Markov Perfect Equilibria in Differential Games with Regime Switching Strategies

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April 15, 2015

Abstract

We propose a new methodology exploring Markov perfect equilibrium strategies in differential games with regime switching. We develop a general game with two players. Players choose an action that influences the evolution of a state variable, and decide on the switching time from one regime to another. Compared to the optimal control problem with regime switching, necessary optimality conditions are modified for the first player to switch. When choosing her optimal switching strategy, this player considers the impact of her choice on the other player's actions and consequently on her own payoffs. In order to determine the equilibrium timing of regime changes, we derive conditions that help eliminate candidate equilibrium strategies that do not survive deviations in switching strategies. We then apply this new methodology to an exhaustible resource extraction game.

Key words: differential games; regime switching strategies; technology adoption; non-renewable resources

JEL classification: C61, C73, Q32.

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1 Introduction

Several decision making problems in economics concern the timing of switching between alternative and consecutive regimes. Regimes may refer to technological and/or institutional states of the world. For instance, a firm with an initial level of technology may find it optimal to either adopt a new technology or to stick with the old one (Boucekkine et al. 2004). Another example is the decision to phase out existing capital controls in a given economy (Makris, 2001). In all non-trivial problems, the switching decision involves a trade-off, since adopting a new regime brings with it immediate costs as well as potential future benefits. Given these considerations, multi-stage optimization is generally used for the analysis of regime switching (Tomiyama, 1985, Makris, 2001), which endogenously determines switching times.

In this article, we consider regime switching strategies in differential games. The game theoretic literature involving regime switching choice is sparse. Early papers on dynamic games of regime change do not involve a stock variable. In these models, the only relevant state of the system is the identity of the players who have adopted the new technology. An example is Reinganum (1981)'s model of technological adoption decisions of two ex ante identical firms. She assumed that firms adopt pre-commitment (open-loop) strategies. That is, it is as if a firm enters a binding commitment on its date of technology switch, knowing the adoption date of the other firm. Reinganum's primary finding is that, with two ex-ante identical firms using open-loop strategies, the equilibrium features diffusion: One firm will innovate first and the other will innovate at a later date. The first player to switch earns higher profits. Fudenberg and Tirole (1985) revisited Reinganum's study by using the concept of pre-emption equilibrium. Focusing on Markov perfect equilibrium as the solution concept, they noted that the second player to switch may try to preempt its rival and become the first to adopt. At the preemption equilibrium, the first player to switch advantage vanishes (see Long, 2010, for a survey of this literature).

A second strand of literature pertains to the strategic interaction of agents in relation to the dynamics of a given stock. For instance, Tornell (1997) presented a model that explores the relationship between economic growth and institutional change. Infinitely-lived agents solve a differential game that drives the changes in property-rights regimes for the economy's capital stock, e.g. common property versus private property. It was shown that a potential equilibrium of the game involves multiple switching between regimes. However, only the symmetric equilibrium was considered, such that the players always choose to switch at the same instant. Consequently, the question of the timing between switching points was not addressed. In addition, even though Tornell explicitly defined the Markov perfect equilibrium for the class of differential games with regime switching, a rigorous modelling of these strategies, for switching time, is missing in his analysis. A more recent example is the analysis by Boucekkine et al. (2011). They analyzed the trade-off between environmental quality and economic performance using a two-player differential game. Assuming that pollution results from the sum of consumption levels and there is no decay, they proved the existence of an open-loop Nash equilibrium. They found that each player chooses the technology without considering the choice made by the other player. There was no interior switching instant. At the open-loop Nash equilibrium, either a player adopts a new technology immediately, or he sticks to the old one.

To our knowledge, there seems to be no existing study which formally defines Markov perfect equilibrium in differential games with regime switching strategies. This is where the first theoretical contribution of this paper lies. We develop a general differential game with two players having two kinds of strategies. First, players have to choose at each point in time an action that influences the evolution of a state variable. Second, they may decide on the timing of switching between alternative and consecutive regimes that differ both in terms of the payoff function and the state equation. For simplicity, we assume that each player can affect a regime change only once. Focusing on Markov perfect equilibrium, we define the switching or timing strategy as a function of the state of the system, which is described by the level of the stock variable, and the regime that is in current operation.

For any possible timing, we characterize the necessary optimality conditions for switching times, both for interior and corner solutions. One interesting finding is that, compared to the necessary conditions characterizing optimal switching in the standard (one-player) optimal control problem, we find that, with two players, the necessary optimal switching conditions are substantially modified for the player who finds it optimal to move first. Indeed, when choosing the optimal date and level of the state variable for switching, this player must take into account that *(i)* her decision will influence the other player's equilibrium switching strategy in the subgame that follows, and *(ii)* the other player's switching time will impact on her own welfare. Depending on the particular economic problem at hand, the interaction through switching times may provide an incentive to either postpone or expedite regime switching. Another important issue is how to determine the equilibrium switching sequence in the Markov perfect equilibrium. We resolve this issue by providing conditions that help eliminate candidate switching sequences that do not survive deviations in switching strategies.

The second contribution of this paper is the application of this new game theoretic material to study a model of management of an exhaustible resource. By incurring a lumpy cost, players can make use of a more efficient extraction technology. Not only do players choose their consumption levels, they also decide whether to adopt the new technology and when. To date, there are only a few papers that have studied the relationship between natural resource exploitation and the timing of technology adoption. Using a finite horizon two-stage optimal control problem, Amit (1986) explored the case of a petroleum producer who considers switching from a primary to a secondary recovery process. He observed that a technological switch occurs if the desired extraction rate is larger than can be obtained by the natural drive, or when the desired final output is more than can be obtained using the primary process. In a more recent paper, Valente (2011) analyzed a two-phase endogenous growth model which concerns a switch from an exhaustible resource input into a backstop technology. He showed that adoption of new technology implies a sudden fall in consumption, but an increase in the growth rate. Finally, Boucekkine et al. (2013) explored a general control problem with both technological and ecological regime switches. They applied it to address the issue of optimal resource extraction under ecological irreversibility, and with the possibility to adopt backstop technology. It was observed that the opportunity to switch to a backstop technology may lead to an irreversible ecological regime.

While the above-mentioned studies have explored resource management and regime switching, they only do so using single-agent optimization programs. None have conducted an analysis using a differential game approach. Indeed, our section 4 tries to fill this gap in the resource extraction literature. It is assumed that heterogenous players start with a less efficient extraction technology and have to decide: *(i)* whether to switch to a more efficient technology, and *(ii)* when, given that switching involves a direct cost that depends on both the switching date and the level of the state variable.

In the application, our main findings can be summarized as follows. We first identify a meaningful condition that allows us to check a proposed timing's robustness to deviations. This condition involves on the one hand the difference in players' switching costs, and on the other hand the difference in technological gains from switching. Indeed, it is possible that both players have an incentive to deviate from a given proposed timing. This happens when the player who is supposed to be the first to switch has a relative disadvantage in adoption costs that is not compensated by any relative technological advantage. This notably encompasses the situation in which the first player to switch incurs higher switching cost and, at the same time, is the one who benefits the least from adoption at any level of the resource stock. When players do not have an incentive to deviate, we provide sufficient conditions for the existence of an interior solution where both players adopt the new technology in finite time and investigate the impact of Markovian strategic interactions on the first player to switch's equilibrium strategy (as compared to the single-agent case). We emphasize the interplay between two opposite effects. First, the switch made by the second player to switch is costly for the first player because it implies a drop in her consumption of the resource. The switching cost is thus augmented by this term, which gives her an incentive, other things equal, to delay her switch. On the other hand, it turns out that the length of time between the two switches is increasing in the level of the stock at the time of the first switch. Therefore, from the point of view of the first player to switch, who controls this level, switching at a relatively more abundant stock level is a means to postpone the switch of the other. This is an incentive for her to switch at a larger stock level, which means an earlier date. In the particular case where the direct switching cost is zero, we show that this player finds it worthwhile to adopt the new technology, but not immediately at the beginning of the game. This is in sharp contrast to the result that one would obtain in the single-player case, namely immediate adoption.

2 The general problem

We consider a two-player differential game in which the instantaneous payoff of each player and the differential equation describing the stock dynamics depend on what regime the system is in. There are a finite number of regimes, indexed by s, and we assume that under certain conditions, the players are able to take action (at some cost) to affect a change of regime. Let $\mathcal{S}$ be the set of regimes. For simplicity, we assume that each player can make a regime switch only once. This implies that regime changes are irreversible, i.e., switching back is not allowed. In this case, there are four possible regimes and the set $\mathcal{S}$ is simply

$$\mathcal{S} \equiv \{11, 12, 21, 22\}$$

We assume that the system is initially in regime 11. Player 1 can take a "regime switching action" to switch the system from regime 11 to regime 21, if the other player has not made a switch. The first number in any regime index indicates player 1's moves. The second refers to player 2. Once the system is in regime 21, only player 2 can take a regime switching action, and this leads the system to regime 22. From regime 11, player 2 can switch to regime 12 (if player 1 has not made a switch). From regime 12, only player 1 can make a regime change, and this switches the system to regime 22. If the system is in regime 11 and both players take regime change action simultaneously, the regime will be switched to 22. Finally, the system may remain in 11 forever if neither agent takes a regime change action. Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ in which player i can make a regime change. Then $\mathcal{S}_1 = \{11, 12\}$ and $\mathcal{S}_2 = \{11, 21\}$.

The state variable x could be in any space $\mathbb{R}_+^m$, $1 \leq m \leq M$. At each instant, each player chooses an action u_i, with $u_i \in \mathbb{R}^{n_i}$, $1 \leq n_i \leq N_i < \infty$, that affects the evolution of x. To simplify the exposition, we set $m = 1$ and $n_i = n$ for $i = 1, 2$. The instantaneous payoff to player i at time t when the system is in regime s is

$$F_i^s(u_i(t), u_{-i}(t), x(t)).$$

If player i takes a regime change action at time $t_i \in \mathbb{R}_+$, a lumpy cost $\Omega_i(t_i, x(t_i))$ is incurred. Then, if for example $0 < t_1 < t_2 < \infty$, player 1's total payoff is

$$\int_0^{t_1} F_1^{11}(u_1, u_2, x)e^{-\rho t}dt + \int_{t_1}^{t_2} F_1^{21}(u_1, u_2, x)e^{-\rho t}dt$$
$$+ \int_{t_2}^{\infty} F_1^{22}(u_1, u_2, x)e^{-\rho t}dt - \Omega_1(t_1, x(t_1))$$

with ρ the discount rate.

The differential equation describing the evolution of the state variable x in regime s is

$$\dot{x} = f^s(u_1, u_2, x)$$

In the subsequent analysis, we use Markov perfect equilibrium (MPE) as the solution concept. As illustrated by the decomposition above, if the equilibrium timing is such that $0 \leqq t_1 \leqq t_2 \leqq \infty$, there are three sub-games to be considered, each being associated with a particular regime. Indeed, for the timing considered, the sequence of regimes is: 11, 21 and 22. A natural way to proceed, for determining a MPE of this game, is to solve the problem recursively, starting from the regime arising after the final regime switching, here 22. This is a natural extension of the method originally developed by Tomiyama (1985) and Amit (1986) to solve their finite horizon two-stage optimal control problems (for infinite horizon problems, see Makris, 2001).

The next assumption ensures that our problem, seen as a sequence of three sub-games, is well-behaved.

Assumption 1 *The functions $F_i^s(.)$ and $f^s(.)$, for any $s \in \mathcal{S}$, belong to the class C^1. Moreover, the sub-game obtained by restricting the general problem to any regime s, satisfies the Arrow-Kurz sufficiency conditions.*

These conditions will allow us to use some envelope properties that require the differentiability of the value function (see Boucekkine et al. 2013, for a detailed discussion).

Let us now define a MPE strategy in our model. Each player has two types of controls, the set of controls being given by $\mathcal{C}_i = \{u_i, t_i\}$. A MPE strategy consists of an action policy and a switching rule describing the actions undertaken by each player at every possible state of the system, $(x, s) \in \mathbb{R}_+ \times S$. Again, for the sake of exposure, we restrict attention to those strategies that are not time-dependent. This requires that the function $\Omega_i(t_i, x(t_i))$ takes the form $e^{-\rho t_i}\omega_i(x(t_i))$.

The *action strategy* of player i is a mapping Φ_i from the state space $\mathbb{R}_+ \times \mathcal{S}$ to the set $\mathbb{R}^n$.

The *switching rule* can be defined as follows: Suppose player 1 thinks that if player 2 finds herself in regime 21 at date t, with $x(t)$ (which implies that he switched at an earlier date $t_1 < t$), she will make a switch at a date $t_2 \geq t$. Then player 1 should think that the interval of time between the current date and the switching date, $t_2 - t$, is a function of the state of the system. More generally, we define the *time-to-go strategy (before switching)* of player i, given that $s \in \mathcal{S}_i$, as a mapping θ_i from $\mathbb{R}_+ \times \mathcal{S}$ to $\mathbb{R}_+ \cup \{\infty\}$. For instance, from the state $(x, 21)$, $\theta_2(x, 21)$ is the length of time that must elapse before player 2 takes her regime switching action. If $\theta_2(x, 21) = \infty$ for all x, it means she does not want to switch at all from regime 21.

Then we say that

Definition 1
- *A **strategy vector** of player i (as guessed by player $-i$) is a pair $\psi_i \equiv (\Phi_i, \theta_i)$, $i = 1, 2$.*
- *A **strategy profile** is a pair of strategy vectors, (ψ_1, ψ_2).*
- *A strategy profile (ψ_1^*, ψ_2^*) is called a **Markov-perfect Nash equilibrium**, if given that player i uses the strategy vector ψ_i^*, the payoff of player j, starting from any state $(x, s) \in \mathbb{R}_+ \times \mathcal{S}$, is maximized by using the strategy vector ψ_j^*, where $i, j = 1, 2$.*

The topic of the paper is all about regime switching strategies in differential games. This means that the core of the analysis is devoted to a presentation of the optimality conditions associated with these strategies and a discussion on the impact of this new source of interaction on players behaviors, and no more. We then get rid of other issues that typically arise in differential games, like existence and uniqueness of the MPE. In other words, hereafter we basically assume that a solution exists and focus on the original part of the problem. The next section presents the set of necessary optimality conditions that characterize a MPE of the differential game with regime switching strategies.

3 Necessary Conditions for switching strategies

The analysis to follow is presented for a particular timing: $0 \leqq t_1 \leqq t_2 \leqq \infty$. Necessary optimality conditions for the other general timing, $0 \leqq t_2 \leqq t_1 \leqq \infty$, can easily be derived by symmetry. First, we state and interpret optimality conditions for an interior solution (a solution with t_i positive and finite and $t_1 \neq t_2$), which allows us to emphasize the impact of the interaction through switching strategies on the solution. Next, we want to know whether a player has or not an incentive to deviate from the timing considered. For that purpose, corner solutions are carefully studied.

3.1 Interior solution

Assume that there exists a solution $(u_1^*(t), u_2^*(t), x^*(t))$ to the differential game defined above and for given (t_1, t_2). In any regime s, Player i's present value Hamiltonian, $H_i^s = F_i^s(u_i, \Phi_{-i}(x,s), x)e^{-\rho t} + \lambda_i^s f^s(u_i, \Phi_{-i}(x,s), x)$ with λ_i^s the co-state variable, evaluated at this solution is denoted by H_i^{s*} and we refer to θ_2' as the derivative w.r.t the state variable x. Our first theorem states the necessary optimality conditions related to the switching strategies at the interior solution, if it exists. All the proofs are displayed in the Appendix A.

Theorem 1 *The necessary optimality conditions for the existence of a MPE featuring the timing $0 < t_1^* < t_2^* < \infty$ are:*

- *For player 2:*

$$H_2^{21*}(t_2) - \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2} = H_2^{22*}(t_2) \tag{1a}$$

$$\lambda_2^{21}(t_2) + \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial x_2} = \lambda_2^{22}(t_2) \tag{1b}$$

$$\lambda_2^{11}(t_1) = \lambda_2^{21}(t_1). \tag{1c}$$

- *For player 1:*

$$H_1^{11*}(t_1) - \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial t_1} = H_1^{21*}(t_1) - [H_1^{21*}(t_2) - H_1^{22*}(t_2)], \tag{2a}$$

$$\lambda_1^{11}(t_1) + \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial x_1} = \lambda_1^{21}(t_1) + \theta_2'(x^*(t_1))[H_1^{21*}(t_2) - H_1^{22*}(t_2)], \tag{2b}$$

$$\lambda_1^{21}(t_2) = \lambda_1^{22}(t_2). \tag{2c}$$

To understand these switching conditions for an interior solution, let us focus on the difference between the optimality conditions of the first player to switch (player 1) and the second (player 2) for the particular timing considered. Player 2's conditions (1) are similar to the ones derived in multi-stage optimal control literature. Condition (1a) states that it is optimal to switch from the penultimate to the final regime when the marginal gain of delaying the switch, given by the difference $H_2^{21*}(.) - H_2^{22*}(.)$, is equal to the marginal cost of switching, $\frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2}$. Condition (1b) equalizes the marginal benefit from an extra unit of the state variable $x(t_2)$ with the corresponding marginal cost. It basically says that the value of the co-state, when approached from the intermediate regime, plus the incremental switching cost must just equal the value of the co-state, approached from the final regime. Hence, as long as a player finds it optimal to be the second player to switch, her optimality conditions are similar to the standard switching conditions of an optimal control problem. Finally, according to condition (1c), the marginal benefit from an extra unit of the state variable x at player 1's switching time must be equal to the corresponding marginal cost.

The novel part of Theorem 1 stems from the problem faced by the player who opts to switch first. Indeed, player 1's optimality conditions are modified (compared to the single agent framework). The first condition (2a) implies that player 1 cares about changes in his situation induced by the switch of player 2. Player 1 decides on his optimal switching time by equalizing the marginal gain of delaying the switch, which is given by the difference $H_1^{11*}(.) - H_1^{21*}(.)$ to the marginal switching cost, $\frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial t_1} - [H_1^{21*}(t_2) - H_1^{22*}(t_2)]$. The extra-term $[H_1^{21*}(t_2) - H_1^{22*}(t_2)]$ is the marginal impact of player 2's switch on player 1. So, player 1 anticipates the impact of player 2's switch on his payoff. Depending on the nature of the problem, the additional term can either be positive or negative. The second optimality condition (2b) is also modified. The cost of a marginal increase in x at t_1 now includes an extra-term: $\theta_2'(x^*(t_1), 21)[H_1^{21*}(t_2) - H_1^{22*}(t_2)]$. This term reflects the fact that player 1 takes into account his influence on player 2's timing strategy, through the level of the state variable at the switching time $x^*(t_1)$. Put differently, player 1 knows that modifying $x^*(t_1)$ is a means to delay or accelerate player 2's regime switching. In sum, the modified switching conditions of player 1 illustrate the existence of a two-way interaction through switching strategies.

A couple of comments are in order here:

First, when deriving the conditions of Theorem 1, we implicitly assume that players follow their MPE strategies for the action policy, i.e., that the triplet

$(u_1^*(t), u_2^*(t), x^*(t))$ is the path followed by each player's action policy and the state variable at a MPE. This boils down to considering that optimal switching conditions are conditional on the optimal action policies. Then, the question is: Is the switching rule robust to deviations in the action policy? Consider the problem of player 1, once he is already in regime 21. Player 1's problem is to choose the time path $\{u_1\}$ that maximizes

$$\int_{t_1}^{t_1+\theta_2(x_1,21)} e^{-\rho t} F_1^{21}(u_1, \Phi_2(x,21), x)dt + V_1^{22*}(x_2, t_1 + \theta_2(x_2, 21))$$

with $V_1^{22*}(.)$ the continuation payoff (resulting from the play of the MPE actions in the final regime) and subject to,

$$\dot{x} = f^{21}(u_1, \Phi_2(x,21), x)$$

$$x(t_1) = x_1,\ x(t_1 + \theta_2(x_1, 21)) = x_2$$

where he takes as given x_1, x_2, $\Phi_2(x, 21)$ and $\theta_2(x_1, 21)$. If he deviates from the equilibrium from time t_1 to some time $t_1 + \epsilon$, with $\epsilon > 0$, what would be his optimization problem at time $t_1 + \epsilon$? The point is that he should expect that player 2 still continues to use the strategy $(\Phi_2(x, 21), \theta_2(x, 21))$, with the switching point x_2, because he knows that $\Phi_1(x, 21)$ will be played by him from time $t_1 + \epsilon$ onward. The deviation will be reflected in the value of the state variable at $t_1 + \epsilon$, $x(t_1 + \epsilon) \neq x^*(t_1 + \epsilon)$. This will in turn affect the length of time before the next switch by Player 2, $\theta_2(x(t_1 + \epsilon), 21)$.

Second, in (2), the term $\theta_2'(x^*(t_1), 21)$ may look like a kind of Stackelberg-leadership consideration: Player 1 knows the function $\theta_2^{*\prime}(x, 21)$, and hence he knows that when he chooses t_1 and the level $x(t_1)$ he is indirectly influencing t_2. But this is not really Stackelberg leadership in a global sense. The situation is just like any standard game tree with sequential moves. If a player moves first, he knows how the second player to switch will move at each subgame that follows, and therefore he will take that into account in choosing which subgame he is going to induce.

3.2 Corner solutions

Still for the same timing, we now examine the conditions for one player to choose a corner strategy.

Theorem 2 *1. Suppose player 1 switches at some instant $t_1 \in (0, \infty)$.*

- *Necessary conditions for player 2 to choose a corner solution with immediate switching, i.e., $t_2 = t_1$ (instead of $t_2 > t_1$) are (1b), (1c), and*

$$H_2^{21*}(t_2) - \frac{\partial \Omega_2(t_2, x^*(t_2)))}{\partial t_2} \leq H_2^{22*}(t_2) \text{ if } t_1 = t_2 < \infty \tag{3}$$

2. *Suppose player 2's switching problem has an interior solution* t_2.

   • *Necessary conditions for player 1 to choose a corner solution with immediate switching* $0 = t_1$ *are (2b), (2c), and*

$$H_1^{11*}(t_1) - \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial t_1} \leq H_1^{21^*}(t_1) - [H_1^{21*}(t_2) - H_1^{22*}(t_2)] \text{ if } 0 = t_1 < t_2 \tag{4}$$

   • *Necessary conditions for player 1 to choose a corner solution of the never switching type* $t_1 = t_2$ *are (2b), (2c), and*

$$H_1^{11*}(t_1) - \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial t_1} \geq H_1^{21^*}(t_1) - [H_1^{21*}(t_2) - H_1^{22*}(t_2)] \text{ if } 0 < t_1 = t_2 \tag{5}$$

The corner solution $t_1 = 0$ and corresponding necessary conditions have already been discussed in literature. For $t_1 = 0$, it must be the case that player 1 wants to escape from regime 11 as soon as possible. According to condition (4), this happens when a delay in switching yields a marginal gain that is not greater than the marginal loss of foregoing for an instant the benefit of the new regime. Of further interest is the interpretation of players' "corner solutions" $t_1 = t_2$. Quotes are needed because those solutions actually correspond to artificial situations where the timing is (pre)specified (here $t_1 \leqq t_2$). The conditions for them to occur deserve much attention since they provide a clear way to check if a candidate equilibrium in switching strategies is robust to deviations. As an illustration, consider player 1's problem. Conditional on player 1 being the first to make a switch and on player 2 being the second, we can derive a necessary condition for t_1 to be at the corner $t_1 = t_2$. If this condition (5) is satisfied, which means that at t_2 a delay in switching yields a marginal gain that is at least as high as the marginal loss of foregoing for an instant the benefit of the new regime, then we suspect that, when we remove the artificial requirement that $t_1 \leqq t_2$, there will be an incentive for player 1 to choose to make a regime switch in second place. In such a case, a candidate solution with $t_1 \leqq t_2$ does not survive the incentive for player 1 to deviate from it. In other words, condition (5) is necessary for the timing not to be robust to deviations in player 1's switching strategy. The same reasoning applies to player 2's corner solution $t_2^* = t_1^*$.

Of course, the timing is not fixed in our differential game with regime switching strategies and the most important task is precisely to determine what will be the timing at the equilibrium, or under which conditions a particular timing will occur. The analysis of situations where one player may have an incentive to deviate is of crucial importance to address this non-trivial issue. Indeed, it should allow us to understand which timing, between $0 \leqq t_1 < t_2 \leqq \infty$ and $0 \leqq t_2 < t_1 \leqq \infty$, is consistent with the MPE requirement. Therefore, in any particular application, such an analysis should be conducted first in order to reduce the set of potential candidates for MPEs, before having a look at other interior or corner solutions.

Let us conclude this section with a brief overview of the other possible combinations between t_1 and t_2. First, note that there is no counterpart to the necessary conditions (3)-(5) for the corner solution $t_2 = \infty$ (see the discussion in Makris, 2001, page 1941). However, the inequality

$$H_2^{21*}(t_2) - \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2} > H_2^{22*}(t_2) \text{ for all } t_1 \leq t_2 < \infty$$

is sufficient to establish that player 2 will never find it optimal to switch regime. Next, it is highly unlikely that heterogeneous players decide on the same switching time. So, the timing $0 < t_1 = t_2 < \infty$ should not give a MPE candidate. However, it's quite easy to derive the optimality conditions is this case. Suppose that it is optimal for the two players to switch at the same date $t_1 = t_2 \in (0, \infty)$, for the same level of the state $x^*(t) = x^*(t_1) = x^*(t_2)$, then the following conditions must hold, for $i = 1, 2$:

$$\begin{array}{l} H_i^{11*}(t) - \frac{\partial \Omega_i(t, x^*(t))}{\partial t_i} = H_i^{22*}(t) \\ \lambda_i^{11}(t) + \frac{\partial \Omega_i(t, x^*(t))}{\partial x_i} = \lambda_i^{22}(t). \end{array} \tag{6}$$

Finally, conditions corresponding to the cases $t_1 = t_2 = 0$ and $t_1 = t_2 = \infty$ can easily be derived from the material presented above.

The next section is devoted to an application of the theory to an exhaustible resource problem. Our purpose is to illustrate how the reasoning above works in a simple example from which we can extract analytical results.

4 Application: A resource extraction game

We consider a differential game of extraction of a non-renewable resource. In the related literature (for extensive surveys on dynamic games in resource economics, refer to Long, 2010, 2011). it is generally argued that the presence of rivalry among multiple agents tends to result in inefficient outcomes, e.g. overextraction of natural resources. Another common feature of the frameworks developed in this literature is the assumption that players cannot adopt new technology that will improve their extraction efficiency. It is usually assumed that consumption is a fixed fraction of the extraction level. In this section, we relax this assumption and consider the possibility of technological adoption among players. That is, players not only choose their consumption, but they also decide when to adopt the more efficient extraction technology. This consideration represents another contribution of this paper.

Our resource extraction game comprises $I = 2$ players. Let $u_i(t)$ denote the consumption rate of player i, $i = 1, 2$, at time $t \geq 0$. Meanwhile, let $e_i(t)$ be player i's extraction rate from the resource at time $t \geq 0$. Extraction is converted into consumption according to the following technology: $\gamma_i u_i(t) = e_i(t)$, where γ_i^{-1} is a positive number that reflects a player's degree of efficiency in transforming the extracted natural resource into a consumption good.

Two production technologies, described only by the parameter γ_i, are available to player i from $t = 0$. Because players' technological menus may differ, one needs to introduce a specific index for the player's actual technology. It is assumed that player 1 starts with technology $l = 1$ and has to decide: *(i)* whether she switches to technology $l = 2$, and *(ii)* when. The state of technology of the other player, 2, is labelled as k and a technological regime is represented by $s = lk$, with $l, k = 1, 2$. For each player i, the ranking between the parameters satisfies: $\gamma_i^1 > \gamma_i^2$, which means that the second new technology is more efficient than the old one. A possible indicator of technological gain for player i from adoption is the ratio $\frac{\gamma_i^2}{\gamma_i^1} \in (0, 1)$, such that the smaller is the ratio, the higher is the gain.

Let $x(t)$ be the stock of the exhaustible resource, with the initial stock x_0 given. As in section 2, t_1 and t_2 are the switching times. Suppose $0 < t_1 < t_2$, then the evolution of the stock is given by the following differential equation:

$$\dot{x} = \begin{cases} -\gamma_1^1 u_1 - \gamma_2^1 u_2 \text{ if } t \in [0, t_1) \\ -\gamma_1^2 u_1 - \gamma_2^1 u_2 \text{ if } t \in [t_1, t_2) \\ -\gamma_1^2 u_1 - \gamma_2^2 u_2 \text{ if } t \in [t_2, \infty) \end{cases}$$

At the switching time, if any, player i incurs a cost that is defined in terms of the level of the state variable at which the adoption occurs, $x(t_i) = x_i$. Let $\omega_i(x(t_i))$ be this cost, with $\omega_i'(.) \geq 0$. The direct switching cost is discounted at rate ρ. As seen from the initial period, if a switch occurs at t_i, the discounted cost amounts to $e^{-\rho t_i}\omega_i(x_i)$ (this is our $\Omega_i(x(t_i), t_i)$ of Section 2). It takes the form: $\omega_i(x_i) = \chi_i + \beta_i x_i$, where $\chi_i > 0$, $\beta_i > 0$, and χ_i is the fixed cost related to technology investment. These may include initial outlay for machinery, etc. On the other hand, β_i represents the sensitivity of adoption cost to the level of the exhaustible resource at the instant of switch. Our assumption implies that the cost of adopting new technology is increasing in x_i. This assumption conveys the idea that the lower the level of the (remaining) stock of resource, the lower the cost of adopting the new technology. It could reflect the fact that scientific progress on installation of resource-saving technology is continually made as the scarcity becomes more acute. Finally, each player's gross utility function depends on her consumption only and takes the logarithmic form: $F(u_i, u_{-i}, x) = \ln(u_i)$.

In the next subsections, attention is paid first to the corner solutions, which allows us to address the issue of the equilibrium timing. Then an analysis of the interior solution – for the correct timing – is conducted with the aim to discuss the features of the equilibrium with regime switching. From now on, the timing considered is $0 \leqq t_1 \leqq t_2$. Results for the other timing are obtained by symmetry. It is well-known that there may exist multiple feedback equilibria in extraction strategies. But, assessing the issue of uniqueness is beyond the scope of the paper. That's why, in the subsequent analysis, we are looking for solutions within the class of linear feedback strategies.

4.1 Equilibrium timing

The first part of the subsequent analysis examines the conditions under which the timing under scrutiny is not robust to deviations in players' switching strategies, i.e., at least one player would prefer to swap position. This will be followed by identifying necessary conditions for corner solutions. Once these tasks are done, we can solve for interior solutions. All the proofs for the application are relegated in Appendix B. Hereafter we shall denote $x^*(t_i)$ the level of the stock of resource at which player i decides to switch at the MPE as x_i^*) and call it the switching point.

4.1.1 Incentives to deviate from the specified timing

A player may find the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$ *non-optimal.* For instance, guessing that player 1 will switch at $t_1 (< \infty)$, player 2 may prefer switching at a date *no later* than t_1. Even though the analysis of non-optimal timing comes logically before considering any other solution, it should be noted that in the Appendix B, the proof of proposition 1 cannot be read independently of the other parts. This is also true for the proofs devoted to corner solutions. As far as non-optimal timing are concerned, it can be shown that

Proposition 1 *If player i, $i = 1, 2$, finds the above timing non-optimal (wants to deviate from it) then it must hold that*

$$\rho\omega_2(x_{-i}^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \leq \rho\omega_1(x_{-i}^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right), \tag{7}$$

As mentioned in Section 3, following Theorem 2, condition (7) characterizes a situation that is more than a simple corner solution. Let's consider for instance player 2's situation. Assume that player 1's switching problem has an interior solution $0 < t_1 < \infty$ given, with $x_1^* = x^*(t_1)$, when he anticipates that player 2 will stick to his timing strategy. In order to obtain (7), we have determined under which conditions it is "optimal" for player 2, who maximizes the discounted value between t_1 and ∞, to switch immediately. This means that the necessary conditions are similar to the usual conditions of the multi-stage optimal control theory for immediate switching (see Amit, 1986, Theorem 1 page 537). However, this particular situation cannot be interpreted as a corner solution because the framework under scrutiny is a differential game. This implies that t_1 (the beginning of the planning period for player 2) is not given. So, we should interpret this artificial corner solution as a situation where it is not optimal for player 2 to adopt after player 1. Player 1, who is supposed to be the first player to switch, may find the timing non-optimal as well. What is worth noting is that the necessary condition for the non robustness to deviations is the same for the two players. There is no single condition however because the reference point in (7), that is given by the switching point of the other player, matters. Therefore, as long as condition (7) holds for at least one of the player, the only timing, that is a MPE candidate, is $0 \leqq t_2 \leqq t_1 \leqq \infty$.

Next, we can exploit the result above to establish that:

Corollary 1 *A sufficient condition for the equilibrium timing (i.e., the timing such that none of the players have an incentive to deviate) to be* $0 \leqq t_1 \leqq t_2 \leqq \infty$ *is:*

$$\rho[\omega_1(x) - \omega_2(x)] < -\left[\ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) - \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] \text{ for all } x \in [0, x_0] \tag{8}$$

If condition (8) holds, then the timing $t_2 < t_1$ cannot occur in equilibrium. This condition can easily be interpreted in economic terms. First note that $-\ln\left(\frac{\gamma_i^2}{\gamma_i^1}\right)$, for $i = 1, 2$, is a measure of the gain from switching. Then, this condition basically states that for player 1, the relative advantage of adoption (RHS), measured in terms of the differential of gains, is greater than the relative disadvantage in terms of adoption costs (LHS). Put differently, player 1 has a relative disadvantage in adoption costs that is compensated by a relative technological advantage. Of course, this inequality is satisfied when player 1 incurs a lower direct switching cost and, at the same time, derives a higher benefit of adoption. But, it might also hold in intermediate situations where player 1's adoption cost is higher provided that the differential in technological gains is largely favorable to player 1.

From now on, let's assume that (8) holds. In the next section, we briefly review the corner solutions associated with the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$.

4.1.2 Corner solutions

First, we emphasize the conditions under which the MPE may feature a corner solution. Next, we tackle the issue of the occurrence of a simultaneous switch.

Proposition 2

- *Assume that player 1's switching problem has an interior solution* t_1*. A sufficient condition for player 2 to choose the "never switching strategy," so that* $0 < t_1 < t_2 = \infty$ *is that*

$$\rho\omega_2(0) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \geq 0. \tag{9}$$

- *Assume that player 2's switching problem has an interior solution* t_2*. A necessary condition for player 1 to switch immediately at the beginning, so that* $0 = t_1 < t_2 < \infty$ *is*

$$\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq e^{-\rho\theta_2(x_0,21)} \ln(1 - \beta_2\rho x_2^*). \tag{10}$$

- *If condition (9) is satisfied then a combination of immediate and never switching* $0 = t_1 < t_2 = \infty$ *may arise only if:*

$$\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq 0. \tag{11}$$

Conditions for being at a corner solution have very simple interpretations. For instance, according to condition (9), a player never finds it worthwhile to adopt the new technology if the fixed cost of adoption, weighted by the rate of time preference, is larger than the gain from switching even when the resource gets close to exhaustion (recall that in our setting, the stock of resource is asymptotically exhausted). In the same vein, for a player to be willing to adopt the new technology immediately it must hold that the switching cost at the initial resource level is lower than the gain from adoption. In the latter case, the particular tradeoff is influenced by the other player's switching decision to switch in finite time (10) or keep the old technology forever (11).

Beyond corner solutions, there are three remaining cases: *(i)* Players might wish to adopt their new technology at the same date and for the same stock of resource. Or, *(ii)* they might both prefer switching instantaneously; or *(iii)* on the contrary they might prefer sticking to the first technology forever. If there is heterogeneity in switching costs, case *(i)* cannot be an equilibrium outcome. The conditions for having the two other possibilities can easily be derived from proposition 2 (see the Appendix B.4).

With this information in mind, we now turn to the analysis of the interior solution.

4.2 Impact of the interaction through switching strategies on the equilibrium

At the interior solution ($0 < t_1 < t_2 < \infty$), our differential game can be divided into three subgames. We proceed backward by examining first the solution to the last stage problem, i.e., to the subgame arising after player 2's regime switch. This is a standard infinite horizon differential game. Recalling that we restrict our attention to linear feedback strategies, we find that the consumption strategies $\Phi_i(x, 22)$ ($i = 1, 2$) at the MPE satisfy

$$\gamma_1^2 \Phi_1(x, 22) = \gamma_2^2 \Phi_2(x, 22) = \rho x. \tag{12}$$

More generally, in each regime, the two players have the same equilibrium extraction rates, but generally not the same consumption rates. This feature is due to the logarithmic utility.

From these strategies, we can easily compute players' present values corresponding to the last period problem, which are used as scrap value functions for the preceding problem. Indeed, the next step is to examine the second period problem in which player 2 has now to take her regime switching action. The resolution consists in determining not only consumption strategies valid in regime 21 but also the switching time and switching point of player 2 at the MPE. Results are summarized in the proposition below. For simplicity, we assume that: $x_0 > (\rho\beta_2)^{-1}$.

Proposition 3 • *In regime 21, consumptions strategies are given by*

$$\gamma_1^2 \Phi_1(x,21) = \gamma_2^1 \Phi_2(x,21) = \Gamma + \rho x \text{ with } \Gamma = \frac{\rho^2 \beta_2 (x_2^*)^2}{1-\beta_2 \rho x_2^*}. \quad (13)$$

- *The optimal switching point, x_2^*, is defined by*

$$\rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) = \ln(1-\beta_2\rho x_2^*). \quad (14)$$

Sufficient conditions for the existence of a unique x_2^* are:

$$\begin{aligned} -\ln\left(\tfrac{\gamma_2^2}{\gamma_2^1}\right) &> \rho\omega_2(0), \\ \rho\omega_2(x_1) + \ln\left(\tfrac{\gamma_2^2}{\gamma_2^1}\right) &> \ln(1-\beta_2\rho x_1). \end{aligned} \quad (15)$$

- *The time-to-go (before switching) strategy is $t_2 - t_1 = \theta_2(x_1, 21)$ with*

$$\theta_2(x_1,21) = \frac{1}{2\rho}\ln\left[(1-\rho\beta_2 x_2^*)\frac{x_1}{x_2^*} + \rho\beta_2 x_2^*\right] = \frac{1}{2\rho}\ln\left[\frac{\Phi_i(x_1,21)}{\Phi_i(x_2^*,21)}\right]. \quad (16)$$

where (t_1, x_1) can be any solution to the switching problem of player 1.

Several comments are in order here. First, from equations (12) and (13), one can observe that $\gamma_2^1 u_2^{21*}(t_2) = \gamma_2^2 u_2^{22*}(t_2)$ if and only if $\beta_2 = 0$. Thus, if $\beta_2 > 0$, players' resource extraction experiences a jump at the switching date of player 2. This results from the dependence of the direct switching cost on the level of the state variable at the switching date. A similar pattern is observed by Valente (2011) and Prieur et al. (2013), in different frameworks. Second, the first sufficient condition (for the existence of a unique switching point, x_2^*) in (15) is nothing else than a necessary condition for having $t_2 < \infty$ (take the converse of condition (9) in Proposition 2). It basically states that player 2 will switch in finite time as long as the technology differential – gain from switching – is large enough compared to the fixed cost of switching, when the stock of resource approaches zero. Third, the time-to-go before switching (16) is defined in terms of player 1's switching point, x_1, the discount rate and some parameters characterizing regime 21, that players leave, and regime 22, that players reach. Hence, player 1 is able to affect player 2's switching strategy and will take this influence into account in the first period problem. Note also that the optimal switching date of player 2 is increasing in x_1. The larger the resource stock at which player 1 decides to switch, the later the adoption of player 2. In other words, switching rapidly for player 1 tends to delay the adoption time of player 2.

Adopting the same methodology as before (notably by computing player 1's present value from regime 21 on), we can finally have a look at the first period problem. In regime 11, player 1 guesses that player 2 has a time-to-go strategy $\theta_2(x_1, 21)$ and a corresponding switching point x_2. He also guesses that player 2's consumption strategy in regime 11 takes the form $\Phi_2(x, 11)$. Now we characterize the MPE in consumption strategies in regime 11 and provide sufficient conditions for having a unique solution to the first player's switching problem. Note that at the MPE, player 1's guesses have to be consistent with player 2's actual strategies.

Proposition 4 • *In regime* 11 *consumption strategies satisfy*

$$\gamma_1^1 \Phi_1(x, 11) = \gamma_2^1 \Phi_2(x, 11) = \Lambda + \rho x, \text{ with } \Lambda = \frac{\Gamma + \rho x_1^*[1 - \zeta(x_1^*; x_2^*)]}{\zeta(x_1^*; x_2^*)},$$

and

$$\zeta(x_1^*; x_2^*) = 1 - \frac{e^{-\rho\theta_2(x_1^*,21)}}{2} \ln(1 - \rho\beta_2 x_2^*) - \beta_1(\Gamma + \rho x_1^*),$$

where Γ *is given in Proposition 3.*

- *The optimal level of the stock for switching* x_1^* *solves*

$$\rho\omega_1(x_1^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = e^{-\rho\theta_2(x_1^*,21)} \left[\rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] + \ln[\zeta(x_1^*; x_2^*)] \tag{17}$$

- *If the following conditions hold:* $\zeta(x_2^*; x_2^*) \geq 1$, $\zeta(x_0; x_2^*) \in (0, 1]$ *and*

$$\begin{aligned} \rho\omega_1(x_0) + \ln\left(\tfrac{\gamma_1^2}{\gamma_1^1}\right) &> e^{-\rho\theta_2(x_0,21)} \left[\rho\omega_2(x_2^*) + \ln\left(\tfrac{\gamma_2^2}{\gamma_2^1}\right)\right], \\ \rho\omega_1(x_2^*) + \ln\left(\tfrac{\gamma_1^2}{\gamma_1^1}\right) &< \rho\omega_2(x_2^*) + \ln\left(\tfrac{\gamma_2^2}{\gamma_2^1}\right), \end{aligned} \tag{18}$$

then there exists a unique $x_1^* \in (x_2^*, x_0)$.

- *With the pair* (x_1^*, x_2^*) *being determined above, the optimal switching time* $t_1 = \theta_1(x_0, 11)$ *is*

$$\theta_1(x_0, 11) = \frac{1}{2\rho} \ln\left(\frac{x_0 + \frac{\Lambda}{\rho}}{x_1^* + \frac{\Lambda}{\rho}}\right) = \frac{1}{2\rho} \ln\left[\frac{\Phi_i(x_0, 11)}{\Phi_i(x_1^*, 11)}\right].$$

Recalling that $\gamma_1^2 \Phi_1(x_1^*, 21) = \Gamma + \rho x_1$ (see 13), we observe that $\zeta(.)$ defined above provides information on the direction ($\lesseqgtr 1$) and magnitude of the jump in the extraction rate at the switching time t_1. The third item of Proposition 4 basically enumerates the sufficient conditions for the existence of an interior solution to player 1's switching problem. At first glance, these boundary conditions may seem difficult to interpret. But, it is clear that conditions in (18) are necessarily satisfied if *(i)* the (sufficient) conditions for the timing to be robust to deviations hold (see Corollary 1) and *(ii)* the opposite of conditions for corner solutions (that can easily be derived from Proposition 2) are met.

In the remainder of this section, we further address the impact of MPE strategies for switching times on the equilibrium. Indeed, given that player 2's switching strategy is based on the state of the system and player 1 is able to affect this state, it is crucial to understand how player 1 adapts his strategy to player 2's switching decision. It is also required to scrutinize the impact of player 2's future switch on player 1's situation. Recall that at the MPE, the guess of player 1 must be consistent with the switching strategy actually adopted by player 2. For the sake of interpretation, player 1's switching can be rewritten as (by substitution of the specific functional forms in conditions (2) of Theorem 1):

$$\begin{array}{l} \ln\left[\frac{u_1^{11*}(t_1)}{u_1^{21*}(t_1)}\right] = -\rho\omega_1(x_1^*) + e^{-\rho\theta_2(x_1^*,21)}\ln\left[\frac{u_1^{22*}(t_2)}{u_1^{21*}(t_2)}\right] \\ \left[\gamma_1^2 u_1^{21*}(t_1)\right]^{-1} - \left[\gamma_1^1 u_1^{11*}(t_1)\right]^{-1} = \omega_1'(x_1^*) + \theta_2'(x_1^*,21)e^{-\rho\theta_2(x_1^*,21)}\ln\left[\frac{u_1^{22*}(t_2)}{u_1^{21*}(t_2)}\right] \end{array} \tag{19}$$

Compared to the single-agent problem, both conditions are modified. The LHS of the first equation in (19) reflects the marginal gain from extending the horizon of the first regime. If there exists $0 < t_1 < t_2$ then this marginal gain must be equal to the marginal cost of switching at t_1. Now, the marginal switching cost (RHS) is augmented (in absolute magnitude) by the extra-term $e^{-\rho\theta_2(x_1^*,21)}\ln[\frac{u_1^{22*}(t_2)}{u_1^{21*}(t_2)}]$. Player 1 anticipates that his switching decision will be followed by the switch (in finite time too) of the second player and that player 2's switch will be costly to him. Why is it so? Adopting a new technology translates into a downward jump in the extraction rate at time t_2: $\gamma_2^1 u_2^{21*}(t_2) > \gamma_2^2 u_2^{22*}(t_2)$. Intuitively, with the new technology, one needs less resource to produce a given amount of the consumption good. The impact of player 2's adoption on her own consumption after time t_2 and thereafter must be positive (for otherwise, she would not make the switch). It is clear that player 1 is worse off after player 2's switch because he bears the decrease in extraction and is not able to compensate this loss because his technology is fixed after time t_1. So, it means that player 1's marginal cost of switching at time t_1 is higher than it would be in the absence of player 2. Other things equal (x_1 constant), this would imply that the switch should occur at a later date, i.e., player 1, when interacting with player 2, has an incentive to postpone adoption.

The second equation in (19) equalizes the marginal benefit from an extra unit of the state variable x_1 (LHS) with the corresponding marginal cost (RHS). The marginal cost is lower in the game than in the control problem because, from (16), $\theta_2'(x_1^*,21)e^{-\rho\theta_2(x_1^*,21)}\ln[\frac{u_1^{22*}(t_2)}{u_1^{21*}(t_2)}] < 0$. Indeed, changing x_1 marginally yields an additional benefit here. Other things equal (t_1 constant), it allows player 1 to induce player 2 to delay the instant of her switch. The impact of player 2's switch will then be felt less acutely because of discounting. This in turn implies that player 1's adoption should occur at a higher x_1^*. This second effect tends to make it worthwhile for player 1 to adopt at an earlier date (because the trajectory of x is monotone non-increasing).

In summary, as a result of the interaction with player 2, player 1 has an incentive to delay the adoption of the new technology (first-order effect corresponding to the first condition in (19)). It does not mean, however, that he will not adopt before player 2. According to the second condition in (19), the sooner the adoption of player 1, the lower the negative impact of player 2's adoption on his welfare (second-order effect).

To conclude this analysis, let us highlight a striking result that can be obtained by focusing on the special case where $\omega_1(.) \equiv 0$: the switching cost is identically zero, so that it is independent of the stock of resource. In this case, player 1 does not bear any direct cost when he switches. Then, we know that the solution of the optimal control problem is $t_1 = 0$: One adopts instantaneously because the new technology is more efficient than the old one. But, it is clear that if the equations in (19) have a solution (this can be guaranteed by deriving the existence conditions for this special case from the analysis of Appendix B.2.2), then conclusions will be very different in the switching game. Player 1 incurs an indirect marginal cost when player 2 adopts. Then, it is optimal for player 1 to switch at a date t_1 such that $0 < t_1 < t_2$ because it allows him to delay the loss caused by player 2's switch.

5 Conclusion

In this paper, we have developed a general two-player differential game with regime switching strategies. The interaction between players is assumed to be governed by two kinds of strategies. At each point in time, they have to choose an action that influences the evolution of a state variable. In addition, they may decide on the switching time between alternative and consecutive regimes. We pay attention to the Markov perfect equilibrium: The switching strategy is defined as a function of the state of the system. Compared to the standard optimal control problem with regime switching, necessary optimality conditions are modified only for the first player to switch. When choosing the switching strategy, this player must take into account that *(i)* his decision will influence the other player's strategy, and *(ii)* the other player's switch will affect his welfare. Furthermore, we have exhibited and interpreted the conditions characterizing the timing at the Markov perfect equilibrium, i.e., the timing that is robust to deviations in switching strategies.

In the second part of this paper, we applied this new theoretical framework to solve a game of exhaustible resource extraction with technological regime switching. It was assumed that, at a given cost, players have the option to adopt a more efficient extraction technology. We then obtained sufficient conditions guaranteeing that both players switch in finite time. Moreover, we investigated the impact of this new source of interaction on the first player to switch adoption strategy. There is an interplay between two conflicting effects. First, adoption by the second player to switch is costly for the first to adopt because it implies a drop in his consumption. Thus, the first player to switch may opt to delay adoption. Meanwhile, because of discounting, delaying the switch of the other play will allow the player that adopts first to incur a lower indirect cost. This is an incentive for this player to adopt at an earlier date.

Overall, the methodology presented in this paper may pave the way to handle a wider class of problems in economics. Potential extensions include the analysis of technology adoption in a climate change game, the consideration of the interaction between the elites and the citizens in a game of institutional regime changes (Acemoglu and Robinson, 2006), and the analysis of conflict between rival groups for the management of natural resources (van der Ploeg and Rohner, 2012). These issues will be addressed in the authors' future research endeavors.

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Appendix

A Proof of Theorems 1&2

Let the triplet $(u_1^*(t), u_2^*(t), x^*(t))$ be the path followed by each player's strategy and the stock variable at a Markov perfect equilibrium (MPE), for every $t \in [0, +\infty)$. A restriction of this path to $[t_{j-1}, t_j]$, which corresponds to a particular regime say s, $j = 1, 2, 3$ with $t_0 = 0$ and $t_3 = \infty$, continues to characterize the solution of the subgame with $x^*(t_j - 1) = x_{j-1}$, t_{j-1} and t_j given and with the maximization of $\int_{t_{j-1}}^{t_j} F_i^s(u_1, u_2, x)e^{-\rho t}dt$ as player i's objective, $i = 1, 2$. The proof consists of the construction of the first order variation of the value function resulting from one player's small deviation from the equilibrium path. Hereafter, we shall go through the main steps of the proof because our approach extensively relies on Amit (1986) and we refer the reader to this paper for more details. We focus on the timing $0 \leqq t_1 \leqq t_2 \leqq \infty$, i.e., on the case where player 1 is the first to switch, followed by player 2.[1] If t_1 is player 1 switching time then from the definition of the switching rule, we have $t_2 = t + \theta_2(x^*(t))$ for $t \geq t_1$.

Player 1's payoffs evaluated along the MPE, the strategies of player 2 being given, can be written as:

$$V_1^* = \begin{array}{l} \int_0^{t_1} F_1^{11}(u_1^*, \Phi_2(x^*), x^*)e^{-\rho t}dt + \int_{t_1}^{t_1+\theta_2(x^*(t_1))} F_1^{21}(u_1^*, \Phi_2(x^*), x^*)e^{-\rho t}dt \\ + \int_{t_1+\theta_2(x^*(t_1))}^{\infty} F_1^{22}(u_1^*, \Phi_2(x^*), x^*)e^{-\rho t}dt - \Omega_1(t_1, x^*(t_1)) \end{array}$$

Assume that player 1 considers the opportunity to change his switching time by a small amount δt_1. Let $(u_1(t), u_2(t), x(t))$ be the vector of feasible controls and state associated with this change. Player 1's switching time becomes $t_1 + \delta t_1$ whereas the one of player 2 (as anticipated by player 1) is now given by $t + \delta t_1 + \theta_2(x(t))$ for $t \geq t_1 + \delta t_1$. Then, player 1's payoffs become:

$$V_1 = \begin{array}{l} \int_0^{t_1+\delta t_1} F_1^{11}(u_1, \Phi_2(x), x)e^{-\rho t}dt + \int_{t_1+\delta t_1}^{t_1+\delta t_1+\theta_2(x(t_1+\delta t_1))} F_1^{21}(u_1, \Phi_2(x), x)e^{-\rho t}dt \\ + \int_{t_1+\delta t_1+\theta_2(x(t_1+\delta t_1))}^{\infty} F_1^{22}(u_1, \Phi_2(x), x)e^{-\rho t}dt - \Omega_1(t_1 + \delta t_1, x(t_1 + \delta t_1)) \end{array}$$

[1] Necessary optimality conditions for the other timing $0 \leqq t_2 \leqq t_1 \leqq \infty$ can be obtained by symmetry. For notational convenience, we do not make the dependence of decision rules on the regime explicit.

Rearranging the terms and introducing the notations $F_1^s(u_1,\Phi_2(x),x) = G_1^s(u_1,x)$, $G_1^{s*} = G_1^s(u_1^*,x^*)$, $f^s(u_1,\Phi_2(x),x) = g^s(u_1,x)$, and $g^{s*} = g^s(u_1^*,x^*)$, the variation of player 1's payoffs $\delta V_1 = V_1 - V_1^*$ is equal to:

$$\delta V_1 = \begin{array}{l} \int_0^{t_1} \left\{ \left[G_1^{11}(u_1,x)e^{-\rho t} + \lambda_1^{11} g^{11}(u_1,x)\right] - \left[G_1^{11}(u_1^*,x^*)e^{-\rho t} + \lambda_1^{11} g^{11}(u_1^*,x^*)\right] - \lambda_1^{11}\dot{h}^{11} \right\} dt \\ + \int_{t_1}^{t_1+\theta_2(x^*(t_1))} \left\{ \left[G_1^{21}(u_1,x)e^{-\rho t} + \lambda_1^{21} g^{21}(u_1,x)\right] - \left[G_1^{21}(u_1^*,x^*)e^{-\rho t} + \lambda_1^{21} g^{21}(u_1^*,x^*)\right] - \lambda_1^{21}\dot{h}^{21} \right\} dt \\ + \int_{t_1+\theta_2(x^*(t_1))}^{\infty} \left\{ \left[G_1^{22}(u_1,x)e^{-\rho t} + \lambda_1^{22} g^{22}(u_1,x)\right] - \left[G_1^{22}(u_1^*,x^*)e^{-\rho t} + \lambda_1^{22} g^{22}(u_1^*,x^*)\right] - \lambda_1^{22}\dot{h}^{22} \right\} dt \\ + \int_{t_1}^{t_1+\delta t_1} \left[G_1^{11}(u_1,x) - G_1^{21}(u_1,x)\right] e^{-\rho t} dt + \int_{t_1+\theta_2(x^*(t_1))}^{t_1+\delta t_1+\theta_2(x(t_1+\delta t_1))} \left[G_1^{21}(u_1,x) - G_1^{22}(u_1,x)\right] e^{-\rho t} dt \\ -\Omega_1(t_1+\delta t_1, x(t_1+\delta t_1)) + \Omega_1(t_1, x^*(t_1)) \end{array}$$

where we have added to the integrant of the first three lines the term $\lambda_1^s(g^s(u_1,x) - \dot{x}) - \lambda_1^s(g^s(u_1^*,x^*) - \dot{x}^*)$, for any differentiable functions λ_1^s, $s = 11, 21, 22$, and defined, in any regime s, the deviation h^s as $h^s = x - x^*$.

Integrating by parts the terms $\int -\lambda_1^s \dot{h}^s dt$ and using appropriate boundary – including the initial and the transversality – conditions yield:

$$\delta V_1 = \begin{array}{l} \int_0^{t_1} \left\{ \left[H_1^{11}(t,u_1,x) - H_1^{11*}(t)\right] + \dot{\lambda}_1^{11} h^{11} \right\} dt + \int_{t_1}^{t_1+\theta_2(x^*(t_1))} \left\{ \left[H_1^{21}(t,u_1,x) - H_1^{21*}(t)\right] + \dot{\lambda}_1^{21} h^{21} \right\} dt \\ + \int_{t_1+\theta_2(x^*(t_1))}^{\infty} \left\{ \left[H_1^{22}(t,u_1,x) - H_1^{22*}(t)\right] + \dot{\lambda}_1^{11} h^{11} \right\} dt + \int_{t_1}^{t_1+\delta t_1} \left[G_1^{11}(u_1,x) - G_1^{21}(u_1,x)\right] e^{-\rho t} dt \\ + \int_{t_1+\theta_2(x^*(t_1))}^{t_1+\delta t_1+\theta_2(x(t_1+\delta t_1))} \left[G_1^{21}(u_1,x) - G_1^{22}(u_1,x)\right] e^{-\rho t} dt - \Omega_1(t_1+\delta t_1, x(t_1+\delta t_1)) + \Omega_1(t_1, x^*(t_1)) \\ -\lambda_1^{11}(t_1)h^{11}(t_1) + \lambda_1^{21}(t_1)h^{21}(t_1) - \lambda_1^{21}(t_2)h^{21}(t_2) + \lambda_1^{22}(t_2)h^{22}(t_2) \end{array} \tag{20}$$

with $t_2 = t_1 + \theta_2(x^*(t_1))$, $H_1^s(t,u_1,x) = G_1^s(u_1,x)e^{-\rho t} + \lambda_1^s g^{11}(u_1,x)$ regime s Hamiltonian and $H_1^{s*}(t)$ the same Hamiltonian evaluated along the equilibrium trajectory.

Now we want to obtain a linear approximation of δV_1. For δt_1 close to zero and h^s, $\delta u_1 = u_1 - u_1^*$ small, we first compute the following first order Taylor series:

$$H_1^s(t,u_1,x) \simeq H_1^{s*}(t) + \frac{\partial H_1^{s*}(t)}{\partial u_1}\delta u_1 + \frac{\partial H_1^{s*}(t)}{\partial x} h^s$$

and,

$$\int_{t_1}^{t_1+\delta t_1} \left[G_1^{11}(u_1,x) - G_1^{21}(u_1,x)\right] e^{-\rho t} dt \simeq \left[G_1^{11*} - G_1^{21*}\right] e^{-\rho t_1} \delta t_1$$

$$\int_{t_1+\theta_2(x^*(t_1))}^{t_1+\delta t_1+\theta_2(x(t_1+\delta t_1))} \left[G_1^{21}(u_1,x) - G_1^{22}(u_1,x)\right] e^{-\rho t} dt \simeq \left[G_1^{21*} - G_1^{22*}\right] e^{-\rho(t_1+\theta_2(x^*(t_1)))} (\delta t_1 + \theta_2(x(t_1+\delta t_1)) - \theta_2(x^*(t_1)))$$

Making use of these approximations, (20) can be rewritten as:

$$\delta V_1 = \begin{array}{l} \int_0^{t_1} \left\{ \frac{\partial H_1^{11*}(t)}{\partial u_1}\delta u_1 + \left[\frac{\partial H_1^{11*}(t)}{\partial x} + \dot{\lambda}_1^{11}\right] h^{11} \right\} dt + \int_{t_1}^{t_1+\theta_2(x^*(t_1))} \left\{ \frac{\partial H_1^{21*}(t)}{\partial u_1}\delta u_1 + \left[\frac{\partial H_1^{21*}(t)}{\partial x} + \dot{\lambda}_1^{21}\right] h^{21} \right\} dt \\ + \int_{t_1+\theta_2(x^*(t_1))}^{\infty} \left\{ \frac{\partial H_1^{22*}(t)}{\partial u_1}\delta u_1 + \left[\frac{\partial H_1^{22*}(t)}{\partial x} + \dot{\lambda}_1^{22}\right] h^{22} \right\} dt + \left[G_1^{11*} - G_1^{21*}\right] e^{-\rho t_1} \delta t_1 \\ + \left[G_1^{21*} - G_1^{22*}\right] e^{-\rho(t_1+\theta_2(x^*(t_1)))} (\delta t_1 + \theta_2(x(t_1+\delta t_1)) - \theta_2(x^*(t_1))) \\ -\Omega_1(t_1+\delta t_1, x(t_1+\delta t_1)) + \Omega_1(t_1, x^*(t_1)) \\ -\lambda_1^{11}(t_1)h^{11}(t_1) + \lambda_1^{21}(t_1)h^{21}(t_1) - \lambda_1^{21}(t_2)h^{21}(t_2) + \lambda_1^{22}(t_2)h^{22}(t_2) \end{array} \tag{21}$$

Next we take the linear parts of the following Taylor expansions:

$$\begin{array}{l} \theta_2(x(t_1+\delta t_1)) \simeq \theta_2(x^*(t_1)) + \theta_2'(x^*(t_1))\delta x_1 \\ \Omega_1(t_1+\delta t_1, x(t_1+\delta t_1)) \simeq \Omega_1(t_1, x^*(t_1)) + \frac{\partial \Omega_1(t_1,x^*(t_1))}{\partial t_1}\delta t_1 + \frac{\partial \Omega_1(t_1,x^*(t_1))}{\partial x_1}\delta x_1 \end{array}$$

with $\delta x_1 = x(t_1+\delta t_1) - x^*(t_1)$ the difference between the value taken by the state variable at the *new* switching time $t_1 + \delta t_1$ and the equilibrium value $x^*(t_1)$. Again, let's consider the following approximation:

$$x(t_1+\delta t_1) \simeq x(t_1) + \dot{x}^*(t_1)\delta t_1$$

where $\dot{x}(t_1)$ has been replaced with $\dot{x}^*(t_1)$, which is possible if $x(t_1)$ is close enough to $x^*(t_1)$. We also need to use the same difference of state values at player 2's switching times, $\delta x_2 = x(t_1+\delta t_1+\theta_2(x(t_1+\delta t_1)) - x^*(t_1+\theta_2(x^*(t_1))$, together with:

$$x(t_1+\delta t_1+\theta_2(x(t_1+\delta t_1)) \simeq x(t_1+\theta_2(x^*(t_1))+\dot{x}^*(t_1+\theta_2(x^*(t_1))(\delta t_1+\theta_2'(x^*(t_1))\delta x_1)$$

Observing that $\dot{x}^*$ in any regime s is equal to g^{s*}, the deviation h^s can be expressed in terms of the variations δt_1, δx_1 and δx_2:

$$\begin{array}{l} h^s(t_1) = \delta x_1 - g^{s*}\delta t_1 \text{ for } s = 11, 21, \\ h^s(t_2) = \delta x_2 - g^{s*}(\delta t_1 + \theta_2'(x^*(t_1))\delta x_1) \text{ for } s = 21, 22. \end{array}$$

Putting all these elements together allows us to get the expression of δV_1, assuming that Pontryagin conditions are satisfied:

$$\delta V_1 = \begin{array}{l} \left[H_1^{11*}(t_1) - H_1^{21*}(t_1) + H_1^{21*}(t_2) - H_1^{22*}(t_2) - \frac{\partial \Omega_1(t_1,x^*(t_1))}{\partial t_1}\right]\delta t_1 \\ + \left[\lambda_1^{21}(t_1) - \lambda_1^{11}(t_1) + \theta_2'(x^*(t_1))(H_1^{21*}(t_2) - H_1^{22*}(t_2)) - \frac{\partial \Omega_1(t_1,x^*(t_1))}{\partial x_1}\right]\delta x_1 \\ + \left[\lambda_1^{22}(t_2) - \lambda_1^{21}(t_2)\right]\delta x_2 \end{array} \tag{22}$$

For the trajectory $(u_1^*(t), u_2^*(t), x^*(t))$, with switching times t_1 and t_2, to be optimal for player 1 we must have $\delta V_1 \leq 0$. From (22), we can extract the necessary conditions associated with player 1's switching strategy.

Interior solution $0 < t_1 < t_2$: If δt_1, δx_1 and δx_2 are completely free and independent variables, then it must hold that:

$$\begin{array}{l} H_1^{11*}(t_1) - \frac{\partial \Omega_1(t_1,x^*(t_1))}{\partial t_1} = H_1^{21*}(t_1) - [H_1^{21*}(t_2) - H_1^{22*}(t_2)] \\ \lambda_1^{11}(t_1) + \frac{\partial \Omega_1(t_1,x^*(t_1))}{\partial x_1} = \lambda_1^{21}(t_1) + \theta_2'(x^*(t_1))[H_1^{21*}(t_2) - H_1^{22*}(t_2)] \\ \lambda_1^{21}(t_2) = \lambda_1^{22}(t_2) \end{array} \tag{23}$$

Corner solutions: If feasible variations are only of the type $\delta t_1 \geq 0$, which corresponds to the solution $t_1 = 0$, then the first necessary condition in (23) is replaced with:

$$H_1^{11*}(t_1) - \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial t_1} \leq H_1^{21*}(t_1) - [H_1^{21*}(t_2) - H_1^{22*}(t_2)] \tag{24}$$

whereas if we consider deviations $\delta t_1 \leq 0$, for the corner solution $t_1 = t_2$, then the inequality in (24) has to be reversed.

Remark 1. In some economic problems δt_1 and δx_1 are not independent variables. This is the case when the state variable follows a monotone trajectory. For example, assume that x is the stock of an exhaustible resource, with $\dot{x}(t) \leq 0$ for all t. At the corner solution $t_1 = 0$, the only possible variations are $\delta t_1 \geq 0$ and at the same time $\delta x_1 \leq 0$. So in this particular case, the second condition in (23) must also be replaced with the following one:

$$\lambda_1^{11}(t_1) + \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial x_1} \leq \lambda_1^{21}(t_1) + \theta_2'(x^*(t_1))[H_1^{21*}(t_2) - H_1^{22*}(t_2)], \quad (25)$$

this condition looks like the initial transversality conditions presented in Seierstad and Sydsaeter (1987) for control problems in which the initial state is a decision variable (Theorem 5, p. 185). The same logic applies to the other corner solution.

The same methodology can be displayed in order to characterize player 2's necessary conditions for switching. The analysis is actually simpler when we solve player 2's problem because by definition of the switching rule, any small deviation δt_2 by player 2 from the switching time t_2 has no impact on player 1's switching strategy when the timing is and remain $0 \leqq t_1 \leqq t_2 \leqq \infty$. This means that the first subgame still runs from the initial time $t_0 = 0$ to the switching time $t_1 = \theta_1(x_0)$ after a deviation. Player 2 switching problem is very much the same as an optimal control problem. So we can skip all the technical details and directly present the expression of the variation $\delta V_2 = V_2 - V_2^*$:

$$\begin{aligned} &\delta V_2 = \left[H_2^{21*}(t_2) - H_2^{22*}(t_2)\right] \delta t_2 + \left[\lambda_1^{22}(t_2) - \lambda_2^{21}(t_2) - \tfrac{\partial \Omega_2(t_2, x^*(t_2))}{\partial x_2}\right] \delta x_2 \\ &+ \left[\lambda_2^{21}(t_1) - \lambda_2^{11}(t_1)\right] \delta x_1 \end{aligned} \quad (26)$$

and the necessary conditions are: For the interior solution $t_1 < t_2 < \infty$:

$$\begin{aligned} &H_2^{21*}(t_2) - \tfrac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2} = H_2^{22*}(t_2) \\ &\lambda_2^{21}(t_2) + \tfrac{\partial \Omega_2(t_2, x^*(t_2))}{\partial x_2} = \lambda_2^{22}(t_2) \\ &\lambda_2^{11}(t_1) = \lambda_2^{21}(t_1) \end{aligned} \quad (27)$$

For the corner solution $t_1 = t_2$, the first condition in (27) becomes

$$H_2^{21*}(t_2) - \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2} \leq H_2^{22*}(t_2) \quad (28)$$

Note that remark 1. is also valid here. In addition, there is no necessary condition for the corner solution $t_2 = \infty$ (see Makris, 2001). However, a sufficient condition for this case is

$$H_2^{21*}(t_2) - \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2} \geq H_2^{22*}(t_2) \text{ (for any } t_2 > t_1) \quad (29)$$

.

Remark 2. In some (actually many) *tractable* differential games, there may exist a relationship between players' co-state variables at any instant, including the switching times. In particular, it is possible that these co-states are linked through a continuous functional $k(.)$: $\lambda_i^s(t) = k(\lambda_j^s(t))$. In this case, the conditions involving the co-states in (23) and (27) cannot hold all together because they imply that one co-state is continuous whereas the other jumps at each switching time. This means that the variation δx_2 in (22) and δx_1 in (26) must be equal to zero, i.e., player i deviation δt_i has no impact of level of the state variable at player j' switching time $x(t_j)$.

B Application

We restrict attention to linear feedback strategies: $\Phi_j(x,s) = a_j^s + b_j^s x$. In any regime s, player i's present value Hamiltonian is given by:

$$H_i^s = \ln(u_i^s)e^{-\rho t} - \lambda_i^s(\gamma_i^l u_i^s + \gamma_j^k(a_j^s + b_j^s x))$$

The FOCs are:

$$\begin{array}{l} (u_i^s)^{-1}e^{-\rho t} = \gamma_i^l \lambda_i^s \\ \dot{\lambda}_i^s = \gamma_j^k b_j^s \lambda_i^s \\ \dot{x} = -\gamma_i^l u_i^s - \gamma_j^k(a_j^s + b_j^s x) \end{array} \tag{30}$$

and have to be combined with the appropriate transversality condition, which depends on whether the regime is terminal, or not. Solving (30), it can easily be checked that players' extraction strategies are the same, whatever the regime:

$$\gamma_i^l \Phi_i(x,s) = \gamma_j^k \Phi_j(x,s), \tag{31}$$

and, when regime s is terminal, we obtain: $\gamma_1^l \Phi_1(x,s) = \gamma_2^k \Phi_2(x,s) = \rho x$.

This property means also that players' co-state variables are identical. According to Remark 2, conditions (1c) and (2c) are no longer necessary conditions and we shall check that player i's switching point does not depend on t_{-i}. Moreover, since we work with an exhaustible resource, Remark 1 also applies for corner solutions. Finally note that the Hamiltonian in any regime reduces to $H_i^s = \ln(u_i^s)e^{-\rho t} - 2$.

B.1 Player 2's switching problem

B.1.1 Interior solution (proof of Proposition 3)

Switching conditions: Assume player 1 has switched at some $t_1 \in (0,\infty)$, for a switching point x_1. For an interior solution (t_2, x_2), using (30), (31) and noticing that $s = 22$ is the terminal regime, condition (1a) of Theorem 1 simplifies to

$$\ln(u_2^{21}(t_2)) + \rho\omega_2(x_2) = \ln\left(\frac{\rho x_2}{\gamma_2^2}\right). \tag{32}$$

and (1b) is simply given by

$$u_2^{21}(t_2) = \frac{\rho x_2}{\gamma_2^1(1-\beta_2\rho x_2)}, \tag{33}$$

In addition, the consumption strategies in regime 21 are:

$$\gamma_1^2\Phi_1(x,21) = \gamma_2^1\Phi_2(x,21) = \frac{\rho^2\beta_2(x_2)^2}{1-\beta_2\rho x_2} + \rho x = \Gamma + \rho x. \tag{34}$$

From (30)-(33) and (34), (32) can be rewritten as

$$\rho\omega_2(x_2) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) = \ln(1-\beta_2\rho x_2). \tag{35}$$

This equation defines the optimal level for switching, x_2^*, which is indeed independent on the switching time of player 1.

Characterization of the solution: The LHS of (35) is defined for all $x_2 \in \left[0,(\rho\beta_2)^{-1}\right)$, increasing in x_2 and varying from zero to ∞ as x goes from zero to $(\rho\beta_2)^{-1}$. The RHS is strictly positive at $x_2 = 0$ iff $\ln\left(\frac{\gamma_2^1}{\gamma_2^2}\right) > \rho\omega_2(0)$. Since $\beta_2 > 0$, the RHS is strictly decreasing in x_2. Thus, if $x_1 \geq (\rho\beta_2)^{-1}$ and

$$\rho\omega_2(0) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) < 0 \tag{36}$$

then, there exists a unique solution x_2^* in $\left[0,(\rho\beta_2)^{-1}\right)$. Otherwise ($x_1 < (\rho\beta_2)^{-1}$), another boundary condition is

$$\rho\omega_2(x_1) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) > \ln(1-\rho\beta_2 x_1).$$

Replacing consumptions with the expressions given by (34) in the state equation, and solving the resulting differential equation (with the boundary condition $x(t_1) = x_1$) yield the expression of the state variable for any $t \in [t_1, t_2]$:

$$x^{21*}(t) = \left[x_1 + \frac{\rho\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*}\right]e^{-2\rho(t-t_1)} - \frac{\rho\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*}.$$

Evaluating this equation at t_2 and solving for $\theta_2 = t_2 - t_1$, one obtains

$$\theta_2(x_1,21) = \frac{1}{2\rho}\ln\left[(1-\rho\beta_2 x_2^*)\frac{x_1}{x_2^*} + \rho\beta_2 x_2^*\right] = \frac{1}{2\rho}\ln\left[\frac{\Phi_i(x_1,21)}{\Phi_i(x_2^*,21)}\right], \tag{37}$$

which gives the time-to-go (before switching) strategy of player 2 for any switching point (and more generally, any level of the state variable) x_1.

B.1.2 Never switching condition (proof of Proposition 2, first item)

Still assuming that there exists $t_1 \in (0, \infty)$, the sufficient condition for a never switching solution ($t_2 = \infty$) is given by:

$$\ln[u_2^{21}(t_2)] + \rho\omega_2(x_2) \geq \ln[u_2^{22}(t_2)], \tag{38}$$

for all $t_2 \in (t_1, \infty) \cup \{\infty\}$. When $t_2 \to \infty$ (and $x_2 \to 0$ because the stock of resource is exhausted asymptotically), we use the feature that regime 21 becomes the final regime and $\gamma_1^2 u_1^{21}(t) = \gamma_2^1 u_2^{21}(t) = \rho x$, and take the limit of (38) to obtain

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(0) \geq 0. \tag{39}$$

This condition for a never switching solution is also sufficient to have

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(x_2) \geq \ln(1 - \beta_2\rho x_2) \text{ for all } t_1 < t_2 \leqq \infty.$$

The analysis of the last artificial corner case ($t_1 = t_2$) is postponed to Appendix B.3 because it requires player 1's switching problem be examined first.

B.2 Player 1's switching problem

B.2.1 Interior solution (proof of Proposition 4)

Switching conditions: Suppose that player 2's regime switching takes place at some $t_2 \in (0, \infty)$. Direct manipulations of (30), (31), (34), (37) and the first switching condition (2a) of Theorem 1 yields:

$$\ln(u_1^{11}(t_1)) + \rho\omega_1(x_1) = \ln\left(\frac{\Gamma + \rho x_1}{\gamma_1^2}\right) + e^{-\rho\theta_2(x_1)} \ln(1 - \rho\beta_2 x_2). \tag{40}$$

whereas condition (2b) can be written as

$$\gamma_1^1 u_1^{11}(t_1) = \frac{\Gamma + \rho x_1}{\zeta(x_1; x_2)}, \tag{41}$$

with Γ defined in (34) and

$$\zeta(x_1; x_2) = 1 - \frac{e^{-\rho\theta_2(x_1,21)}}{2} \ln(1 - \rho\beta_2 x_2) - \beta_1(\Gamma + \rho x_1). \tag{42}$$

Solving for the MPE in consumption strategies in regime 11, one finds

$$\gamma_1^1 \Phi_1(x, 11) = \gamma_2^1 \Phi_2(x, 11) = \Lambda + \rho x \text{ with } \Lambda = \frac{\Gamma + \rho x_1[1 - \zeta(x_1; x_2)]}{\zeta(x_1; x_2)}.$$

Substituting $u_1^{11}(t_1)$ with the expression in (41), using (35) and $\gamma_1^2 u_1^{21}(t_1) = \Gamma + \rho x_1$, the optimality condition (40) can be rewritten as:

$$\rho\omega_1(x_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = e^{-\rho\theta_2(x_1,21)} \left[\rho\omega_2(x_2) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] + \ln[\zeta(x_1; x_2)]. \tag{43}$$

At the MPE, player 1's guess must be consistent with player 2's actual strategy, which implies that x_2 and θ_2 are given by x_2^* and θ_2, defined by (35) and (37). Thus, equation (43), that defines player 1's switching point x_1^*, has to be evaluated at this particular point and for this particular strategy. Note also that x_1^* is independent of t_2.

Characterization of the solution: $\zeta(x_1; x_2^*)$ is defined over (x_2^*, x_0) with $\zeta'(x_1; x_2^*) < 0$. Let us assume that $\zeta(x_0; x_2^*) > 0$, which requires x_0 be high enough. The LHS of (43) increases with x_1 on the interval $[x_2^*, x_0]$ whereas the RHS is non monotone because the time-to-go (before switching), θ_2^*, is increasing in x_1. Therefore, imposing $\zeta(x_2^*; x_2^*) \geq 1$, with

$$\zeta(x_2^*; x_2^*) = \frac{1 - \rho(\beta_1 + \beta_2)x_2^*}{1 - \rho\beta_2 x_2^*} - \frac{\ln(1 - \rho\beta_2 x_2^*)}{2},$$

$\zeta(x_0; x_2^*) \leq 1$, and

$$\begin{aligned} \rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) &> \rho\omega_2(x_0) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right), \\ \rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) &< \rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right), \end{aligned}$$

guarantees the existence of a unique $x_1^* \in (x_2^*, x_0)$ that satisfies (43). Note that $\zeta(x_2^*; x_2^*) \geq 1$ holds under specific assumptions. Assuming that $\beta_2 > 2\beta_1$, it is pretty easy to show that $\exists! \bar{x}_2^* \in (0, (\rho\beta_2)^{-1})$ such that $\zeta(x_2^*; x_2^*) > 1$ for all $x_2^* < \bar{x}_2^*$. From now on, we will assume that this technical condition holds.

Using all the material above, the resource stock is given by: $x^{11}(t) = \left(x_0 + \frac{\Lambda}{\rho}\right) e^{-2\rho t} - \frac{\Lambda}{\rho}$. Evaluating this expression at t_1, one obtains:

$$t_1 = \theta_1(x_0, 11) = \frac{1}{2\rho} \ln\left(\frac{x_0 + \frac{\Lambda}{\rho}}{x_1^* + \frac{\Lambda}{\rho}}\right).$$

Remark 3. There is no reason for player 1's switching point to be the same when $t_2 < \infty$ than when $t_2 = \infty$. Indeed, when $t_2 = \infty$, it can easily be shown that x_1 solves: $\ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) + \rho\omega_1(x_1) = \ln(1 - \beta_1\rho x_1)$.

B.2.2 Immediate switching (proof of Proposition 2, second item)

Still assuming that player 2's switching problem has a solution t_2 (with x_2^* that solves 35), if player 1 finds it optimal to switch instantaneously then, according to Theorem 2 condition (4) must hold. In our application, it is given by:

$$\ln[u_1^{11}(t_1)] + \rho\omega_1(x_1^*) \leq \ln[u_1^{21}(t_1)] + e^{-\rho\theta_2^*(x_1, 21)} \ln(1 - \beta_2\rho x_2^*) \tag{44}$$

if $0 = t_1 < t_2$.

Recall that according to Remark 1., when the state follows a monotone trajectory, we cannot use condition (2b) since the equality has to be replaced with an inequality. However, at the particular date $0 = t_1$ (implying that $x_1^* = x_0$), exploiting the fact that regime 11 vanishes in regime 21, which implies that $\gamma_1^1 u_1^{11}(0) = \gamma_2^1 u_1^{21}(0) = \Gamma + \rho x_0$, condition (44) reduces to:

$$\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq e^{-\rho\theta_2(x_0,21)} \ln(1 - \beta_2\rho x_2^*) \tag{45}$$

B.3 Robustness to deviations (proof of Proposition 1)

B.3.1 For player 2

The situation where player 2 has an incentive to deviate from the timing $t_1 \leqq t_2$ corresponds to the case $t_1 = t_2$. Condition (3) of Theorem 2 simplifies to:

$$\ln[u_2^{21}(t_2)] \leq \ln[u_2^{22}(t_2)] - \rho\omega_2(x_2^*) \text{ if } t_1 = t_2 \tag{46}$$

with $u_2^{22}(t_2) = \rho x_2^*$. Next, we use the relationship (31) which, combined with the fact that regime 21 actually vanishes into regime 11, i.e., $t_1 = t_2$, implies that: $\gamma_2^1 u_2^{21}(t_1) = \gamma_2^1 u_2^{11}(t_1)$. From the resolution of player 1's switching problem in this hypothetical case, we first obtain $\gamma_1^1 u_1^{11}(t_1) = \frac{\rho x_1^*}{\gamma_1^1(1-\beta_2\rho x_1^*)} \frac{1}{\zeta(x_1^*;x_1^*)}$. Thus, (46) simplifies to:

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(x_1^*) \leq \ln(1 - \beta_2\rho x_1^*) + \ln[\zeta(x_1^*;x_1^*)]. \tag{47}$$

Moreover, player 1's second switching condition is $\rho\omega_1(x_1^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = \ln(1 - \rho\beta_2 x_1^*) + \ln\left[\zeta(x_1^*;x_1^*)\right]$, which implies that (47) can be rewritten as:

$$\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho\omega_2(x_1^*) \leq \rho\omega_1(x_1^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right). \tag{48}$$

B.3.2 For player 1

Assume now that player 2's switching problem has an interior solution, t_2. Applying the condition (5) of Theorem 2 to our example, the timing is not robust to deviations in player 1's switching strategy only if:

$$\ln[u_1^{11}(t_1)] + \rho\omega_1(x_1^*) \geq \ln[u_1^{21}(t_1)] + e^{-\rho\theta_2(x_1^*,21)} \ln(1 - \beta_2\rho x_2^*) \tag{49}$$

if $t_1^* = t_2^*$. Making use of $x_1^* = x_2^*$ (and $\theta_2^*(x_2^*, 21) = 0$), $\gamma_1^1 u_1^{11}(t_2^*) = \gamma_2^1 u_2^{21}(t_2^*) = \Gamma + \rho x_2^*$, condition (49) is equivalent to: $\rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \ln(1 - \beta_2\rho x_2^*)$, which, from (35), can be rewritten as:

$$\rho\omega_1(x_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \rho\omega_2(x_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \tag{50}$$

B.4 Remaining cases

• **Immediate and never switching:** $0 = t_1 < t_2 = \infty$. From Appendix B.1.2, we know that (39) is a sufficient condition for player 2 to be at the corner $t_2 = \infty$. In this case, player 1 compares the (marginal) value he would obtain under the permanent regime 11 with the corresponding value he would get by switching directly to 21. Given that $\gamma_1^l u_1(0) = \rho x_0$ for $l = 1, 2$, the condition for an immediate switching is: $\rho\omega_1(x_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq 0$.

• **Simultaneous interior switches:** $0 < t_1 = t_2 = t < \infty$. From (30) and (31), we have $\lambda_1^s = \lambda_2^s$ in any regime s. It is clear that the last switching condition in (6) cannot be simultaneously satisfied for the two players whenever $\omega_1'(x) \neq \omega_2'(x)$ for all x (recall that $\omega_i'(x)e^{-\rho t} = \frac{\partial \Omega_i(x,t)}{\partial x}$ for $i = 1, 2$).

• **Simultaneous instantaneous switches:** $t_1 = t_2 = 0$: In this case, it must be true that $\rho\omega_i(x_0) + \ln\left(\frac{\gamma_i^2}{\gamma_i^1}\right) \leq 0$ for $i = 1, 2$.

• **Never switching for both players:** $t_1 = t_2 = \infty$. This case occurs when condition (39) holds for the two players.