The Pricing of Recyclable Exhaustible Resources

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Abstract

We consider a model of extraction of an exhaustible resource by a monopolist, in which current demand increases future demand. We aim mainly at analyzing the effect of recycling on the rate of extraction of the monopoly, on the exhaustion date of phosphorus, on the dynamic of the price of the resource and on consumers' surplus. Toward these ends, we use an optimal control model and show four main results: (i) the price increases through time if the level of recyclability is low, (ii) the price decreases then increases if the level of recyclability is high, (iii) the higher the recyclability rate, the more extraction and the exhaustion date are delayed, (iv) a higher recyclability rate leads to an increase in price in the short-run (a decrease of consumers'surplus in the short run) while it decreases after.

Exaustible Resource, Hotelling's Rule, Optimal Control.

1 Introduction

In the current context of the depletion[1] of phosphorus, recycling has attracted increasing attention. Indeed, many countries including Germany, Netherlands, Sweden, United States of America, Canada, China, Japan, etc. engage in recycling. In these countries, recycling has become a part of everyday life (Blomberg and Söderholm, 2009). Some papers have argued that recycling will delay the exhaustion of this important resource (Cordell and al., 2009 and Cordell and al., 2011). To the best of our knowledge, only two papers have theoretically[2] dealt with the issue of recycling of phosphorus. Weikard and Seyhan (2009) and Seyhan and al. (2012) have investigated whether recycling contributes to the prolongation of the lifetime of phosphorus or not. Using optimal control models, both papers find that recycling delays the depletion of phosphorus. Also, Weikard and Seyhan (2009) show that recycling increases the short-run extraction, whereas it decreases the long-run extraction. Dealing with a continuous-time model, our paper shows, unlike Weikard and Seyhan (2009), that the short-run extraction decreases in the recycling rate and the long-run extraction increases with recycling.

[1] For instance, see Tweeten (1989): assuming that demand will increase at the rate of 3.6%, he stresses that phosphate reserves will be depleted in 61 years, i.e. in 2050; Smill (2000): he argues that phosphate reserves will be depleted in 80 years, i.e. in 2080; IDFC (2010): this report outlines that phosphate reserves will be exhausted in $300 - 400$ years.

[2] Notice that there are many papers which have, empirically, explored the issue of recycling of phosphorus (see Cordell and al.,). In line with the theoretical papers, they have stressed that recycling delays the depletion of phosphorus.

Thank to the consideration of a continuous-time model, a number of interesting questions emerges. Does the price of phosphorus increase continuously over time, as suggested by Hotelling (1931) in the case of exhaustible resources ? Is the path of extraction of the monopolist sensitive to the length of the period ? Is recycling always beneficial to consumers ? Does the discount factor affect the pace of extraction of phosphorus ? Does recycling prolong the lifetime of phosphorus or can the announced dates of the depletion of phosphorus be modified by the implementation of recycling? Is the level of the reserves crucial in the postponement of the depletion of the resource ? In the present paper, we address these and related questions.

In order to answer the questions above, we postulate an optimal control model where two firms compete. We consider a monopolist that holds phosphate reserves. He is faced with a recycling sector that recycles one part of the consumed phosphorus. We assume that both firms do not bear costs.

Summarizing some of our main findings, we show that (i) the price increases through time if the level of recyclability is low, (ii) the price decreases then increases if the level of recyclability is high, (iii) the higher the recyclability rate, the more extraction and the exhaustion date are delayed, (iv) a higher recyclability rate leads to an increase in price in the short-run (a decrease of consumers'surplus in the short-run) while it decreases after. It is worth pointing out that point (iv) results from the consideration of the continuous dynamic framework. Indeed, through a two-period model (Chapter II of our thesis), we have shown, in the case where phosphorus is exhausted over the two periods, that recycling increases the first-period production of the monopolist, meaning that it decreases the initial price of the monopoly. This contrasts sharply with point (iv).

This paper is also related to another strand of the literature, which concerns the pricing of exhaustible or durable[3] resource/good. Coase (1972) focused on the issue of the durable good monopolist. He argued that the monopolist of a durable good would be forced to provide the competitive equilibrium because the monopoly, having sold some output previously, would have the incentive to always maximize profits against the residual demand, and that this would entail producing more output so long as the total quantity was below the competitive quantity. Hence, the global stock would be close to the competitive stock, and the price would want to wait until the competitive price was reached, forcing the monopolist to produce the competitive quantity right away. Knowing that the price of the durable good will decline, rational consumers prefer to postpone their purchase in the future. In order to incite consumers to buy in the present period, some solutions have been identified by the economists. Indeed, the monopolist must rent the durable good instead of selling it (Bullow, 1982), must commit that he will not increase the output in the subsequent period for not to reduce the future price, has to establish a dynamically consistent plan. Investigating the case of a durable-good monopolist who cannot precommit, Bullow (1982) shows that the optimal dynamically consistent plan

[3]Notice that a durable Good is a good that does not quickly wear out, or more specifically, one that yields utility over time rather than being completely consumed in one use (Wikipedia). Phosphorus is taken as a durable good in the sense that it does not disappear completely, thank to the possibility of recycling. Other types of durable good include gold, diamonds, aluminum and silver and (Malueg and Solow, 1998; Levhari and Pindyck, 1981).

consists of producing a quantity which is below the monopoly quantity in the first period, and then, optimizing against the residual demand, producing some additional output in the second period. Bond and Samuelson (1984) find that depreciation of the good and replacement sales reduce the monopolist's tendency to cut down price. Assuming constant marginal costs, Kahn (1986) echoes the conclusion of Bond and Samuelson (1984). Focusing on a durable resource, Stewart (1980) stresses that the monopolist sets higher prices in the first periods, whereas prices decline over time. Van den Berg and al. (2012) consider two asymetric firms, which compete in quantity. They are asymetric in the sense that one firm holds a large intitial supply and the other firm has a medium-sized stock. They assume that the larger firm faces no capacity constraint, whereas the smaller firm is capacity-constrained in the second-period. They find that the equilibrium price strictly decreases over time. Istemi (2014) considers a resource duopoly model with two firms, competing in quantity for two consecutive periods. In the first-period, each firm is endowed with a fixed amount of exhaustible resource stock and is then allowed to invest in capacity in between the two periods of production in order to increase its resource stock. Thus, their second-period capacity constraints become endogenous. It finds that the equilibrium price weakly decreases over time with endogenous capacity constraints. In contrast to this earlier literature, we show that price decreases in the first periods before rising after reaching its minimum at a certain date. Contrary to the previous literature, the following authors highlight that the price of an exhaustible resource follows an upward direction. Using a non-cooperative Cournot Oligopoly model, Loury (1986) shows that aggregate production of oil decreases over time, meaning that its price increases over time. Van den Berg and al. (2012) find that the price weakly increases over time. Our result is consistent with theirs in the case where the recycling rate is low. Nevertheless, if the recycling rate is high, our paper states that the price does not increase continuously, in the sens that the price curve is U-shaped.

The primary contribution of this paper is that the price of an exhaustible resource can take a downward phase. This result is not in line with Hotelling-based reasoning, wich states that the scarcity of rent of the exhaustible resources would cause the prices to increase over time (Hotelling, 1931). This result captures the short-run stylized characteristics of exhaustible resource markets in which price drops are observed from time to time (Istemi, 2014). For example, the price may decline for several reasons. First, it can be the case due to discovery of new reserves, which increases the supply of the exhaustible resource. Second, it can be explained by the improvement of the technologies of production. Such an improvement reduces the cost of production of the exhaustible resource, resulting in the decline of its price. It is noteworthy to mention that we aware of two papers which show that the price can decrease from time to time, namely Levhari and Pindyck (1981) and Gaskins (1974). Levhari and Pindyck (1981) aim at investigating whether Hotelling's r-percent rule (which states that the rent or the price of exhaustible resources must rise at the interest rate) holds or not. They distinguish the case where the durable resource is perfectly durable and that where the resource is partially[4] durable. In the context of a competitive market and constant demand, they find that price always falls, and the price profile is U-shaped for a partially durable resource or growing demand or both.

[4] They define a partially durable resource as a resource that depreciates over time.

By analogy, we can say that the resource is perfectly durable in our case in that it does not depreciate. Although dealing with different market structures (competitive market for them and monopoly for us), we find the same result. Where they don't identify the profile of the price in the monopoly[5] case, we find that the price curve is always U-shaped.

Empirically, Martinez-de-Albéniz and Talluri (2011) investigate price competition for an oligopoly in a dynamic setting, where each of the sellers has a fixed number of units avalaible for sale over a fixed number of periods. They assume that demand is stochastic and find that prices decrease in the first periods and increase in the last periods.Hnyilicza and Pindyck (1976) charactrize the optimal price trajectories for a cartel (OPEC). For some discount rates[6], they show that the price of the exhaustible resource decreases[7] the first periods (from 1975 to 1980) before rising slowly the last periods (from 1980 to 2010). For these authors, the curve of price is U-shaped.

The remainder of the paper is organized as follows. The next section introduces Hotelling's standard model where the monopolist of the exhaustible resource faces a recycling fringe. Section 3 presents the optimal recycled quantity. Section 4 describes the optimal extraction path and price dynamics. The role of recyclability is established in secton 5. The main conclusions and some further research lines are given in section 6 and all proofs are relegated to the appendix.

2 The Hotelling Model with a Recycling Fringe

Consider an economy in which consumption is given by Q. The consumption good is a nonrenewable resource which can be recycled and the virgin and recycled materials are assumed to be perfect substitutes in demand. There is a unique resource owner and a competitive fringe of recycling firms.

Nonrenewable resource and scrap dynamics: The resource is characterized by its initial stock, $X^0 \geq 0$. Let $X \geq 0$ be the residual stock at time t and $x \geq 0$ the extraction rate. Then,

$$\dot{X} = -x. \tag{1}$$

Let the cost of extraction of the resource be zero. Let $r \geq 0$ be the quantity of recycled materials marketed at time t. The total quantity consumed at time t is then $Q = x + r$. Let $S(0) = S^0 \geq 0$ be the initial quantity of scrap. Let $S \geq 0$ be the stock of scrap at time t. Let $\alpha \in [0, 1]$ the proportion of resource that is not recycled that becomes recyclable scrap. The dynamic of scrap writes $\dot{S} = \alpha (Q - r)$ or,

$$\dot{S} = \alpha x, \tag{2}$$

where parameter α represents the "recyclability" rate of the nonrenewable resource.

The recycling sector: The recycling sector is a competitive fringe. We assume that the marginal cost of recycling is decreasing in the stock of scrap net of the quantity of recycled

[5]They find that Hotelling's r-percent rule does not hold in the case of monopoly.

[6]For instance $\delta = 0.02$, $\delta = 0.05$, $\delta = 0.10$.

[7]Except for the initial date where the cartel charges higher price.

materials (i.e. the remaining stock of scrap):

$$c(S,r) = 1 - b - \beta(S - r), \tag{3}$$

with $\beta > 0$ and $b \in (0,1)$.

We assume that the inverse demand is linear, $p(Q) = 1 - Q$. Thus, b is a measure of the added value of recycled material compared to scrap.

In equilibrium, the price must equal the marginal cost of recycling:

$$p(Q) = c(S,r). \tag{4}$$

The extraction sector: The owner of the resource chooses the optimal level of extraction that maximizes its discounted profits with discount rate $\delta \geq 0$,

$$\underset{\{x\}}{Max} \int_{0}^{+\infty} e^{-\delta t} p(Q)\, x dt, \tag{5}$$

subject to (4), (1), (2), $X, S, x \geq 0$ and X^0 given and $S(0) = 0$.

3 Optimal Extraction of the Recyclable Resource

Solving the recycling sector equilibrium condition (4), we characterize the equilibrium quantity of recycled material at time t as follows:

$$r = \frac{b + \beta S - x}{1 + \beta}. \tag{6}$$

Thus, the quantity of recycled material at time t increases with the quantity of scrap and (proportionally) decreases with the quantity of extracted resource. We focus on the cases in which the right hand side is nonnegative.

The current value Hamiltonian H and Lagrangian $\mathcal{L}$ are defined as follow:

$$H = p(Q)\, x + \lambda_X(-x) + \lambda_S(\alpha x), \tag{7}$$

and,

$$\mathcal{L} = H + \mu_X X + \mu_S S + \mu_x x, \tag{8}$$

where λ_X and λ_S are the co-state variables associated with the stocks X and S, and, μ_X, μ_S, μ_x the multipliers associated with the nonnegativity constraints $X \geq 0$, $S \geq 0$ and $x \geq 0$. The total quantity writes $Q = x + r = [b + \beta(S + x)] / (1 + \beta)$.

The necessary conditions include:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\beta}{1+\beta} p'(Q)\, x + p(Q) - \lambda_X + \alpha \lambda_S + \mu_x = 0, \tag{9}$$

$$\dot{\lambda}_X = \delta\lambda_X - \frac{\partial \mathcal{L}}{\partial X} = \delta\lambda_X - \mu_X, \tag{10}$$

$$\dot{\lambda}_S = \delta\lambda_S - \frac{\partial \mathcal{L}}{\partial S} = \delta\lambda_S - \mu_S - \frac{\beta}{1+\beta} p'(Q)\, x, \tag{11}$$

$$x \geq 0, \mu_x \geq 0, \mu_x x = 0, \tag{12}$$

$$X \geq 0, \mu_X \geq 0, \mu_X X = 0, \tag{13}$$

$$S \geq 0, \mu_S \geq 0, \mu_S S = 0, \tag{14}$$

and two transversality constraints:

$$\lim_{t \to +\infty} e^{-\delta t} \lambda_X(t)\, X(t) = 0, \tag{15}$$

$$\lim_{t \to +\infty} e^{-\delta t} \lambda_S(t)\, S(t) = 0, \tag{16}$$

and $S(0) \geq 0$ and $X^0 > 0$ given.

In the rest of the paper, we focus on the solution of the problem such that the optimal exhaustion date of the nonrenewable resource is finite, $T^* < +\infty$. Let $p^*(t)$ be the optimal price path.

4 Optimal Extraction Path and Price Dynamics

Proposition 1 [Optimal Path]: *The optimal extraction path is such that the extracted quantity decreases while the recycled material quantity increases through time:*

$$\dot{x}^*(t) \leq 0 \text{ and } \dot{r}^*(t) \geq 0.$$

Proposition 1 states that the optimal level of extraction decreases through time. This result is in line with the standard Hotelling model: since the extractor discount time, he chooses to extract larger quantities of resource today and less tomorow. The quantity of recycled material increases through time. This is a quite intuitive result. Indeed, extracted quantities become scrap and increase the stock of scrap. Hence, the stock of scrap increases and then the unit cost of recycling decreases. At the same time, extracted quantities decrease, which increases the market price and encourages recycling.

Proposition 2 [Price Dynamics]: *The price of the final good is never decreasing if and only if the recyclability rate is sufficiently low. Formally, there exists $\widehat{\alpha} > 0$ such that $\partial p^*/\partial t \geq 0$ for all t if and only if $\alpha \leq \widehat{\alpha}$.*

Corollary 1 [Non Monotonic Price]: *If the recyclability rate is sufficiently large compared to the discount rate and the initial stock of the resource is sufficiently large, then the price of the final good first decreases and then increases. Formally, if $\alpha > \widehat{\alpha}$ there exists $0 < \widehat{t} < T^*$ such that $\partial p^*/\partial t < 0$ if $t \in [0, \widehat{t})$ and $\partial p^*/\partial t \geq 0$ if $t \in [\widehat{t}, T^*]$.*

Proposition 2 and Corollary 1 can be illustrated thanks to numerical Example XX:

Numerical Example a: Let $X^0 = 1$, $\beta = 1$ and $\delta = 0.02$. Figure XX shows the optimal extraction path for different values of α.

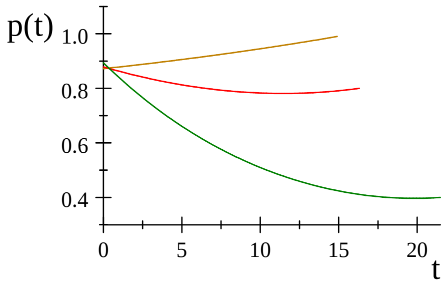

Figure 1: Non Monotonic Price Dynamics ($\alpha = 0.01$**;** 0.2; 0.6)

This figure shows that, in the case where the recycling rate is high, Hotelling's rule which states that the price of exhaustible resources increases over time is called into question. This behavior of the price is more consistent with what happens in many markets of exhaustible resources where the price usually follows cyclical phases. For instance, a cyclical phase has been observed through the market of phosphorus between 2006 and 2010. In fact, from January 2006 to January 2008, one has observed the rise of the price of phosphorus. At this point in time, the price has reached its maximum. From January 2008 to January 2010, the price has declined (Cordell and White, 2011). The peak of the price in 2008 results from the fact that China had imposed a 135% export tariff on phosphates (Ashley and al., 2011). Thus, this effectively halted China's exports (Fertilzer Week, 2008).

5 The Role of Recyclability

Proposition 3 [Exhaustion date and Recyclability]: *The optimal exhaustion date increases with the initial stock and the recyclability rate of the resource. Formally:*

$$\frac{\partial T^*}{\partial X^0} > 0 \textit{ and } \frac{\partial T^*}{\partial \alpha} > 0$$

Proposition 3 states that the date of exhaustion of the resource increases with the initial stock, which is intuitive. It also states that the exhaustion date increases with the recyclability rate of the resource. This suggests that recyclability delays extraction, which is confirmed the result of Proposition 3.

Proposition 4 [Extraction and Recyclability]: *Early extraction decreases while latter extraction increases with recyclability. Formally, there exists a date $0 < \tilde{t} < T^*$ such that*

$$\frac{\partial x^*}{\partial \alpha} < 0 \textit{ for } t \in [0, \tilde{t}) \textit{ and } \frac{\partial x^*}{\partial \alpha} \geq 0 \textit{ for } t \in \left(\tilde{t}, T^*\right).$$

Proposition 4 states that when the level of recyclability of the resource increases, then extraction is delayed. It first decreases and increases latter. The intuition of this result is as follows. When the resource is not exhausted ($X > 0$), the dynamic of its shadow price follows the Hotelling's rule, the (shadow) price of the resource grows at a rate equal to the discount rate, $\dot{\lambda}_X/\lambda_X = \delta > 0$. This means that the extracting firm has incentives to extract the resource early. However, unlike in a standard Hotelling model, the extractor faces the recycling fringe sector. When there is a stock of scrap ($S > 0$) the dynamic of the shadow price of scrap is $\dot{\lambda}_S = \delta\lambda_S + \frac{\beta}{1+\beta}x$. If there is no extraction ($x = 0$), the $\dot{\lambda}_S/\lambda_S = \delta > 0$, then the owner of the resource has an incentive to delay extraction (in order to maintain a small scrap stock), as long as $\alpha > 0$. If the level of extraction is positive, $\frac{\beta}{1+\beta}x > 0$, which leads to a tendancy for the shadow price of scrap to increase, which reinforce the incentives to delay extraction, as long as $\alpha > 0$. Hence, the higher the recyclability rate of the final good, the larger the resource owner incentives to postpone extraction. Since the resource is exhausted in finite time and the initial stock is fixed, extraction increases with the recyclability rate at some point in time.

Numerical Example a: Let $X^0 = 1$, $\beta = 1$ and $\delta = 0.02$. Figure XX shows the optimal extraction path for different values of α.

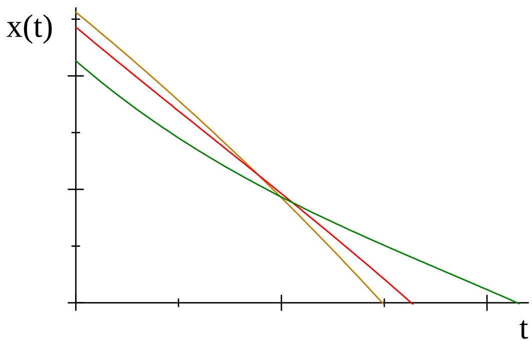

Figure 2: Extraction Path ($\alpha = 0.01$**;** 0.2; 0.6)

Proposition 5 [Price and Recyclability]: *The price first increases and then decreases with the recyclability rate . Formally, there exists a date* $0 < t' < T^*$ *such that*

$$\frac{\partial p^*}{\partial \alpha} > 0 \text{ for } t \in [0, t') \text{ and } \frac{\partial p^*}{\partial \alpha} \leq 0 \text{ for } t \in (t', T^*).$$

This result shows that a higher recyclability of the resource is not always beneficial to consumers. In the short term, consumers are worse off while they are better off in when the resource is sufficiently depleted. The intuition of this result is as follows. The price of the final good negatively depends on the extraction level and the stock of scrap. When the recyclability rate increases, the stock of scrap tends to increase. However, it increases with extraction with a factor $\alpha < 1$, the increase in recycling cannot compensate for the short run decrease in extraction. This explains why the price of the final good increases in the short run. When going close to the exhaustion date, an increase in the recyclability rate increases both the scrap stock and the level of extraction. This explains why the price of the final good decreases when going close to the exhaustion date.

This pricing behavior is in line with what happens in the market of phosphorus. Through the following graphic, one can observe that the price of phosphorus increases between January 2006 and January 2008, reaches its maximum, and then declines until January 2010.

This result can be illustrated by the following numerical example.

Numerical Example a: Let $X^0 = 1$, $\beta = 1$ and $\delta = 0.02$. Figure XX shows the optimal extraction path for different values of α.

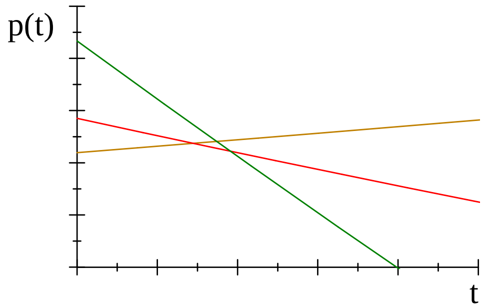

Figure 4: Recyclability and Price Dynamics ($\alpha = 0.01$; 0.2; 0.6)

6 Discussion and conclusion

Thank to the importance of phosphorus in the agricultural process and due the announced depletion of the resource in a near future, recycling becomes a major challenge for many Countries. In fact, in order to reduce their dependence on phosphate imports, many Countries of european union (like Germany, Netherlands, Sweden, etc.) have implemented recycling factories. Recycling is also undertaken in United States of America, in Canada, in China, in Japan. With this renewed interest, many papers have investigated whether recycling of phosphorus delays the depletion of the resource or not. Cordell and al. (2009) have emprically shown that this is particularly true. It is noteworthy to mention that only two papers have theoretically dealt with this issue. Weikard and Seyhan (2009) and Seyhan and al. (2012) have analyzed the effect of recycling on the path of extraction of phosphate reserves. They show that recycling increases the short-run extraction, while it decreases the long-run extraction. Since the decreasing effect dominates the increasing effect, they conclude that recycling delays the exhaustion of phosphorus. Our final conclusions which state that recycling contributes to prolong the lifetime of phosphorus are identical, but in contrast to them, we show clearly that recycling decreases the short-run extraction, resulting then in the decline of consumers'surplus, whereas it increases the long-run extraction (consumers'surplus increases). The decline of the price, observed in our model in the case where the recycling rate is high, contradicts Hotelling-based reasoning, which states that the scarcity of rent of the exhaustible resources would cause the prices to increase over time. Indeed, in contrast to Hotelling (1931), the dynamic of the price follows, in our case, a downward phase in the first periods. After reaching its minimum, the price follows an upward phase in the last periods. Accordingly, the curve of the price is U-shaped in our case, whereas it is monotonic in the case of Hotelling. Our result may explain why the markets of exhaustible resources dont present the features previously highlighted by Hotelling. We have also demonstrated other interesting results. In fact, our paper shows that recycling contributes to the posteponement of the exhaustion of phosphorus in the sense that extraction decreases in the recycling rate, whereas the date of the depletion of the resource increases with the latter. Also, we have found that consumers' surplus decreases with recyclability in the short term. Naturally, the following question can arise: May consumers oppose the implementation of the recycling activites? At the first glance, we may reply by the positive. But may be for other reasons related to a willing of conserving this precious resource or to the altruism, they can accepte the implementation of the recycling activities, in spite the fact that they are worse off in terms of surplus. Among other results established in this paper, we can stress the fact that higher is the recycling rate or higher is the level of the stock of phosphorus, later is the resource depleted. Such a result shows that, the announced dates of the depletion of phosphorus can be modified by recycling and by new discoveries which increase phosphate reserves. We also show that higher is the discount factor, higher is the rate of extraction. This last result indicates that our monopolist prefers present to the future, meaning that he is impatient.

In this paper, we have assumed that recycled phosphorus and extracted phosphorus are substitutes. The challenging question consists of investigating whether the established results still hold in the case where both resources are complements or not. Note that some papers have argued that recycled phosphorus and extracted phosphorus are not perfectly substitutable in the sense that the quality[8] of the former is lower than that of the latter. Naturally, it will be interesting to set up a model which takes into account the vertical differentiation of these two types of phosphorus.

[8]No that other papers show that recyled phosphorus is better than extracted phosphorus in the sense that it reduces eutrophication, at least in the short run, whereas extracted phosphorus increases it due to water run-off and soil erosion.

Appendix

Proof of Proposition 1: Using the equilibrium recycling condition (6) and substituting, we have $p(Q) = \frac{\beta}{1+\beta}(a - x - S)$, where $a = (1 - b + \beta)/\beta$. The maximization problem then formally writes:

$$\underset{\{x\}}{Max} \int_0^{+\infty} \frac{\beta e^{-\delta t}}{1+\beta}(a - x - S)\, x dt, \tag{17}$$

subject to (6), (1), (2), $X, S, x \geq 0$, X^0 and S^0 given.

Thus, for the new problem, the necessary conditions include:

$$\frac{\partial \mathcal{L}}{\partial x} = a - 2x - S - \lambda_X + \alpha\lambda_S + \mu_x = 0, \tag{18}$$

$$\dot{\lambda}_X = \delta\lambda_X - \frac{\partial \mathcal{L}}{\partial X} = \delta\lambda_X - \mu_X, \tag{19}$$

$$\dot{\lambda}_S = \delta\lambda_S - \frac{\partial \mathcal{L}}{\partial S} = \delta\lambda_S - \mu_S + x, \tag{20}$$

$$x \geq 0, \mu_x \geq 0, \mu_x x = 0, \tag{21}$$

$$X \geq 0, \mu_X \geq 0, \mu_X X = 0, \tag{22}$$

$$S \geq 0, \mu_S \geq 0, \mu_S S = 0, \tag{23}$$

and two transversality constraints:

$$\lim_{t \to +\infty} e^{-\delta t}\lambda_X(t)\, X(t) = 0, \tag{24}$$

$$\lim_{t \to +\infty} e^{-\delta t}\lambda_S(t)\, S(t) = 0. \tag{25}$$

Let us assume that the solution is such that $x(t) > 0$ and $X(t) > 0$ over $[0, T)$ and $x(t) = X(t) = 0$ for $t \geq T$. We also assume that $S(t) > 0$ for all $t > 0$.

First consider the first phase where $t \in [0, T)$. Since $x(t) > 0$, $X(t) > 0$ and $S(t) > 0$, using (21), (22), and (23), we have $\mu_x = \mu_X = \mu_S = 0$. Then (19) writes

$$\dot{\lambda}_X = \delta\lambda_X, \tag{26}$$

and then

$$\lambda_X = c_1 e^{\delta t}, \tag{27}$$

where c_1 is a constant to be determined latter.

Conditions (18), and (20) write

$$a - 2x - S - c_1 e^{\delta t} + \alpha\lambda_S = 0, \tag{28}$$

and,

$$\dot{\lambda}_S = \delta \lambda_S + x, \tag{29}$$

Differentiating (28) with respect to time, we find

$$-2\dot{x} - \dot{S} - \delta c_1 e^{\delta t} + \alpha \dot{\lambda}_S = 0. \tag{30}$$

Using (28) and (30), we find

$$-2\dot{x} - \dot{S} - \delta c_1 e^{\delta t} - \delta \left(a - 2x - S - c_1 e^{\delta t} \right) + \alpha \left(\dot{\lambda}_S - \delta \lambda_S \right) = 0, \tag{31}$$

or,

$$-2\dot{x} - \dot{S} + \delta S + (\alpha + 2\delta)\, x - \delta a = 0, \tag{32}$$

Differentiating (2) with respect to time, we obtain

$$\ddot{S} = \alpha \dot{x}. \tag{33}$$

Substituting (33) and (2) into (32), and rearranging, we have

$$\ddot{S} - \delta \dot{S} - \frac{\alpha\delta}{2} S + \frac{\alpha\delta}{2} a = 0. \tag{34}$$

Solving for the stock of scrap S we find

$$S = c_2 e^{\gamma^+ t} + c_3 e^{\gamma^- t} + a, \tag{35}$$

where $\gamma^+ = \frac{\delta + \sqrt{\delta(2\alpha+\delta)}}{2}$ and $\gamma^- = \frac{\delta - \sqrt{\delta(2\alpha+\delta)}}{2}$.

Differentiating (35) with respect to time, we obtain

$$\dot{S} = \gamma^+ c_2 e^{\gamma^+ t} + \gamma^- c_3 e^{\gamma^- t}. \tag{36}$$

Using (2), we have

$$x = \frac{\gamma^+}{\alpha} c_2 e^{\gamma^+ t} + \frac{\gamma^-}{\alpha} c_3 e^{\gamma^- t}. \tag{37}$$

Substituting (37) into (29), we obtain

$$\dot{\lambda}_S - \delta \lambda_S = \frac{\gamma^+}{\alpha} c_2 e^{\gamma^+ t} + \frac{\gamma^-}{\alpha} c_3 e^{\gamma^- t}. \tag{38}$$

Solving for the shadow price of the stock of scrap λ_S we find

$$\lambda_S = D e^{\delta t} + \frac{\gamma^+}{\alpha\left(\gamma^+ - \delta\right)} c_2 e^{\gamma^+ t} - \frac{\gamma^-}{\alpha\left(\delta - \gamma^-\right)} c_3 e^{\gamma^- t}. \tag{39}$$

Using $X^0 - X(t) = \int\limits_0^t x dt$ and integrating (37) between 0 and t, we find

$$X^0 - X(t) = \frac{1}{\alpha}\left(c_2\left(e^{\gamma^+ t} - 1\right) + c_3\left(e^{\gamma^- t} - 1\right)\right). \tag{40}$$

Now consider the second phase where $t \geq T$. We have $x(t) = 0 = X(t) > 0$ and $S(t) > 0$. Using (23), we have $\mu_S = 0$. Condition (20) writes

$$\dot{\lambda}_S = \delta \lambda_S, \tag{41}$$

and then

$$\lambda_S = c_5 e^{\delta t}, \tag{42}$$

where c_5 is a constant to be determined latter.

Notice that $\dot{S} = \alpha x = 0$, and then

$$S = c_4, \tag{43}$$

where c_4 is a constant to be determined latter.

Using (43) and (42), transversality constraint (25) becomes

$$c_4 c_5 = 0. \tag{44}$$

Assume $c_5 \neq 0$. Then, using (43) at $t = T$, we have $S(T) = c_4 = 0$. Combining (35) and (40) and taking $t = T$, we have $\alpha X^0 = S(T)$. Hence, we must have $X^0 = 0$, which is false. We conclude that $c_5 = 0$ and then, for $t \geq T$, we have

$$\lambda_S = 0. \tag{45}$$

In order to solve for c_1, c_2, c_3, c_4, D and T, let us to focus on solutions such that λ_S is continuous. Using (39) and (45) at $t = T$, we obtain

$$De^{\delta T} + \frac{\gamma^+}{\alpha(\gamma^+ - \delta)} c_2 e^{\gamma^+ T} - \frac{\gamma^-}{\alpha(\delta - \gamma^-)} c_3 e^{\gamma^- T} = 0. \tag{46}$$

Using $x(T) = 0$ and (37), we have

$$\gamma^+ c_2 e^{\gamma^+ T} + \gamma^- c_3 e^{\gamma^- T} = 0. \tag{47}$$

Using $X(T) = 0$ and (40), we have

$$\alpha X^0 = c_2\left(e^{\gamma^+ T} - 1\right) + c_3\left(e^{\gamma^- T} - 1\right). \tag{48}$$

Using (35) at $t = 0$, we have

$$S^0 = c_2 + c_3 + a. \tag{49}$$

Using (28) and (35) at $t = T$, we obtain

$$c_2 e^{\gamma^+ T} + c_3 e^{\gamma^- T} + c_1 e^{\delta T} = 0. \tag{50}$$

Using (35) and (43) at $t = T$, we have

$$c_4 = c_2 e^{\gamma^+ T} + c_3 e^{\gamma^- T} + a. \tag{51}$$

Solving for c_1, c_2, c_3, c_4, D and T thanks to conditions (46)-(51), we obtain

$$\begin{aligned}
c_1 &= \frac{\gamma^+ - \gamma^-}{\gamma^+ e^{\gamma^+ T} - \gamma^- e^{\gamma^- T}} \left(a - S^0\right), \\
c_2 &= -\frac{\gamma^- e^{\gamma^- T}}{\gamma^+ e^{\gamma^+ T} - \gamma^- e^{\gamma^- T}} \left(S^0 - a\right), \\
c_3 &= \frac{\gamma^+ e^{\gamma^+ T}}{\gamma^+ e^{\gamma^+ T} - \gamma^- e^{\gamma^- T}} \left(S^0 - a\right), \\
c_4 &= \left(S^0 - a\right) \frac{\left(\gamma^+ - \gamma^-\right) e^{\delta T}}{\gamma^+ e^{\gamma^+ T} - \gamma^- e^{\gamma^- T}} + a, \\
D &= \frac{a - S^0}{\alpha} \frac{\gamma^+ - \gamma^-}{\gamma^+ e^{\gamma^+ T} - \gamma^- e^{\gamma^- T}},
\end{aligned}$$

and the exhaustion date T^* is implicitly characterized by :

$$\alpha X^0 = \left(a - S^0\right) \left(1 - \frac{\gamma^+ - \gamma^-}{\gamma^+ e^{\gamma^+ T^*} - \gamma^- e^{\gamma^- T^*}} e^{\delta T^*}\right). \tag{52}$$

We conclude that the optimal extraction path is, for $t \in [0, T]$:

$$x^*(t) = \frac{\left(a - S^0\right) \delta}{2} \left(\frac{e^{\gamma^+ T^*} e^{\gamma^- t} - e^{\gamma^- T^*} e^{\gamma^+ t}}{\gamma^+ e^{\gamma^+ T^*} - \gamma^- e^{\gamma^- T^*}}\right), \tag{53}$$

the stock of scrap is, for $t \in [0, T]$,

$$S^*(t) = \left(a - S^0\right) \left(1 - \frac{\gamma^+ e^{\gamma^+ T^*} e^{\gamma^- t} - \gamma^- e^{\gamma^- T^*} e^{\gamma^+ t}}{\gamma^+ e^{\gamma^+ T^*} - \gamma^- e^{\gamma^- T^*}}\right), \tag{54}$$

and the market price, for $t \in [0, T]$,

$$p^*(t) = S^0 + \left(a - S^0\right) \left(\frac{\gamma^+ - \gamma^-}{2}\right) \frac{e^{\gamma^+ T^*} e^{\gamma^- t} + e^{\gamma^- T^*} e^{\gamma^+ t}}{\gamma^+ e^{\gamma^+ T^*} - \gamma^- e^{\gamma^- T^*}}. \tag{55}$$

Since $\gamma^+ > 0 > \gamma^-$, the extraction level $x^*(t)$ characterized in (53) decreases through time, while the stock of scrap, since increases through time, $\dot{S}^*(t) = \alpha x^*(t) \geq 0$. Recycling is given by

$$r^*(t) = \frac{b}{\beta} + a\left(1 - \frac{\left(\gamma^+ + \frac{\delta}{2\beta}\right)e^{\gamma^+T^*}e^{\gamma^-t} - \left(\gamma^- + \frac{\delta}{2\beta}\right)e^{\gamma^-T^*}e^{\gamma^+t}}{\gamma^+e^{\gamma^+T^*} - \gamma^-e^{\gamma^-T^*}}\right), \tag{56}$$

and it is increasing through time.□

Proof of Proposition 2: From (55) we know that the price is

$$p^*(t,\alpha) = \frac{a}{2}\sqrt{\delta(2\alpha+\delta)}\frac{e^{\gamma^+T^*}e^{\gamma^-t} + e^{\gamma^-T^*}e^{\gamma^+t}}{\gamma^+e^{\gamma^+T^*} - \gamma^-e^{\gamma^-T^*}}, \tag{57}$$

The sign of the derivative with respect to time is given by

$$\frac{\partial p^*}{\partial t} \propto \gamma^-e^{\gamma^+T^*}e^{\gamma^-t} + \gamma^+e^{\gamma^-T^*}e^{\gamma^+t}, \tag{58}$$

which is positive if and only if

$$t \geq T^* + \frac{1}{\gamma^+ - \gamma^-}\ln\left(1 - \frac{\delta}{\gamma^+}\right). \tag{59}$$

Hence, $\frac{\partial p^*}{\partial t} \geq 0$ for all $t \in [0,T]$ if and only if

$$T^* \leq \frac{1}{\gamma^+ - \gamma^-}\ln\left(\frac{\gamma^+}{\gamma^+ - \delta}\right). \tag{60}$$

We know from Proposition 3 that the left hand side of condition (60) is increasing with α. The derivative of the right hand side with respect to γ^+ is $\frac{-1}{(2\gamma^+-\delta)^2}\left(2\ln\frac{\gamma^+}{\gamma^+-\delta} + \frac{\delta(2\gamma^+-\delta)}{\gamma^+(\gamma^+-\delta)}\right) < 0$, thus it is decreasing with α. When α goes to 0, γ^+ goes to δ and then the right hand side in (60) goes to $+\infty$. A first order approximation of (52) at $\alpha = 0$ leads to $\left(\frac{X^0}{a}\delta + 1 - \delta T^*\right)e^{\delta T^*} \simeq 1$ and the solution of this equation is $T^* < +\infty$ because the left hand side is $\frac{X^0}{a}\delta + 1 > 1$ at $T^* = 0$, it increases up to $T^* = \frac{X^0}{a}$ and then decreases and goes to $-\infty$ when $T^* \to +\infty$. This concludes the proof.□

Proof of Corollary 1: The result directly follows from the proof of Proposition 2.□

Proof of Proposition 3: The optimal exhaustion date is implicitly characterized by (52), which can be rewritten as:

$$f\left(\gamma^+, T^*, \alpha, X^0, \delta\right) \equiv 1 - \frac{2\gamma^+ - \delta}{\gamma^+e^{\gamma^+T^*} + (\gamma^+ - \delta)e^{(\delta-\gamma^+)T^*}}e^{\delta T^*} - \frac{\alpha X^0}{a} = 0, \tag{61}$$

where $\gamma^+ = \left(\delta + \sqrt{\delta(2\alpha+\delta)}\right)/2$. Its derivative with respect to T^* is given by

$$\frac{\partial f}{\partial T^*} = \frac{\gamma^+(\gamma^+ - \delta)(2\gamma^+ - \delta)e^{\delta T^*}}{\left(\gamma^+e^{\gamma^+T^*} + (\gamma^+ - \delta)e^{(\delta-\gamma^+)T^*}\right)^2}\left(e^{\gamma^+T^*} - e^{-(\gamma^+-\delta)T^*}\right). \tag{62}$$

Since $\gamma^+ \geq \delta$, we have

$$\frac{\partial f}{\partial T^*} > 0. \tag{63}$$

The derivative of f with respect to γ^+ is given by

$$\frac{\partial f}{\partial \gamma^+} = -\frac{\delta\left(e^{\gamma^+ T^*} - e^{-\left(\gamma^+ - \delta\right)T^*}\right) + \left(\gamma^+ e^{\gamma^+ T^*} - \left(\gamma^+ - \delta\right) e^{-\left(\gamma^+ - \delta\right)T^*}\right)\left(\gamma^+ - \delta\right) T^*}{\left(\gamma^+ - \delta\right)^2 \left(\frac{\gamma^+}{\gamma^+ - \delta} e^{\gamma^+ T^*} + e^{-\left(\gamma^+ - \delta\right)T^*}\right)^2} e^{\delta T^*}. \quad (64)$$

Since $\gamma^+ > \delta$, we have

$$\frac{\partial f}{\partial \gamma^+} < 0. \quad (65)$$

The derivative of f with respect to α is

$$\frac{\partial f}{\partial \alpha} = -X^0 < 0 \quad (66)$$

Using (61) and the implicit function theorem, we have:

$$\frac{\partial T^*}{\partial X^0} = -\frac{\partial f / \partial X^0}{\partial f / \partial T} = \frac{\alpha / a}{\partial f / \partial T} > 0. \quad (67)$$

Using the implicit function theorem again, and $\partial \gamma^+ / \partial \alpha > 0$, (65), (63) and (66), the derivative of the exhaustion date with respect to the recyclability rate is such that:

$$\frac{\partial T^*}{\partial \alpha} = -\frac{\left(\partial f / \partial \gamma^+\right)\left(\partial \gamma^+ / \partial \alpha\right) + \partial f / \partial \alpha}{\partial f / \partial T} > 0. \quad (68)$$

□

Proof of Proposition 4: The proof proceeds in three steps. We first show that the growth rate of extraction is decreasing through time. Second, we show that the growth rate is increasing with the recyclability rate. We then combine these properties in order to prove the result.

Differentiating (53), we can write the growth rate of extraction:

$$\tau\left(\gamma^+, T^*\right) \equiv \frac{\dot{x}^*(t)}{x^*(t)} = -\frac{\left(\gamma^+ - \delta\right) e^{\left(2\gamma^+ - \delta\right)\left(T^* - t\right)} + \gamma^+}{e^{\left(2\gamma^+ - \delta\right)\left(T^* - t\right)} - 1}. \quad (69)$$

The derivative of the growth rate with respect to T^* is

$$\frac{\partial \tau}{\partial T^*} = \frac{\gamma^+ + \delta + \left(2\gamma^+ - \delta\right)\gamma^+}{\left(e^{\left(2\gamma^+ - \delta\right)\left(T^* - t\right)} - 1\right)^2} \delta e^{\left(2\gamma^+ - \delta\right)\left(T^* - t\right)} > 0. \quad (70)$$

The derivative of the growth rate with respect to γ^+ is

$$\frac{\partial \tau}{\partial \gamma^+} = \delta \frac{G(t)}{\left(e^{\left(2\gamma^+ - \delta\right)\left(T^* - t\right)} - 1\right)^2},$$

where $G(t) = 2\left(T^* - t\right)\left(2\gamma^+ - \delta\right) e^{\left(2\gamma^+ - \delta\right)\left(T^* - t\right)} + 1 - e^{2\left(2\gamma^+ - \delta\right)\left(T^* - t\right)}$. Notice that $G'(t) =$

$-2(T^* - t)(2\gamma^+ - \delta)^2 e^{\left(2\gamma^+ - \delta\right)(T^* - t)} < 0$ and $G(T^*) = 0$. Hence $G(t) > 0$ and then

$$\frac{\partial \tau}{\partial \gamma^+} > 0. \tag{71}$$

We know from Proposition 3 that T^* increases with α and we also know that γ^+ increases with α. Using (70) and (71), we conclude that

$$\frac{d\tau}{d\alpha} > 0. \tag{72}$$

In other words, we have

$$\frac{\partial^2 \ln(x)}{\partial t \partial \alpha} > 0. \tag{73}$$

Hence $\ln x$ has the single crossing property with respect to time and the recyclability rate.

According to Proposition 3, the exhaustion date increases with recyclability, $\frac{\partial T^*}{\partial \alpha} > 0$. Hence recyclability necessarely increases extraction when time gets close to the exhaustion date. Since the initial stock does not depend on the recyclability rate, recyclability necessarely decreases extraction at some point in time. Thanks to the single-crossing property, there exists a date $0 < \tilde{t} < T^*$ such that $\frac{\partial x^*}{\partial \alpha} < 0 \Longleftrightarrow t < \tilde{t}$.□

Proof of Proposition 5:

The stock of scrap can be rewritten as follows:

$$S^*(t) = F\left(\gamma^+, T^*\right) = a\left(1 - \frac{\gamma^+ e^{\gamma^+(T^*-t)} + (\gamma^+ - \delta)\, e^{-\left(\gamma^+ - \delta\right)(T^* - t)}}{\gamma^+ e^{\gamma^+ T^*} + (\gamma^+ - \delta)\, e^{-(\gamma^+ - \delta)T^*}} e^{\delta t}\right).$$

The derivative of this function with respect to T^* is:

$$\frac{\partial F}{\partial T^*} = -a \frac{\gamma^+(\gamma^+ - \delta)(2\gamma^+ - \delta)\left[e^{\delta T - \gamma t} - e^{\delta T + \left(\gamma^+ - \delta\right)t}\right]}{\left(\gamma^+ e^{\gamma^+ T^*} + (\gamma^+ - \delta)\, e^{-(\gamma^+ - \delta)T^*}\right)^2} e^{\delta t} \geq 0, \tag{74}$$

and its derivative with respect to γ^+ is

$$\frac{\partial F}{\partial \gamma^+} = -a \frac{-t(\gamma^+)^2 e^{\gamma^+(2T^* - t)} + \left[(2T^* - t)(\gamma^+)^2 - \delta\right] e^{\delta T - \gamma^+ t} - \left[2T^*(\gamma^+)^2 - \delta\right] e^{\delta T + \left(\gamma^+ - \delta\right)t}}{\left(\gamma^+ e^{\gamma^+ T^*} + (\gamma^+ - \delta)\, e^{-(\gamma^+ - \delta)T^*}\right)^2 e^{-\delta t}} \geq 0. \tag{75}$$

Since T^* and γ^+ both increase with α, using (74) and (75) we conclude that S^* increases when α increases.

Now consider the growth rate of the price. Using (55), it can be written as follows:

$$\frac{\dot{p}^*}{p^*} = \frac{\gamma^+ - (\gamma^+ - \delta)\, e^{\left(2\gamma^+ - \delta\right)(T^* - t)}}{1 + e^{(2\gamma^+ - \delta)(T^* - t)}} \equiv H\left(\gamma^+, T^*\right). \tag{76}$$

Its derivative with respect to γ^+ is given by:

$$\frac{\partial H}{\partial \gamma^+} = \frac{1 - e^{2\left(2\gamma^+ - \delta\right)(T^* - t)} - 2\left(2\gamma^+ - \delta\right)\left(T^* - t\right) e^{\left(2\gamma^+ - \delta\right)(T^* - t)}}{\left(1 + e^{(2\gamma^+ - \delta)(T^* - t)}\right)^2} \leq 0. \tag{77}$$

Since H is also decreasing with T^* and both T^* and γ^+ both increase with α, we conclude that

$$\frac{\partial^2 \ln p^*}{\partial t \partial \alpha} < 0. \tag{78}$$

This means that $\ln p^*$ has the single-crossing property with respect to t and α. We know that

$$\begin{aligned} p^*\left(T^*\right) &= 1 - S^*\left(T\right) - x^*\left(T\right) \\ &= 1 - \alpha X^0. \end{aligned}$$

Then $p^*\left(T^*\right)$ decreases with α. Moreover, we have

$$\begin{aligned} p^*\left(0\right) &= 1 - S^*\left(0\right) - x^*\left(0\right) \\ &= 1 - x^*\left(0\right). \end{aligned} \tag{79}$$

Using Proposition 4, we know that x^* decreases with α at $t = 0$. Hence p^* increases with α at $t = 0$. Using the single-crossing property (78), we conclude that there exists $t' \in (0, T^*)$ such that $\partial p^* / \partial \alpha > 0 \Longleftrightarrow t' \in [0, t')$.□

References

[1] **Ashley, K., D. Cordell and D. Mavinic, 2011**. A Brief Story of Phosphorus: From the Philosopher's Stone to Nutrient Recovery and Reuse, Chemosphere, xxx, xxx-xxx.

[2] **Blomberg, J. and P. Söderholm, 2009.** The economics of secondary aluminium supply: An econometric analysis based on European data, Resources, Conservation and Recycling, 53, 455 – 463.

[3] **Bond, E. and L. Samuelson, 1984**. Durable Good Monopolies with Rational Expectations and Replacement Sales, Rand Journal of Economics, Vol. 15, No. 3, pp. 336-345.

[4] **Bullow, J., 1982**. Durable-goods Monopolist, Journal of Political Economics, 90, pp. 314-332.

[5] **Coase, R., 1972**. Durability and Monopoly, J. Law Econ., 15, 143-149.

[6] **Cordell, D. and S. White, 2011**. Peak of Phosphorus: Clarifying the Key Issues of a Vigorous Debate about Long-Term Phosphorus Security, Sustainability, 3, 2027-2049, doi: 10.3390/su3102027.

[7] **Cordell, D. and S. White, 2013**. Sustainable Phosphorus Measures: Strategies and Technologies For Achieving Phosphorus Security, Agronomy, 3, 86-116; doi: 10.3390/agronomy3010086.

[8] **Cordell, D., J.-O. Drangert and S. White, 2009**. The Story of Phosphorus: Global Food Security and Food For Thought, Global Environement Change, 19, 292-305.

[9] **Cordell, D., A. Rosemarin, J. J. Schröder and A. L. Smit, 2011.** Towards Global Phosphorus Security: A Systems Framework for Phosphorus Recovery and Reuse Options, Chemosphere, xxx, xxx-xxx.

[10] **Fertilizer Week, 2008**. Industry Ponders the Impact of China's Trade Policy. In: Thusday Markets Reports, 24th April 2008. British Sulfure Consultants, CRU.

[11] **Gaskins, D., 1974**. Alcoa revisited: The welfare implications of a secondhand market, Journal of Economic Theory, 7, 254-271.

[12] **Hotelling, H.,** 1931. The Economics of Exhaustible Resources, Journal of Political Economy, 39, 2, 137-175.

[13] **Hnyilicza, E. and R. S. Pindyck,** 1976. Pricing Policies for a Two-Part Exhaustible Resource Cartel, MITEL76-008WP.

[14] **IFDC, 2010**. World Phosphate Reserves and Resources, International Fertilizer Development Center, Washington DC.

[15] **Istemi, B**., 2014. Two-Period Resource Duopoly with Endogenous Intertemporal Capacity Constraints, EWI Working Paper, No. 14/13.

[16] **Kahn, C., 1986**. The Durable Good Monopolist and Consistency with Increasing Costs, Econometrica, Vol. 15, No. 2, pp. 275-294.

[17] **Loury, G. C.,** 1986. A theory of Oil'igopoly: Cournot Equilibrium in Exhaustible Resource Markets with Fixed Supplies, Internatonal Economic Review, 27 (2), pp. 285-301.

[18] **Malueg, D. A. and J. L. Solow, 1998**. Exhaustibility and The Durable Goods Monopolist, Mathl Comptu. Modelling, Vol. No. 6, pp. 419-427.

[19] **Levhary, D. and R. Pindyck, 1981.** The Pricing of Durable Exhaustible Resources, Quaterly Journal of Economics, Vol. XCV1, No. 3.

[20] **Smil, V., 2000**. Phosphorus in the Environment : Natural Flows and Human Interferences, Ann. Rev. Energy Environement, 25, 53-88.

[21] **Stewart, M. B., 1980**. Monopoly and the Intertemporal Production of a Durable Extractable Resource, The Quaterly Journal of Economics.

[22] **Tweeten, L., 1989**. Agricultural Policy Analysis Tools for Economic Development, Westview Press: Bulder CO, USA, ISBN: 0813377463.

[23] **Van den Berg, A., Bos, I., Herings, P. J.-J., and Peters, H**., 2012. Dynamic Cournot Duoply with Intertemporal Capacity Constraints, International Journal of Industrial Organisation, 30 (20), 174-192.

[24] **https://en.wikipedia.org/wiki/Durable_good.**