(\delta^L+1)}{\delta-1} N \left(x_i(e_i) - x_j(e_j) \right) \right] \\
&- 2C^2 \left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) \right)^T S^{-1} \left[f\left(|g_j - x_j(e_j)|\right) \right] \\
&+ 2C^2 \left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) \right)^T S^{-1} \left[\frac{\delta^{L+1}-1}{\delta-1} \left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k) \right) \right] \\
&+ 2C^2 \left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1} x_j(e_j) \right)^T S^{-1} \left[\frac{\delta^{L+1}-1}{\delta-1} \left(\sum_{k\in\Lambda_i^f} x_k(e_k) - \sum_{k\in\Lambda_j^f} x_k(e_k) \right) \right]
\end{aligned} \quad (9)$$

The condition for nodes to evolve connected being $\dot{m}(C)_{ij} \leq 0$, we fall on a configuration dependent on $\sum_{k\in\Lambda_i^l} x_k(e_k) + \sum_{k\in\Lambda_i^f} x_k(e_k) > \sum_{k\in\Lambda_j^l} x_k(e_k) + \sum_{k\in\Lambda_j^f} x_k(e_k)$. The inequality holds when the follower nodes are more densely connected to the network than the leader nodes.

Corollary 2 *Necessary and sufficient condition for arbitrary leaders and followers to evolve connected in a graph is that the leader nodes be more sparsely connected in the network than the follower nodes.*

The proof is provided in the appendix.

It may be noted, through the two corollaries, that similar constraints, based upon the density of connectivity of followers, which can be understood as the magnitude of cohesiveness among them, ensure connectivity in the graph.

2.4 Optimal control problem

To guarantee the maintenance of the network connectivity, we apply the optimal control method. It drives the states of the nodes by adjusting the values of control inputs. Following the methodology by Mesbahi and Egerstedt (2010), let us introduce the performance function J, which measures the preservation of the network weighted by the opportunity costs. Because of the constraints relative to the connectivity, it is in form of a standard cost function that is integrated over finite time, such that for all $0 \leq t \leq T$

$$J = \int_0^T \sum m(C)_{ij} \; dt \tag{10}$$

The resource manager, or the network administrator, decides to sustain the network topology, via control variables $m(C)_{ij}$ in which control is assumed to be the creation of edges associated with the graph (Sengupta and Lafortune, 1992), such that the continuous time optimal control problem can be seen as that of maintaining connectivity, defined over state variables $x_i(e_i)$ and $x_j(e_j)$, which are subject to consensus dynamics, under the constraint of network equilibrium (Lachner *et al.*, 1998). Provided the cost of doing so, as well as its impact on the alternative option, his or her optimal control problem can be formulated as the minimization of the performance function. Put differently

$$\min_{x_i(e_i),\ x_j(e_j)} \; J \tag{11}$$

subject to two first-order dynamic constraints

$$\dot{x}_i, \; \dot{x}_j \tag{12}$$

Unlike the standard control law, the problem relates together to the choice of the control vector and the presence of constraints on the state vector. Indeed, the updating of the node state being invariably conducted from the rest of the network, which both includes leader nodes and follower nodes, we need to look at the first-order necessary optimality conditions of both the control and state components. Naturally, the states of the leader nodes are used as inputs to the network.

The optimal control problem is solved by means of the current value Hamiltonian, discounted in time up to $t = T$, which represents the impact of evolution of $x_i(e_i)$ and $x_j(e_j)$ on the network topology. The first-order optimality conditions yield

$$\mu = \lambda \frac{(2N(\delta+2)-1)(\delta-1)^2 - 2N(\delta^{L+1}-1)(\delta+\delta^{L+1}-2)}{(\delta-1)\left[(2N(\delta+2)-1)(\delta^{L+1}-1) - 2(N-1)(\delta+\delta^{L+1}-2)\right]} \tag{13}$$

The costate variables, obtained by relaxing the connectivity constraints (Lyon, 1999), are represented by λ and μ. They reveal the shadow prices[5] for keeping the network connected and thus express the network connectivity value: λ for the connectivity between followers and μ for the connectivity between leaders and followers. The former equality is part of the initial conditions on the choices of costate variables for the system control, such that Theorem 1 holds.

Let the initial network state be given by $w_0 = \left[x_{i_0}(e_i)^T, x_{j_0}(e_j)^T, x_{k_0}(e_k)^T, \lambda_0^T, \mu_0^T\right]^T$. In order to control the network, the task consists in fixing λ_0 and μ_0 such that

$$\begin{aligned} f\left(|g_j - x_{j_T}(e_j)|\right) &= \frac{\delta^{L+1}-1}{\delta-1}\left[N\left(x_{i_T}(e_i) - x_{j_T}(e_j)\right)\right] \\ &- \frac{\delta^{L+1}-1}{\delta-1}\left[\sum_{k\in\Lambda_i^l} x_{k_T}(e_k) - \sum_{k\in\Lambda_j^l} x_{k_T}(e_k) + \sum_{k\in\Lambda_i^f} x_{k_T}(e_k) - \sum_{k\in\Lambda_j^f} x_{k_T}(e_k)\right] \end{aligned} \tag{14}$$

where the choices of λ_0 and μ_0 are constrained by (13).

Lemma 3 *The connectivity value between leaders and followers has to be equal to the connectivity value between followers weighted by the spatial discount rate.*

The proof is provided in the appendix.

Thereby, the willingness to pay for maintaining the connectivity between leaders and followers has to be equal to the willingness to pay for preserving the connectivity between followers weighted by spatial discounting up to the graph diagram. Otherwise, controlling the network is non-optimal.

By letting $w = \left[x_i(e_i)^T, x_j(e_j)^T, x_k(e_k)^T, \lambda^T, \mu^T\right]^T$ reflect the network state, where the values of shadow prices comply with (13), the system control is obtained through the following Hamiltonian system.

$$\dot{w} = Pw \tag{15}$$

where $P =$

$$\begin{bmatrix} 0 & \frac{\delta^{L+1}-1}{\delta-1} & \frac{(\delta-1)(4C^2)^{-1}Se^{\delta t}}{2-\delta(\delta^L+1)} & \frac{(1-\delta)(4C^2)^{-1}Se^{\delta t}}{2-\delta(\delta^L+1)} \\ \frac{\delta^{L+1}-1}{\delta-1} & 0 & \frac{(\delta^{L+1}-1)N(4C^2)^{-1}Se^{\delta t}}{(2-\delta(\delta^L+1))N+\frac{1}{2}(\delta-1)} & \frac{(1-\delta)(N-1)(4C^2)^{-1}Se^{\delta t}}{(2-\delta(\delta^L+1))N+\frac{1}{2}(\delta-1)} \\ \frac{(2-\delta(\delta^L+1))N(4C^2)}{(1-\delta)Se^{\delta t}} & \left(\frac{2-\delta(\delta^L+1)}{\delta-1}N+\frac{1}{2}\right)\frac{4C^2}{Se^{\delta t}} & N & \frac{\delta^{L+1}-1}{1-\delta}N \\ \frac{(\delta^{L+1}-1)(2-\delta(\delta^L+1))N(4C^2)}{-(\delta-1)^2Se^{\delta t}} & \left(\frac{2-\delta(\delta^L+1)}{\delta-1}N-\frac{1}{2}\right)\frac{(\delta^{L+1}-1)(4C^2)}{(\delta-1)Se^{\delta t}} & \frac{\delta^{L+1}-1}{\delta-1}N & N \end{bmatrix}$$

[5] In our case, the shadow price is the change in the optimal control solution obtained by relaxing the constraint of connectivity. Economically speaking, it is the maximum price a resource manager, or a network administrator, is willing to pay to maintain the network connected for an additional unit of time.

Theorem 2 *The network state w optimally evolves according to coordinates P.*

The proof is provided in the appendix.

In order to control the network, the costate variables must not invalidate Theorem 2.

3 Simulations

Based on the properties and conditions previously obtained, the aim of this section is to discuss, through simulations, the conditions that guarantee the maintenance of connectivity. By extension, the results make it possible to estimate the connectivity value discounted in space and time.

Consider a network composed of nine nodes such as the one depicted in Fig. 1. Four nodes represent the leaders with dynamics given by (5). The remaining five nodes, expressing various followers, are subject to dynamics given by (4). The vector of states, such that $x(0) = [8 \quad 8 \quad 4 \quad 4 \quad 4 \quad 10 \quad 4 \quad 4 \quad 10]$, where $N_{\mathrm{L}} = \{1,2,6,9\}$ and $N_{\mathrm{F}} = \{3,4,5,7,8\}$, is converted into a square-form distance matrix. The attraction function is defined as $f(|g_j - x_j(e_j)|) = a(|g_j - x_j(e_j)|)$, for $j = 1,2,6,9$, with $a = 2$. Now consider the covariance matrix $e = \{1,2,3,4,5,6,7,8,9\}$ where

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
e_1	0.20	0.50	0.70	0.1	0.20	0.90	0.50	0.40	0.30
e_2	0.50	0.50	0.2	0.80	0.60	0.40	0.10	0.30	0.80
e_3	0.70	0.20	0.40	0.40	0.90	0.30	0.30	0.40	0.10
e_4	0.10	0.80	0.40	0.90	0.80	0.70	0.90	0.80	0.50
e_5	0.20	0.60	0.90	0.80	0.70	0.50	0.70	0.30	0.30
e_6	0.90	0.40	0.30	0.70	0.50	0.10	0.20	0.70	0.60
e_7	0.50	0.10	0.30	0.90	0.70	0.20	0.20	0.50	0.20
e_8	0.40	0.30	0.40	0.80	0.30	0.70	0.50	0.20	0.90
e_9	0.30	0.80	0.10	0.50	0.30	0.60	0.20	0.90	0.50

The Laplacian dynamics applied to leader and follower nodes yield the following evolution of Mahalanobis coordinates. We suppose that the network is connected at $t = 0$. In order to see whether the network connectivity is in jeopardy, let us expose the nodes to the Laplacian laws of motion. The evolution of coordinates is shown in Fig. 2. We can see that Theorem 1 is verified along the time path, for the network barycenters are similar at $t = 0$ and $t = 50$, that is, $10.93 \simeq 11.89$. For instance, at $t = 50$, $x_{\mathrm{L}_j} = (10.93, 12.12)$, for $j = 1,2,6,9$. Likewise, we have $x_{\mathrm{F}_i} \in (11.85, 11.89)$ for $i = 3,4,5,7,8$. Thereby the interval center value is of $11.86 \simeq \frac{12.13+10.93}{2} + \frac{2(12.13-10.93)}{9} = 11.79$, which signifies that the average follower state does depend on the average leader state and its magnitude of attraction.

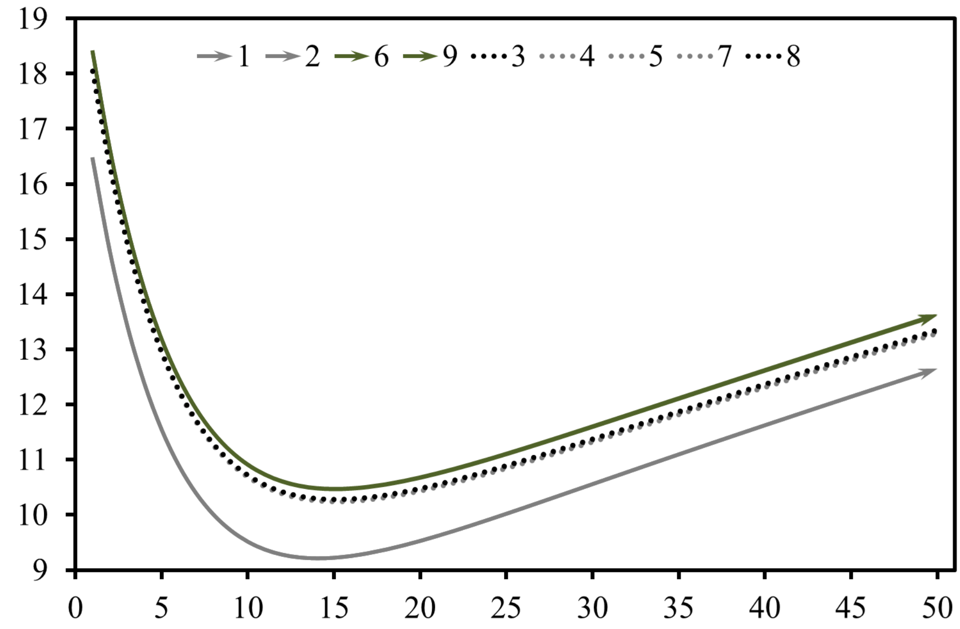

Figure 2: Mahalanobis coordinates (ordinates) as functions of time (abscissa) of two large leaders $(\mathrm{L}_6, \mathrm{L}_9)$, two mid-sized leaders $(\mathrm{L}_1, \mathrm{L}_2)$ and five followers $(\mathrm{F}_3, \mathrm{F}_4, \mathrm{F}_5, \mathrm{F}_7, \mathrm{F}_8)$ such as represented in Fig. 1.

As can be observed, the graph connectedness is stationary throughout the timeline. In detail, the connectivity between followers is ensured by the fact that leader nodes are more sparsely connected between them than to the follower nodes. As a matter of fact, the average degree of leaders connected to other leaders is of $\frac{2\times6}{4} = 3$ while the average degree of connections between leaders and followers is of $\frac{2\times12}{4} = 6$, thus validating Corollary 1. As for Corollary 2, that is, the leader nodes ought to be more sparsely connected in the network than the follower nodes, we observe an average degree of leaders equal to $\frac{2\times18}{4} = 9$ and that of followers equal to $\frac{2\times22}{5} = 8.8$, a result which proves the corollary by contradiction. Indeed, the result explains why the mid-sized leaders, while evolving at a distance, follow the trajectory of large leaders, as well as that of followers – who remain tightly close to large leaders –, upon whom they also try to act. However, due to the consensus protocol, this distance tends to shorten after the initial time steps.

Next to the analysis of Mahalanobis coordinates, let us take a closer look at the optimal control conditions for different levels of opportunity costs. We recall that the optimal control is meant to secure the coordinates of the ecological network, at a certain cost, such that the overall connectivity is preserved. Fig. 3 illustrates the behavior of shadow prices with rising levels of opportunity costs. We observe that $\lim_{C\to0}(\lambda + \mu) = -\infty$ and $\lim_{C\to\infty}(\lambda + \mu) = 604.00$. That is, in absence of opportunity costs, the shadow prices reveal high negative values of connectivity, while with increasing opportunity costs, they attain a threshold level starting from $C = 1$. Therefore, when the opportunity cost is near zero, implying that a resource manager is indifferent between harvesting the resource and managing the ecological network, the value of connectivity is negative, in

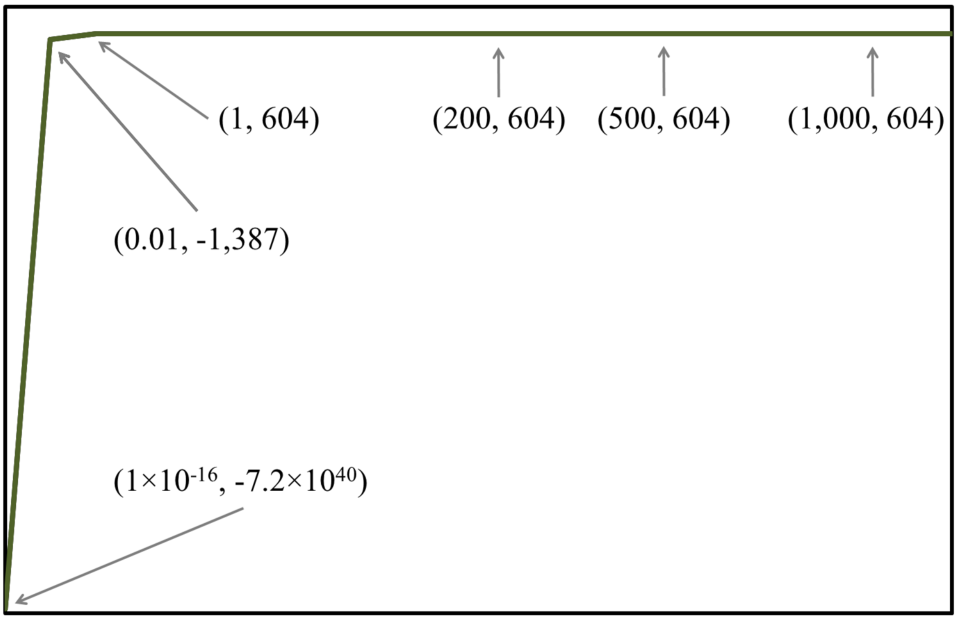

Figure 3: Shadow prices (ordinates) as functions of opportunity costs (abscissa) for equal discounting in space and time of $\delta^{l,t} = 0.02$.

that he or she has to incur costs in building the network. When the opportunity cost of exploiting the resource is sizable, the value of connectivity remains limited, because the role of multifunctional environments is to both provide resource products and ecosystem services.

In case of learning by means of social networks, when an agent is indifferent between trading information with other agents and learning by himself, forming links becomes an obsolete investment, yielding a negative value of connectivity. On the other side, if the opportunity cost is of significant level, the value of connectivity plays its full role but remains upper-bounded, for he or she can always return back to a personal learning. The simulations thus show that maintaining connectivity is optimal in a large array of opportunity costs, but provides with a finite value of connectivity.

Result 1 *The shadow prices are negatively unbounded at near-zero opportunity costs and positively bounded with significant levels of opportunity costs.*

Fig. 4 illustrates the levels of shadow prices discounted, both spatially and temporally, at different levels of factors, for a value of opportunity costs equal to $C = 1,000$. In view of the spatial discounting, where $\delta^l \in [0.00, 0.10]$, we observe that shadow prices increase with the discount factor, meaning that agents value more distant connections. For example, with a fixed level of $\delta^t = 0.10$, the connectivity values are between 672.73 at $\delta^l = 0.00$ and 730.29 at $\delta^l = 0.10$. On the contrary, for a fixed level of discounting in space, the results reveal stationary values of connectivity up to high levels of discounting, which then drastically decrease starting from $\delta^t = 0.40$. In detail, for $\delta^l = 0.01$, subsequent to a stationary level of 622.83, shadow values are in the range of 591.08 to 236.06 for

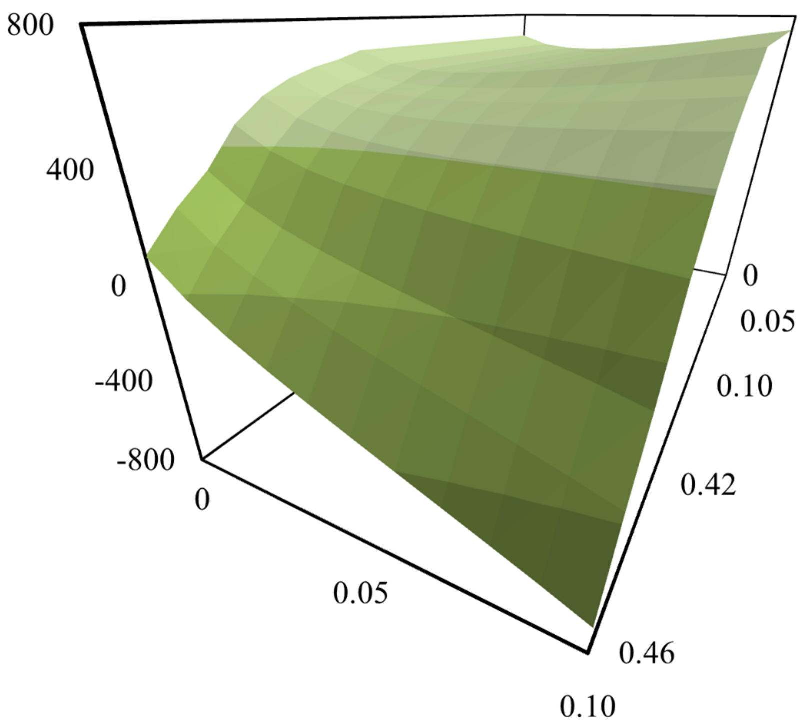

Figure 4: Shadow prices (applicates) with respect to different space (left-sided abscissa) and time (right-sided ordinates) levels of discount rates, with $C = 1,000$.

$\delta^t = [0.40, 0.45]$, be it a decrease of 60%. With $\delta^l = 0.10$, in the same interval of temporal discounting, the values of connectivity go from 660.05 to -125.41, that is, a decrease of 119%.

Therefore, for a fixed level of temporal discounting up to $\delta^t = 0.40$, the connectivity value goes up with increasing spatial discounting, which implies a spacetime discounting complementarity. Above this discounting threshold, the fall in values is all the more accentuated as the level of spatial discounting rises, be it a spacetime discounting substitutability. Provided that high levels of discounting signify low values agents put on distant revenues, both in space and time, the results show that agents are less sensitive to preserving connectivity in time than in space. More accurately, they place an identical value on connectivity, within a large range of temporal discounting rates, which then severely drops. Furthermore, when agents start devaluing connectivity in time, they tend to undervalue the connectivity in space. Due to a net cost of investment in such a configuration, we denote negative values of connectedness far in time and space.

Result 2 *At a fixed level of foregone gains, agents value the safeguard of connections less in time than in space.*

4 Conclusion

Such as stated earlier, we modeled a network of leaders and followers, connected through edges, as a system of inputs and outputs running the consensus protocol. The model outcomes show that the network equilibrium depends on the magnitude of influence of leader nodes, while the network connectivity depends on the density of connectedeness of followers.

Through shadow prices, obtained by solving the optimal control problem, we managed to divulge the economic value of network connectivity discounted in time and space. To the best of our knowledge, this has never been offered nor formalized in the literature. Furthermore, the results achieved via numerical simulations show that connectivity values increase with spatial discounting and decrease with temporal discounting, underway a high threshold level, after being in a stationary state over a long sequence of factor values. The results thus invalidate the usual pattern of hyperbolic discounting observed in a positive orthant, where a very high initial rate of discount is followed by a decline and comparative flattening of the rate of time preference (Frederick et al., 2002). In fact, what we observe is the exact opposite, that is, a reciprocal of a rectangular hyperbola.

In parallel, with regard to the opportunity costs, the fact that, under optimal control, the connectivity value is positively bounded, even with significant levels of foregone gains, tells us that the existence of an outside option, which corresponds to resource exploitation and individual learning in our illustrative examples, does impact the well-known law of increasing opportunity costs (Tucker, 2008). Moreover, our results tie in with the idea, advocated by quantum economists (Schmitt, 1984; Cencini and Rossi, 2015) who refer to David Ricardo's works, that a perfect measure of value consists in finding an invariable one.

With the purpose of broadening the discussion, let us dedicate a couple of words to spacetime theories. According to special relativity, various dimensions satisfy Minkowski spacetime (Skow, 2007). Under this geometry, lengths are assigned to vectors. The vector that points in a spatial direction has a positive sign, unlike the vector pointing in temporal direction, the sign of which is negative. In consequence, those two vectors can be of same magnitude but head toward opposite directions. From the foregoing, it can be noted that a similar pattern to that of connectivity discounted in space as well as in time is observed, for the values tend to increase spatially, which can be translated into a positive sign of change, and decrease temporally, which then matches with a negative sign.

All this considered, we would like to finish by clarifying that this work has to be considered as exploratory, inasmuch as graph theory is a schematized representation of network patterns. Therefore, complementary works, through the concepts developed in behavioral economics (Samson, 2016), notably those on time and space preferences, should be conducted in order to endorse or to disapprove the subject matter designed and discussed in this paper.

Appendix

2 Model

2.2 Dynamics

Equation (4)

$$
\begin{aligned}
\dot{x}_i &= -\sum_{k\in\Lambda^f}\left(x_i(e_i)-\sum_{l=0}^{L}\delta^l x_k(e_k)\right)-\sum_{k\in\Lambda_i^l}\left(x_i(e_i)-\sum_{l=0}^{L}\delta^l x_k(e_k)\right)\\
&+ \sum_{k\in\Lambda^f}\sum_{l=0}^{L}\delta^l\left(x_j(e_j)-x_k(e_k)\right)+\sum_{k\in\Lambda_j^l}\sum_{l=0}^{L}\delta^l\left(x_j(e_j)-x_k(e_k)\right)\\
&= -N_f x_i(e_i)-N_{li}x_i(e_i)+N_f\sum_{l=0}^{L}\delta^l x_j(e_j)+N_{lj}\sum_{l=0}^{L}\delta^l x_j(e_j)\\
&+ \sum_{k\in\Lambda^f}\sum_{l=0}^{L}\delta^l x_k(e_k)+\sum_{k\in\Lambda_i^l}\sum_{l=0}^{L}\delta^l x_k(e_k)-\sum_{k\in\Lambda^f}\sum_{l=0}^{L}\delta^l x_k(e_k)-\sum_{k\in\Lambda_j^l}\sum_{l=0}^{L}\delta^l x_k(e_k)\\
&= -N_f\left(x_i(e_i)-\sum_{l=0}^{L}\delta^l x_j(e_j)\right)-N_l\left(x_i(e_i)-\sum_{l=0}^{L}\delta^l x_j(e_j)\right)\\
&+ \sum_{k\in\Lambda_i^l}\sum_{l=0}^{L}\delta^l x_k(e_k)-\sum_{k\in\Lambda_j^l}\sum_{l=0}^{L}\delta^l x_k(e_k)\\
&= -N\left(x_i(e_i)-\sum_{l=0}^{L}\delta^l x_j(e_j)\right)+\sum_{k\in\Lambda_i^l}\sum_{l=0}^{L}\delta^l x_k(e_k)-\sum_{k\in\Lambda_j^l}\sum_{l=0}^{L}\delta^l x_k(e_k)\\
&= -Nx_i(e_i)+\frac{\delta^{L+1}-1}{\delta-1}\left[Nx_j(e_j)+\sum_{k\in\Lambda_i^l}x_k(e_k)-\sum_{k\in\Lambda_j^l}x_k(e_k)\right]
\end{aligned}
$$

Proof of Lemma 1. Following Bhattacharya *et al.* (2009), let Λ_i, Λ_j and Λ_k be the subsets of Λ. We pose $\Xi = (\Lambda_i \setminus \Lambda_j)\cup\Lambda_k = (\Lambda_i\cup\Lambda_k)\setminus(\Lambda_j\setminus\Lambda_k)$ and $\Omega\times[0,T]=\Xi\subseteq\mathbb{R}^N$, with Ω the convex hull. We thus have a boundary. The consensus problem is well-defined if $(x_i(e_i,0),x_j(e_j,0))\in\Omega(0)$, i.e. if the initial states allow for a proper connection, such that $\underset{j\in\mathbb{R}^N}{\cup} u_{ij}(0) > 0$. Given that the control input compels the network evolution to invariant sets, there exists a fixed point at steady state (Shakarian *et al.*, 2012), and the consensus equilibrium is unique. Accordingly,

$$
\begin{aligned}
0 &= -Nx_i(e_i) + \frac{\delta^{L+1}-1}{\delta-1}\left[Nx_j(e_j) + \sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right] \\
Nx_i(e_i) &= \frac{\delta^{L+1}-1}{\delta-1}\left[Nx_j(e_j) + \sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right] \\
x_i(e_i) &= \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j) + \frac{\delta^{L+1}-1}{N(\delta-1)}\left[\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right]
\end{aligned}
$$

The state of follower node $x_i(e_i)$ converges to an aggregate of the state of leader node $x_j(e_j)$ and the gap between the subnetwork barycenters $\frac{1}{N}\sum_{k\in\Lambda_i^l} x_k(e_k) - \frac{1}{N}\sum_{k\in\Lambda_j^l} x_k(e_k)$.

■

Equation (5)

$$
\begin{aligned}
\dot{x}_j &= -\sum_{k\in\Lambda^l}\left(x_j(e_j) - \sum_{l=0}^{L}\delta^l x_k(e_k)\right) - \sum_{k\in\Lambda_j^f}\left(x_j(e_j) - \sum_{l=0}^{L}\delta^l x_k(e_k)\right) \\
&+ \sum_{k\in\Lambda^l}\sum_{l=0}^{L}\delta^l\left(x_i(e_i) - x_k(e_k)\right) + \sum_{k\in\Lambda_i^f}\sum_{l=0}^{L}\delta^l\left(x_i(e_i) - x_k(e_k)\right) + f\left(|g_j - x_j(e_j)|\right) \\
&= -N_l x_j(e_j) - N_{fj}x_j(e_j) + N_l\sum_{l=0}^{L}\delta^l x_i(e_i) + N_{fi}\sum_{l=0}^{L}\delta^l x_i(e_i) \\
&+ \sum_{k\in\Lambda^l}\sum_{l=0}^{L}\delta^l x_k(e_k) + \sum_{k\in\Lambda_j^f}\sum_{l=0}^{L}\delta^l x_k(e_k) - \sum_{k\in\Lambda^l}\sum_{l=0}^{L}\delta^l x_k(e_k) - \sum_{k\in\Lambda_i^f}\sum_{l=0}^{L}\delta^l x_k(e_k) + f\left(|g_j - x_j(e_j)|\right) \\
&= -N_l\left(x_j(e_j) - \sum_{l=0}^{L}\delta^l x_i(e_i)\right) - N_f\left(x_j(e_j) - \sum_{l=0}^{L}\delta^l x_i(e_i)\right) \\
&+ \sum_{k\in\Lambda_j^f}\sum_{l=0}^{L}\delta^l x_k(e_k) - \sum_{k\in\Lambda_i^f}\sum_{l=0}^{L}\delta^l x_k(e_k) + f\left(|g_j - x_j(e_j)|\right) \\
&= -N\left(x_j(e_j) - \sum_{l=0}^{L}\delta^l x_i(e_i)\right) + \sum_{k\in\Lambda_j^f}\sum_{l=0}^{L}\delta^l x_k(e_k) - \sum_{k\in\Lambda_i^f}\sum_{l=0}^{L}\delta^l x_k(e_k) + f\left(|g_j - x_j(e_j)|\right) \\
&= -Nx_j(e_j) + \frac{\delta^{L+1}-1}{\delta-1}\left[Nx_i(e_i) + \sum_{k\in\Lambda_j^f} x_k(e_k) - \sum_{k\in\Lambda_i^f} x_k(e_k)\right] + f\left(|g_j - x_j(e_j)|\right)
\end{aligned}
$$

Proof of Lemma 2. Let Λ_i, Λ_j and Λ_k be the subsets of Λ. We pose $\Xi = (\Lambda_i \setminus \Lambda_j) \cup \Lambda_k = (\Lambda_i \cup \Lambda_k) \setminus (\Lambda_j \setminus \Lambda_k)$, $\Phi = (\Lambda_i \setminus \Lambda_j) \cap \Lambda_k = (\Lambda_i \cap \Lambda_k) \setminus \Lambda_j$ and $\Omega_g \times [0, T] = \Xi - \Phi \subseteq \mathbb{R}^N$, with Ω the convex hull. We thus have a boundary. The consensus problem is well-defined if $(x_i(e_i, 0), x_j(e_j, 0)) \in \Omega_g(0)$, i.e. if, prior to the effect of the attraction function, the initial states allow for a proper connection, such that $\underset{j \in \mathbb{R}^N}{\cup} u_{ij}(0) > 0$. Given that the control input compels the network evolution to invariant sets, there exists a fixed point at steady state, and the consensus equilibrium is unique. Accordingly,

$$\begin{aligned}
0 &= -Nx_j(e_j) + \frac{\delta^{L+1} - 1}{\delta - 1} \left[Nx_i(e_i) + \sum_{k \in \Lambda_j^f} x_k(e_k) - \sum_{k \in \Lambda_i^f} x_k(e_k) \right] + f\left(|g_j - x_j(e_j)|\right) \\
Nx_j(e_j) &= \frac{\delta^{L+1} - 1}{\delta - 1} \left[Nx_i(e_i) + \sum_{k \in \Lambda_j^f} x_k(e_k) - \sum_{k \in \Lambda_i^f} x_k(e_k) \right] + f\left(|g_j - x_j(e_j)|\right) \\
x_j(e_j) &= \frac{\delta^{L+1} - 1}{\delta - 1} x_i(e_i) + \frac{\delta^{L+1} - 1}{N(\delta - 1)} \left[\sum_{k \in \Lambda_j^f} x_k(e_k) - \sum_{k \in \Lambda_i^f} x_k(e_k) \right] + \frac{1}{N} f\left(|g_j - x_j(e_j)|\right)
\end{aligned}$$

The state of leader node $x_j(e_j)$ converges to an aggregate of the state of follower node $x_i(e_i)$ and the gap between the subnetwork barycenters $\frac{1}{N} \sum_{k \in \Lambda_j^f} x_k(e_k) - \frac{1}{N} \sum_{k \in \Lambda_i^f} x_k(e_k)$ increased by the marginal value of the attraction function $\frac{1}{N} f\left(|g_j - x_j(e_j)|\right)$. ■

Proof of Theorem 1. The proof follows from Lemma 1 and Lemma 2. We know that the follower nodes update their states from the states of the leader nodes, which converge to an aggregate of the network barycenter and the attraction function. We thus have $\lim_{t \to T} x_i(e_i, t) = \lim_{t \to T} x_j(e_j, t) = g_j$, such that $(x_i(e_i, T), x_j(e_j, T)) \in \Omega_g(T)$. Accordingly,

$$\begin{aligned}
x_i(e_i) - x_j(e_j) &= \frac{1}{N} \left[\sum_{k \in \Lambda_i^l} x_k(e_k) - \sum_{k \in \Lambda_j^l} x_k(e_k) + \sum_{k \in \Lambda_i^f} x_k(e_k) - \sum_{k \in \Lambda_j^f} x_k(e_k) \right] \\
&+ \frac{\delta - 1}{N(\delta^{L+1} - 1)} f\left(|g_j - x_j(e_j)|\right)
\end{aligned}$$

■

2.3 Connectivity

Equation (8)

$$
\begin{aligned}
\dot{m}(C)^2_{ij} &= 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[-Nx_i(e_i)\right]C^2 \\
&+ 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(Nx_j(e_j) + \sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right]C^2 \\
&- 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[-Nx_j(e_j)\right]C^2 \\
&- 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(Nx_i(e_i) + \sum_{k\in\Lambda_j^l} x_k(e_k) - \sum_{k\in\Lambda_i^l} x_k(e_k)\right)\right]C^2 \\
&= 2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i) - x_j(e_j)\right)\right] \\
&+ 4C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right]
\end{aligned}
$$

Proof of Corollary 1.

$$
\begin{aligned}
2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i) - x_j(e_j)\right)\right] & \\
+4C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right] &\leq 0 \\
\Leftrightarrow \frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i) - x_j(e_j)\right) + 2\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right) &\leq 0 \\
\Leftrightarrow -\frac{2(\delta^{L+1}-1)\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)}{\left(2-\delta(\delta^L+1)\right)\left(x_i(e_i) - x_j(e_j)\right)} &\leq N
\end{aligned}
$$

because $\frac{1}{\delta-1} < 0$. Provided that $N > 0$ by definition, we have

$$
\begin{aligned}
-\frac{2(\delta^{L+1}-1)\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)}{\left(2-\delta(\delta^L+1)\right)\left(x_i(e_i) - x_j(e_j)\right)} &> 0 \\
\Leftrightarrow \sum_{k\in\Lambda_i^l} x_k(e_k) &> \sum_{k\in\Lambda_j^l} x_k(e_k)
\end{aligned}
$$

■

Equation (9)

$$
\begin{aligned}
\dot{m}(C)^2_{ij} &= 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[-Nx_i(e_i)\right]C^2 \\
&+ 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(Nx_j(e_j) + \sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right]C^2 \\
&- 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[-Nx_j(e_j) + f\left(|g_j - x_j(e_j)|\right)\right]C^2 \\
&- 2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(Nx_i(e_i) + \sum_{k\in\Lambda_j^f} x_k(e_k) - \sum_{k\in\Lambda_i^f} x_k(e_k)\right)\right]C^2 \\
&= 2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i) - x_j(e_j)\right)\right] \\
&- 2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[f\left(|g_j - x_j(e_j)|\right)\right] \\
&+ 2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right] \\
&+ 2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^f} x_k(e_k) - \sum_{k\in\Lambda_j^f} x_k(e_k)\right)\right]
\end{aligned}
$$

Proof of Corollary 2.

$$
\begin{aligned}
&2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i) - x_j(e_j)\right)\right] \\
&-2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[f\left(|g_j - x_j(e_j)|\right)\right] \\
&+2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right] \\
&+2C^2\left(x_i(e_i) - \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^f} x_k(e_k) - \sum_{k\in\Lambda_j^f} x_k(e_k)\right)\right] \leq 0 \\
&\Leftrightarrow \frac{1}{\delta-1}\left[(2-\delta(\delta^L+1))N\left(x_i(e_i) - x_j(e_j)\right)\right] \\
&+\frac{1}{\delta-1}\left[(\delta^{L+1}-1)\left(\sum_{k\in\Lambda_i^l} x_k(e_k) - \sum_{k\in\Lambda_j^l} x_k(e_k) + \sum_{k\in\Lambda_i^f} x_k(e_k) - \sum_{k\in\Lambda_j^f} x_k(e_k)\right)\right] \leq f\left(|g_j - x_j(e_j)|\right)
\end{aligned}
$$

Given that $\frac{1}{\delta-1} < 0$, $f\left(|g_j - x_j(e_j)|\right) \geq 0$ and $N > 0$, we have

$$-\frac{(\delta^{L+1}-1)\left(\sum_{k\in\Lambda_i^l} x_k(e_k)-\sum_{k\in\Lambda_j^l} x_k(e_k)+\sum_{k\in\Lambda_i^f} x_k(e_k)-\sum_{k\in\Lambda_j^f} x_k(e_k)\right)}{(2-\delta(\delta^L+1))\left(x_i(e_i)-x_j(e_j)\right)}>0$$
$$\Leftrightarrow \sum_{k\in\Lambda_i^l} x_k(e_k)+\sum_{k\in\Lambda_i^f} x_k(e_k)>\sum_{k\in\Lambda_j^l} x_k(e_k)+\sum_{k\in\Lambda_j^f} x_k(e_k)$$

■

2.4 Optimal control problem

The Hamiltonian corresponds to

$$H=\begin{cases} 4C^2\left(x_i(e_i)-\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i)-x_j(e_j)\right)\right]e^{-\delta t} \\ -2C^2\left(x_i(e_i)-\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[f\left(|g_j-x_j(e_j)|\right)\right]e^{-\delta t} \\ +2C^2\left(x_i(e_i)-\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^f} x_k(e_k)-\sum_{k\in\Lambda_j^f} x_k(e_k)\right)\right]e^{-\delta t} \\ +6C^2\left(x_i(e_i)-\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l} x_k(e_k)-\sum_{k\in\Lambda_j^l} x_k(e_k)\right)\right]e^{-\delta t} \\ +\lambda^T\left[-Nx_i(e_i)+\frac{\delta^{L+1}-1}{\delta-1}\left[Nx_j(e_j)+\sum_{k\in\Lambda_i^l} x_k(e_k)-\sum_{k\in\Lambda_j^l} x_k(e_k)\right]\right] \\ +\mu^T\left[-Nx_j(e_j)+\frac{\delta^{L+1}-1}{\delta-1}\left[Nx_i(e_i)+\sum_{k\in\Lambda_j^f} x_k(e_k)-\sum_{k\in\Lambda_i^f} x_k(e_k)\right]+f\left(|g_j-x_j(e_j)|\right)\right] \end{cases}$$

The first-order optimality conditions are

$$\begin{aligned}\frac{\partial H}{\partial x_i(e_i)} &= 4C^2\left(x_i(e_i)-\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\right]e^{-\delta t} \\ &+ \lambda^T[-N]+\mu^T\left[\frac{\delta^{L+1}-1}{\delta-1}N\right]=0\end{aligned}$$

from which we obtain

$$x_i(e_i)=\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)+\frac{\lambda(\delta-1)-\mu(\delta^{L+1}-1)}{2-\delta(\delta^L+1)}(4C^2)^{-1}Se^{\delta t}$$

And

$$\begin{aligned}\frac{\partial H}{\partial x_j(e_j)} &= -4C^2\left(x_i(e_i)-\frac{\delta^{L+1}-1}{\delta-1}x_j(e_j)\right)^T S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N+\frac{1}{2}\right]e^{-\delta t} \\ &+ \lambda^T\left[\frac{\delta^{L+1}-1}{\delta-1}N\right]-\mu^T\left[N-1\right]=0\end{aligned}$$

which yields

$$x_i(e_i) = \frac{\delta^{L+1}-1}{\delta-1}x_j(e_j) + \frac{\lambda(\delta^{L+1}-1)N - \mu(\delta-1)(N-1)}{\left(2-\delta(\delta^L+1)\right)N + \frac{1}{2}(\delta-1)}(4C^2)^{-1}Se^{\delta t}$$

The boundary conditions are as follows

$$\begin{aligned}
\dot{\lambda} &= -\left[\frac{\partial H}{\partial x_i(e_i)}\right]^T \\
&= -4C^2S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i)-x_j(e_j)\right)\right]e^{-\delta t} \\
&+ 2C^2S^{-1}\left[f\left(|g_j - x_j(e_j)|\right)\right]e^{-\delta t} \\
&- 2C^2S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^f}x_k(e_k) - \sum_{k\in\Lambda_j^f}x_k(e_k)\right)\right]e^{-\delta t} \\
&- 6C^2S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l}x_k(e_k) - \sum_{k\in\Lambda_j^l}x_k(e_k)\right)\right]e^{-\delta t} \\
&+ \lambda N - \mu\frac{\delta^{L+1}-1}{\delta-1}N
\end{aligned}$$

And

$$\begin{aligned}
\dot{\mu} &= -\left[\frac{\partial H}{\partial x_j(e_j)}\right]^T \\
&= -4C^2\frac{\delta^{L+1}-1}{\delta-1}S^{-1}\left[\frac{2-\delta(\delta^L+1)}{\delta-1}N\left(x_i(e_i)-x_j(e_j)\right)\right]e^{-\delta t} \\
&- 2C^2\frac{\delta^{L+1}-1}{\delta-1}S^{-1}\left[f\left(|g_j - x_j(e_j)|\right)\right]e^{-\delta t} \\
&+ 2C^2\frac{\delta^{L+1}-1}{\delta-1}S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^f}x_k(e_k) - \sum_{k\in\Lambda_j^f}x_k(e_k)\right)\right]e^{-\delta t} \\
&+ 6C^2\frac{\delta^{L+1}-1}{\delta-1}S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(\sum_{k\in\Lambda_i^l}x_k(e_k) - \sum_{k\in\Lambda_j^l}x_k(e_k)\right)\right]e^{-\delta t} \\
&+ \lambda\frac{\delta^{L+1}-1}{\delta-1}N + \mu N
\end{aligned}$$

Proof of Lemma 3. From the optimality conditions, we obtain

$$\frac{\lambda(\delta-1)-\mu(\delta^{L+1}-1)}{2-\delta(\delta^{L}+1)} = \frac{\lambda(\delta^{L+1}-1)N-\mu(\delta-1)(N-1)}{(2-\delta(\delta^{L}+1))\,N+\frac{1}{2}(\delta-1)}$$
$$\Leftrightarrow \mu = \lambda\frac{(2N(\delta+2)-1)(\delta-1)^2-2N(\delta^{L+1}-1)(\delta+\delta^{L+1}-2)}{(\delta-1)\left[(2N(\delta+2)-1)(\delta^{L+1}-1)-2(N-1)(\delta+\delta^{L+1}-2)\right]}$$

■

Proof of Theorem 2. By Theorem 1, we know that the network is at equilibrium. We have determined the first-order necessary optimality conditions to set up the Hamiltonian coordinates. All the necessary and sufficient conditions are met. ■

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