Water Allocation, Crop Choice, and Priority Services (preliminary)

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Abstract

PRELIMINARY AND INCOMPLETE – PLEASE DO NOT CIRCULATE.

Allocating water for irrigation among different farmers should be a simple cake-sharing problem, but the stochasticity of the amount of available water raises some interesting issues. The creation of a spot market is ex-post efficient; but conflicts are harder to solve ex-post than ex-ante, and efficiency also requires the creation of contingent markets for insurance reasons. We also examine a popular system in use in many countries, that we call uniform rationing; and we show that it is inefficient, and leads farmers to overexpose to risk, thus making shortages more severe and more frequent in case of drought. Based on the analysis in Wilson (1989), we propose a system of priority

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classes, for which we derive an efficiency result, and that can easily be complemented by simple insurance contracts. We also characterize the set of farmers that may be hurt by such a reform.

Keywords: Water, Irrigation, Priority Classes.
JEL Classification: .

1 Introduction

The agricultural sector is a major consumer of water, and this input is an essential one for food production.[1] It turns out that in many countries water resources are becoming scarcer, due to growing demand from various categories of users, and maybe also due to climate change. Climate change also introduces an element of scientific uncertainty that supports more flexible systems to allocate water among farmers. Indeed, existing systems are often characterized by important rigidities. Farmers are endowed with rights that are not precisely defined, given that the amount of available water is uncertain. These rights are not tradable, and are often based on past usage or observed needs: as we shall show, this creates incentives for an over-optimal exposure to risk, and in fine makes crisis more severe and more frequent.

This has led different countries to reconsider various mechanisms to improve water use efficiency. There are evidences that water for irrigation is not used efficiently. It is now widely recognized that existing management systems rely too much on administrative rules, and not enough on economic

[1] According to the Millennium Ecosystem Assessment (2005), about 18% of the world cropland receive supplementary water through irrigation.

instruments. For instance, in France irrigation water is currently allocated according to an administrative system. French farmers have to apply for annual irrigation licenses, which are specified for a given location and volume and are renewed each year. In the French river basins that have adopted "volumetric management of water", available water is allocated to users proportionally to their licenses (Lefebvre et al., 2014). Climate change is likely to require more flexible systems which are able to cope with extreme events, as well as to adapt to various climatic scenarios.

Hence, it seems likely that properly implemented policy interventions such as water markets and water pricing may improve water use efficiency. Markets have indeed be created in Australia, in the western states of the USA, and in Chile. Dinar and Mody (2003) survey factors that prevent water markets from becoming more widespread. One of the main disadvantages of spot markets is the high volatility of the spot price. This volatility is high for structural reasons: nobody needs irrigation water when there is plenty, and everybody needs it when there is none. Demand and supply are thus negatively correlated, due to climate shocks. Clearly, highly volatile spot markets may create political difficulties in case of droughts, since a major conflict is to be solved in a period when all actors face a major negative shock. This feature also supports the creation of a system of contingent markets, that allow to provide insurance. But the overall complexity of such a market system may become unsustainable, given that in addition the good which is priced is differentiated by location and period of availability.[2]

[2] In addition, in many countries farmers are historically used to very low prices (Bazzani et al, 2002, Hamdy, 2002).

This paper aims at understanding whether market systems are unavoidable, or whether there exists other management systems that would reach a better balance between efficiency, complexity, and social acceptability. Our main proposal is to create priority classes for irrigation as an alternative way of allocating water. This system requires farmers to subscribe ex-ante to occupy seats in the queue for water: ex-post the highest priority class is served first, then the second highest, and so on until all available water is exhausted. Wilson (1989) shows in a quite general framework that this alternative achieves most of the efficiency gains attributed to spot markets. While such systems are used in the electricity sector, to the best of our knowledge nobody has yet proposed to apply the priority class mechanism to irrigation.

We proceed as follows. We first set up a rather general model of cake-sharing under uncertainty, with the key ingredient that heterogeneous farmers choose ex-ante (before the climatic shock) their exposure to risk, i.e., the share of their land they devote to crops that depend on water availability. Ex-post, the realized amount of available water is allocated among farmers, according to some institutional system. We then examine the properties of three such systems (spot market, uniform rationing, and priority services), and we also derive results on which systems is preferred by which type of farmers.

A key trade-off is characterized between ex-ante and ex-post efficiency. Ex-post efficiency requires giving more water to marginally more productive crops. The ex-ante choices of farmers such as risk exposure (robust or vulnerable crops), or their investments in irrigation devices, take into account how water will be allocated ex-post. So, the mechanism used ex-post matters for ex-ante choices, and thus for overall efficiency.

The uniform rationing system is built on the idea that farmers who grow more irrigated crops should be given more water ex-post. This is inefficient ex-post because such a system does not take into account the heterogeneity of farmers. Hence, it gives too much water to relatively inefficient farmers and not enough to relatively efficient ones. On top of that, the uniform rationing is inefficient ex-post, even if farmers are identical: farmers are led to increase their risk exposure in order to secure more water, a feature that favors the repetition of crisis.

The priority class system is then analyzed. Wilson (1989) described such a system for cases in which demand is 0/1. We extend it to general demand functions, thus making it closer to a forward market for water – or more precisely for seats in the queue for water. We provide conditions under which this system is as efficient as a market system. We also show that some farmers may be hurt by the switch from a uniform rationing system to the priority class system, and we characterize the set of such farmers.

2 Literature and Evidence

TO BE COMPLETED.

3 The Model

Consider a continuum of heterogeneous farmers, indexed by a one-dimensional parameter θ with a cumulative distribution function F (we discuss in Section XX what may happen if farmers are not infinitesimal). Farmers belong to the same hydrological basin. Each farmer owns a given land area, which can be devoted to different crops with different needs for water. The general timing is as follows.

Ex-ante (typically in fall), each farmer chooses the value of a variable x. In what follows x is interpreted as the area devoted by the farmer to the crop whose yield depends on irrigation water, and is thus a one-dimensional measure for risk-exposure. As discussed in Section XX, the case of multiple ex-ante choices can also be handled, at the price of more complex notations.

The economy is then affected by a climatic shock s whose distribution is known. s is a state of nature, i.e., a complete description of technological possibilities. Hence s contains information on, say, temperature, or the amount of rainwater that accrued to each farmer's area. This shock also determines $Q(s)$, the total amount of water (from rivers or groundwater) available for irrigation, assumed positive for all s. In what follows we denote by E the expectation operator with respect to the distribution of s.

Ex-post (in summer) the quantity $Q(s)$ is allocated among the farmers, according to three different institutional systems to be described in the next sections: a market for water, a uniform rationing rule, or a system with priority classes. We shall discuss the relative efficiency of these three systems, based on the farmers' payoffs in each case.

We assume that farmers are endowed with a quasi-linear utility function, and therefore that they are risk-neutral with respect to monetary revenues (wealth effects and risk-aversion are discussed in Section XX). Hence farmer θ gets a gross surplus $B(q,x,s,\theta)$ when he gets q units of water in state s, and his risk exposure level is x. We assume that B is smooth, increasing in q, and strictly concave in (x,q). This last property ensures in particular that the farmer's demand function for water is well-defined:[3]

$$B_q(q,x,s,\theta) = p \Leftrightarrow q = D(p,x,s,\theta).$$

We also assume the following property:[4]

$$qB_q + xB_x \text{ is increasing in } q. \tag{1}$$

This is the property we need to ensure that the unique solution to

$$\max_x \; EB(kxQ(s),x,s,\theta)$$

is increasing in the parameter $k > 0$.[5] In words: if a farmer is promised a share of available water which is proportional to the irrigated area x, then he will choose a higher x, the higher the promised share. This does not seem very demanding, as the second case below illustrates.[6]

[3] Subscripts denote partial derivatives.

[4] Assume that function B is homogeneous of degree k, i.e., $B(\alpha z) = \alpha^k B(z)$ for an integer k and $z = (q,x)$. In such a case the Euler's homogeneous function theorem holds:

$$z \cdot \nabla B(z) = kB(z),$$

i.e., the property (1) is satisfied.

[5] A proof of this claim is given at the beginning of the Proof Appendix.

[6] To be discussed somewhere: with non-infinitesimal farmers, the program becomes $\max_x \; EB(kxQ(s)/(X+x),x,s,\theta)$. It seems sufficient to add the condition $B_{xq} > 0$.

Two particular cases will help to convey intuitions. **In the homogeneous case**, all farmers are identical, so that the surplus B and the demand D do not depend on θ. **In the heterogeneous case**, we assume that farmers are different in that their land is more or less productive when dedicated to the crop that rely on irrigation water. We then measure this productivity in efficiency units: only the product θx matters. Secondly, we assume that every efficiency unit of irrigated land displays the same productivity. This leads to the following specification:

$$B(q, x, s, \theta) = \theta x b\left(\frac{q}{\theta x}, s\right) - c(x)$$

where we assume that the cost function c is strictly convex, and that the function b is strictly concave in its first argument. It is easily checked that this homothetic specification satisfies (1). It also leads to a simple expression for demand:

$$D(p, x, s, \theta) = \theta x d(p, s)$$

where d is the inverse function of the first derivative b_1 of b. In other words, all farmers react identically to a change of price or to the climatic shock s, for each efficiency unit of irrigated land.

To summarize, this setting is designed to allow for quite general specifications, so as to make theoretical predictions as general as possible. Important limitations are that land cannot be exchanged, and that the timing is limited to one period of irrigation.

4 Efficiency and the Spot Market

In the absence of wealth effects and risk-aversion, it is natural to define efficiency as expected surplus maximization. A planner would thus choose ex-ante decisions $x(\theta)$ and ex-post water allocation $q(\theta, s)$ to maximize

$$\int_\theta EB(q(\theta, s), x(\theta), s, \theta) dF(\theta)$$

under the feasibility constraints in each state s:

$$\int_\theta q(\theta, s) dF(\theta) \leq Q(s).$$

Let $p^*(s)$ be the shadow price of water in state s. Under strict concavity, $p^*(s)$ and the solution (x^*, q^*) are uniquely determined by the following equalities:[7]

$$\begin{aligned} q^*(\theta, s) &= D(p^*(s), x^*(\theta), s, \theta) \\ Q(s) &= \int_\theta q^*(\theta, s) dF(\theta) \\ x^*(\theta) &\in \arg\max_x \; E \max_q [B(q, x, s, \theta) - p^*(s) q]. \end{aligned}$$

Without surprise, the first welfare theorem holds for this economy without any externalities. The creation of a spot market that opens ex-post to balance water supply and water demand ensures that both the consumption of water and the ex-ante decisions can be freely chosen by each farmer. The price of water exhausts all trade opportunities ex-post, and exerts the right incentives to choose the optimal decisions ex-ante.

[7] For simplicity, and without loss of generality, we ignore the states of nature in which $Q(s)$ is so high that the shadow price of water is zero.

Nevertheless, one should not underestimate the difficulties associated with the functioning of a spot market for water. The equilibrium price is likely to be highly volatile, as supply and demand are strongly negatively correlated for evident climatic reasons. On the one hand, this remark underlines that the farmers' risk-aversion should not be neglected, and consequently that the creation of contingent markets should accompany the creation of a spot market. This will be discussed in our study of a priority system. On the other hand, it is difficult in practice to ask farmers to pay a high price for water when their revenues are already threatened by a dry climate. We will come back to this acceptability issue later on.

For further reference, we can simplify the above characterization in our two cases. In the homogeneous case, we now have an explicit expression for the equilibrium price of water:

$$\begin{aligned} q^*(s) &= Q(s) \\ p^*(s) &= B_q(Q(s), x^*, s) \\ x^* &\in \arg\max_x \; E \max_q [B(q, x, s) - p^*(s)q]. \end{aligned}$$

The heterogeneous case also yields substantial simplifications:

$$\begin{aligned} q^*(\theta, s) &= \frac{\theta x^*(\theta)}{\int_t t x^*(t) dF(t)} Q(s) \\ p^*(s) &= b_1 \left(\frac{Q(s)}{\int_t t x^*(t) dF(t)}, s \right) \\ x^*(\theta) &\in \arg\max_x \; \theta x E \max_{q'} [b(q'^*(s)q'] - c(x). \end{aligned}$$

From these equations, one immediately sees that farmers with higher types choose a higher risk-exposure, and consume a higher quantity of water. In fact, they even consume a higher quantity of water per unit of land; on the other hand, the quantity of water per efficiency unit of land is a constant, since these units are homogeneous. Therefore, the optimal allocation rule allocates water in proportion to irrigated land area, measured in efficiency units θx. This is a key difference with uniform rationing, to which we now turn.

5 Uniform Rationing

Uniform rationing is used in many countries to allocate water ex-post when it is scarce. It can be seen as an attempt to safeguard efficiency ex-post, since it aims at directing water to those farmers who need it most. The key difficulty is however that while the irrigated area is observed ex-post, the productivity of water in each plot is not observable. This leads us to the following definition:

Definition: *Uniform rationing allocates water in proportion to each observed irrigated area x.*

We can now derive how such a system works, and what are its consequences. Ex-post, given the choices $x(.)$ for irrigated land and the state s, each farmer θ gets a quantity of water equal to

$$\frac{x(\theta)}{X}Q(s)$$

where $X = \int x(t)dF(t)$ is the aggregate irrigated area the other farmers have chosen. By construction, water supply is fully allocated. Ex-ante, given

X each farmer thus chooses x to maximize

$$EB\left(\frac{x}{X}Q(s), x, s, \theta\right).$$

In other words, farmers now play a Nash equilibrium, instead of having their choices coordinated through the spot market. Under our concavity assumption, there exists a unique solution $x(\theta, X)$ to the above maximization problem, and this solution is decreasing with X under (1). As a consequence, there exists a unique Nash equilibrium of this game, that we index with the subscript R; it is such that

$$x^R(\theta) = x(\theta, X^R) \qquad X^R = \int x^R(\theta)dF(\theta).$$

To study this equilibrium outcome in more details, define

$$p^R(\theta, s) = B_q\left(\frac{x^R(\theta)}{X^R}Q(s), x^R(\theta), s, \theta\right)$$

as the implicit price that farmer θ faces in state s. Then the farmer's problem is to maximize on x, given X^R, the expected profit

$$E\max_q\,[B(q, x, s, \theta) - p^R(\theta, s)q] + x\frac{E[p^R(\theta, s)Q(s)]}{X^R}. \tag{2}$$

Compared to the efficient case, in which the farmer had to maximize on x the expected profit

$$E\max_q\,[B(q, x, \theta, s) - p^*(s)q], \tag{3}$$

two differences appear. Firstly, each farmer now faces an idiosyncratic price. If this price indeed depends on θ, then some gains from trades are not exploited, and thus the allocation of water is ex-post inefficient. Secondly, the farmer is reimbursed the amount paid: this is the last term in the profit above. This term can be interpreted as a positive subsidy to x. Therefore, the farmer's choice of x is likely to be too high, compared to the social optimum. However, one has to be careful as this intuition need not hold generally: if all farmers choose a higher x, then their water quotas will be accordingly reduced, and we cannot exclude that some farmers choose a value of x below the social optimum.

To go further, we first deal with the homogenous case. We immediately obtain the following proposition:

Proposition 1 *Under uniform rationing, in the homogenous case the allocation of water is ex-post efficient ($q^*(s) = q^R(s) = Q(s)$), but farmers choose an excessive level for their irrigated area ($x^R > x^*$).*

The first result amounts to say that in the homogeneous case a symmetric allocation of total water supply is optimal. The second result shows in full generality one of the main message of this study: uniform rationing leads to excessive risk-exposure. By trying to allocate water as a function of ex-post needs, this mechanism encourages farmers to irrigate more, and finally leads to more dramatic crisis in dry years.

In the heterogenous case, uniform rationing has differentiated effects, as follows:[8]

Proposition 2 *Under uniform rationing, in the heterogenous case there exists $\theta_1 < \theta_2$ such that:*

[8] Proofs not in the text are relegated to the Appendix.

i) Farmers θ below θ_1 get too much water per unit of irrigated land, and choose to overexpose to risk ($x^R(\theta) \geq x^*(\theta)$);

ii) Farmers $\theta \in [\theta_1, \theta_2]$ get too few water per unit of irrigated land, and choose to overexpose to risk ($x^R(\theta) \geq x^*(\theta)$);

iii) Farmers θ above θ_2 get too few water per unit of irrigated land, and choose to underexpose to risk ($x^R(\theta) \leq x^*(\theta)$).

iv) Assume that $c(x) = \frac{c_0 x^2}{2}$, then:

$$\int_\theta x^R(\theta)dF(\theta) - \int_\theta x^*(\theta)dF(\theta) \leq \frac{Ep^*(s)Q(s)}{c_0 X^R} \tag{4}$$

with the equality if function b is linear in q'.

6 Priority services

We now introduce a third institutional system, in which farmers are invited to pay in advance for registering in a queue for water. In the Walrasian tradition, a price system $P(.)$ is announced ex-ante, and farmers are offered the opportunity to buy at price $P(C)$ the C^{th} unit of water. De jure, this unit will be delivered only if it is available, i.e., if s is such that $Q(s) \geq C$. No other payments or transactions are needed; all action thus takes place ex-ante.

The question we ask is whether we can attain efficiency with such a simple system. This will be the case if one may in addition organize and open a spot market ex-post, without impacting the functioning of the priority system. This thought experiment leads to a no-arbitrage condition, that determines the pricing of priority classes:

$$P(C) = E\left[p^*(s)1_{C \leq Q(s)}\right].$$

At these prices, a farmer would then buy a portfolio with different Cs, that can be represented by an increasing function G. For this portfolio, he would pay:

$$\int_C P(C)dG(C) = E\left[p^*(s)\int_{C\leq Q(s)} dG(C)\right] = E\left[p^*(s)G(Q(s))\right].$$

By doing so he would thus get a quantity of water $G(Q(s))$ in state s. His expected profit would therefore be

$$E\left[B\left(G(Q(s)), x, s, \theta\right) - p^*(s)G(Q(s))\right]$$

to be maximized over G and x, under the constraint that G is non-decreasing. Efficiency obtains if one can find G such that

$$q^*(\theta, s) = G(Q(s)).$$

Indeed, under this condition we are back at the efficient choice of x (see (3)). Overall we have shown:

Proposition 3 *A system of priority classes is efficient if the following two conditions hold:*

- $q^*(\theta, s)$ *is a function of* $Q(s)$ *and* θ *only;*
- *and this function increases with* Q*.*

The second requirement can be relaxed if the buyer can sell in advance the C^{th} unit (provided it has bought a C' unit, with $C' < C$). The first requirement is more demanding; s may matter *per se* to the farmer, even when $Q(s)$ is known. For example, for the same available quantity Q it might have rained on the farmer's parcels, or not.

7 Winners and Losers

The ex-ante and ex-post inefficiencies of uniform rationing motivate a reform aiming at more efficiency. Still, even though the aggregate wealth produced would be higher after the reform, there are constraints to redistributing these gains so that everybody is better off. In this section, we investigate whether this phenomenon is important, and who are the farmers who are most likely to be hurt – hereafter the losers, to make it short.

Interestingly, we are able to provide a general result that will help us identify winners and losers. One can write (omitting the variables s and θ)

$$E[B(q^R, x^R) - p^* q^R] \leq E[B(q^*, x^*) - p^* q^*]$$

because (q^*, x^*) maximizes the right-hand-side. The term on the left-hand-side is the expected profit $\mathcal{B}^R$ of farmer θ under uniform rationing, while the first term on the right-hand-side is his profit under an efficient market system. However, in this latter case the revenues from water sale are available and may be redistributed to farmers. Given that one does not want to perturb efficiency, the only thing one can do is to redistribute these payments in a lump-sum manner, each farmer getting the same amount $E[p^*Q]$. Hence profits of farmer θ under a market system with lump-sum redistribution are

$$\mathcal{B}^* \equiv E[B(q^*, x^*) - p^* q^*] + E[p^* Q].$$

Therefore the above inequality becomes

$$\mathcal{B}^R \leq \mathcal{B}^* + E[p^* q^R - p^* Q].$$

One may notice that this inequality holds not only for uniform rationing but also for any system different from a market one. It implies that if the water consumption of farmer θ under current system, evaluated at the efficient prices, is less than the average water consumption, then this farmer benefits from the switch to the market system with lump-sum redistribution.

Recalling that

$$q^R = \frac{x^R}{X^R} Q,$$

we obtain the following result.

Proposition 4 *Suppose one switches from uniform rationing to a market system with lump-sum redistribution. If under the uniform rationing farmer* θ *chooses the irrigated area* x^R *lower or equal to the average* X^R*, then this farmer benefits from the switch.*

In the heterogeneous case we obtain the following result.

Proposition 5 *Assume that* $c(x) = \frac{c_0 x^2}{2}$*, then in the heterogenous case there exists* $\widehat{\theta} \in [\theta_1, \theta_2]$ *such that farmers with* θ *above* $\widehat{\theta}$ *benefit from the switch from the uniform rationing to a market system (without redistribution) and the farmers with* θ *below* $\widehat{\theta}$ *loose from the switch.*

7.1 Acceptability

TO BE COMPLETED. In this subsection we would like to explore the idea of acceptability of the reform, i.e., the switch from the uniform rationing to the priority services (market system). Logically, if the distribution $F(\theta)$ is such that the number of farmers falling into the category of losers is significant then the switch would be opposed.

8 Extensions

8.1 Risk and Insurance

TO BE COMPLETED. The idea here is to redistribute the revenues from selling seats in the queue to farmers who end up without water, so as to reach efficiency even when farmers are risk-averse. We thus extend a result in Wilson (1989) to the case of general demand functions.

8.2 Incentives to cheat

TO BE COMPLETED. Incentives to cheat can be measured by $B_q(q, x, s, \theta)$, which is equal to p^R under uniform rationing and to p^* under a market system. As we have shown[9] p^R is increasing in θ, while p^* does not depend on θ. Moreover, $p^R \leq p^*$ for $\theta \leq \theta_1$, and the opposite inequality holds for $\theta > \theta_1$.

9 Conclusion

TO BE COMPLETED ... SORRY!

References

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[9] See Appendix, proof of Proposition **2**.

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Proof Appendix:

Proof of the claim in section 3: consider the problem

$$\max_{x} \; EB(kxQ(s), x)$$

where we omit the variables s and θ for clarity. We want it to be supermodular in $(k,\, x)$, so that we write the cross-derivative

$$E[QB_q + xQ(kQB_{qq} + B_{qx})]$$

and by setting $q(s) = kxQ(s)$ we obtain

$$E[Q(B_q + qB_{qq} + xB_{xq})]$$

which indeed is positive under (A1). This concludes the proof.

Proof of Proposition 1: Using (2) and (3) in the homogeneous case we have

$$E\left(B_x(Q(s), x^R, s) - \frac{p^R(\theta, s)Q(s)}{X^R}\right) = 0$$

and

$$EB_x(Q(s), x^*, s) = 0.$$

Therefore, $B_x(Q(s), x^R, s) < B_x(Q(s), x^*, s)$. From the concavity assumption B_x is decreasing in x. This completes the proof.

Proof of Proposition 2: In the heterogeneous case we have

$$p^R(\theta, s) = b_1\left(\frac{Q(s)}{\theta X^R}, s\right)$$

which is increasing with respect to θ. The water quantity per unit of irrigated land is $\frac{q^R(\theta,s)}{x^R(\theta)} = \frac{1}{X^R}Q(s)$, which is (by the very definition of uniform rationing) independent from θ. In the efficient situation, the price $p^*(s)$ is independent from θ, and the water quantity per unit of irrigated land is $\frac{q^*(\theta,s)}{x^*(\theta)} = \frac{\theta}{\int tx^*(t)dF(t)}Q(s)$, which increases with θ. As announced in the Proposition, the two quantities are equal for $\theta = \theta_1 \equiv \frac{\int tx^*(t)dF(t)}{X^R}$.

Moreover farmer θ chooses x to maximize $\theta x Eb(\frac{xQ(s)}{\theta X^R}, s) - c(x)$, and therefore

$$c'(x^R(\theta)) = \theta Eb\left(\frac{Q(s)}{\theta X^R}, s\right). \tag{5}$$

The corresponding expression for the efficient situation is

$$c'(x^*(\theta)) = \theta E[b(d(p^*(s), s) - p^*(s)d(p^*(s), s)] \tag{6}$$

which implies that $A^* \equiv \frac{c'^*(\theta))}{\theta}$ is independent from θ, while $A^R \equiv \frac{c'^R(\theta))}{\theta}$ is decreasing in θ. At $\theta = \theta_1$, we obtain

$$A^R = Eb\left(\frac{Q(s)}{\int tx^*(t)dF(t)}, s\right) = Eb(d(p^*(s), s) > A^* = E[b(d(p^*(s), s) - p^*(s)d(p^*(s), s)]$$

and therefore the two quantities can only be equal at some $\theta_2 > \theta_1$.

iv) The difference (4) can be rewritten as:

$$\frac{\theta}{c_0}\int_\theta E\left[b\left(\frac{Q(s)}{\theta X^R}, s\right) - [b(d(p^*(s), s) - p^*(s)d(p^*(s), s)]\right] dF(\theta)$$

By concavity:

$$b\left(\frac{Q(s)}{\theta X^R}, s\right) - b\left(d\left(p^*(s), s\right), s\right) \leq \left[\frac{Q(s)}{\theta X^R} - d\left(p^*(s), s\right)\right] p^*(s).$$

After rearranging the terms one obtains:

$$b\left(\frac{Q(s)}{\theta X^R}, s\right) - \left[b\left(d\left(p^*(s), s\right), s\right) - d\left(p^*(s), s\right)p^*(s)\right] \leq \frac{Q(s)p^*(s)}{\theta X^R}.$$

This concludes the proof.

Proof of Proposition 5

Under the assumption of quadratic cost function the expression (5) simplifies to

$$x^R = \frac{\theta}{c_0} Eb\left(\frac{q^R}{\theta x^R}\right).$$

Therefore, the profit of farmer θ under the uniform rationing can be written as

$$\mathcal{B}^R = \frac{1}{2} c_0 \left(x^R\right)^2, \tag{7}$$

or equivalently,

$$\mathcal{B}^R = \frac{\theta^2}{2c_0}\left(Eb\left(\frac{q^R}{\theta x^R}\right)\right)^2. \tag{8}$$

Along the same lines, (6) simlifies to

$$x^* = \frac{\theta}{c_0} E\left[b\left(d\left(p^*\right)\right) - p^* d(p^*)\right].$$

Hence, the profit of farmer θ under a market system without redistribution is

$$\mathcal{B}^* = \frac{1}{2} c_0 \left(x^*\right)^2 + E\left[p^* q^*\right], \tag{9}$$

or, equivalently,

$$\mathcal{B}^* = \frac{\theta^2}{2c_0}\left[\left(Eb\left(\frac{q^*}{\theta x^*}\right)\right)^2 - \left(Ep^* d(p^*)\right)^2\right], \tag{10}$$

From (8) and (10) one can obtain the expression for the difference:

$$\frac{2c_0}{\theta^2}\left(\mathcal{B}^* - \mathcal{B}^R\right) = \left[\left(Eb\left(d(p^*)\right)\right)^2 - \left(Ep^* d(p^*)\right)^2\right] - \left(Eb\left(\frac{q^R}{\theta x^R}\right)\right)^2.$$

One may notice that the second term is decreasing in θ and the first one is independent of θ. On top of that, at $\theta = \theta_1$

$$\frac{2c_0}{\theta^2}\left(\mathcal{B}^* - \mathcal{B}^R\right) = -\left(Ep^* d(p^*)\right)^2 < 0$$

and from (7) and (9) at $\theta = \theta_2$

$$\mathcal{B}^* - \mathcal{B}^R = E\left[p^* q^*\right] > 0.$$

This concludes the proof.