Competitive intergenerational altruism*

Geir B. Asheim[1]   Frikk Nesje[2]

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Abstract

Will serious future climate change be avoided through strengthened intergenerational altruism? This paper first shows that normatively attractive outcomes will be implemented in a one-sector model if each generation has sufficient non-paternalistic altruism for its immediate descendants. This conclusion is radically changed in a two-sector model where one form of capital is more productive than the other, but leads to negative atmospheric externalities. In fact, the model shows that, if each dynasty is trying to get ahead in a world threatened by climate change by increasing its intergenerational altruism, then long-term wellbeing will be seriously undermined. This indicates that stronger concern for future generations need not be an effective substitute for collective climate action.

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[1] *Address*: Department of Economics, University of Oslo, P.O. Box 1095 Blindern, N-0317 Oslo, Norway. *Email*: g.b.asheim@econ.uio.no

[2] *Address*: Department of Economics, University of Oslo, P.O. Box 1095 Blindern, N-0317 Oslo, Norway. *Email*: frikk.nesje@econ.uio.no

1 Introduction

Judging from the economic development that has been realized at a world scale during the last 150 years, later generations have been more fortunate in comparison with the earlier ones, but with large intragenerational inequities. In particular, it seems like members of each generation are motivated to bequeath assets to their children, facilitating intergenerational transfers. As climate change concerns to a large extent intergenerational distribution, can we conclude that the altruism of parents towards their children will lead to a good outcome for the long-term development of societies even when faced with climate change? Moreover, is the "nirvana ethics" (Sinn, 2008, pp. 369–370) of criteria of intergenerational equity, leading us to conclude that "the market solution cannot be accepted on ethical grounds", superfluous for this reason?

The basic model of economic growth is the one-sector model where capital and an exogenous amount of labor is used to produce output, which is split into consumption and capital investment. If interpreted as a model where non-overlapping generations follow each other in sequence, and the welfare of any generation depends linearly on the utility that it derives from its consumption and the welfare of the next, then the behavior will be as if the economy is following a discounted utilitarian optimum.

If the altruism for the next generation is sufficiently strong compared to the net productivity of capital, then the realized stream has nice properties: it is increasing and converges to a steady state with constant per-capita consumption.

The model can be interpreted as representing an economy divided into dynasties (families, tribes or countries) where the members of any generation cares for the welfare of the immediate descendants in their own dynasty. Since the welfare of the immediate descendants depends on the welfare of their descendants and so forth, the interests of the members of later generations are indirectly taken into account.

If this model describes the technological opportunities and the preferences of real economies, then the question of implementing an equitable intergenerational distribution is reduced to each generation having sufficient altruism for the immediate descendants of their own dynasty. These conclusions in the standard one-sector model of economic growth might explain many economists' support for discounted utilitarianism as a justifiable normative position.

As conjectured in Asheim (2012), the nice properties might disappear when the asset that the dynasties will choose to accumulate is a public bad. Given the external effects of greenhouse gas emissions, taking care of our own descendants does not necessarily solve the distributional problems that climate change poses, but may add gravity to the problem. If each of us seeks to protect our immediate descendants against the effect of climate change, then reducing our own greenhouse gas emissions is not a productive line of action. Rather, it is likely that we will choose to accumulate the most productive assets, thereby ensuring the next generation in our own dynasty a good start in life at the cost of all other dynasties through increasing greenhouse gas emissions. A similar logic applies to the country level (see, e.g., Foley, 2008, Karp, 2013, and Rezai et al., 2012, for related analyses).

The purpose of this paper is to present a simple model that shows how intergenerational altruism might undermine future wellbeing in the presence of atmospheric externalities. In Section 2 we recapitulate the Ramsey model, on which the analysis is based, and show the consequences of different criteria of intergenerational equity in the setting of this model. Furthermore, we show how the streams promoted by all these criteria can be implemented by sufficient intergenerational altruism, even in a world decentralized into dynasties. In Section 3 we consider a two-sector model, with two capital goods: "brown" and "green" capital. We assume that "brown" capital is more productive, but is also a public bad in the sense of leading to atmospheric externalities that affect all dynasties in the same manner. In Section 4 we show that streams implemented in a world decentralized into dynasties cannot be endorsed by criteria of intergenerational equity, if intergenerational altruism is sufficiently strong. In Section 5 we investigate the scope of collective climate action. In Section 6 we conclude.

There are four appendices. Appendix A presents in more detail the different criteria of intergenerational equity and proves the results of Section 2. Appendix B establishes existence of the equilibrium that of the game that the competitively altruistic dynasties will play, and show the properties of this equilibrium, thereby proving the results of Section 3. Appendix C establishes the existence of an efficient maximin stream which facilitates the normative analysis of Section 4. Appendix D establishes, in a special case, the highest possible productivity difference between "brown" and "green" capital consistent with short-run efficiency in all periods in the case where only "green" capital is accumulated, thereby facilitating the analysis of Section 5.

2 Ramsey model

Assume that the technology is given by an increasing, strictly concave, and twice continuously differentiable production function $f : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $f(0) = 0$,

$\lim_{k\to\infty} f(k) = \bar{c} < \infty$ (implying that $\lim_{k\to\infty} f'(k) = 0$) and $\lim_{k\to 0} f'(k) = \infty$. Labor is constant and normalized to 1, implying that $f(k) = F(k,1)$ for all $k \in \mathbb{R}_+$, where $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a linearly homogeneous production function of capital and labor.

A consumption stream ${}_1c = (c_1, c_2, \dots) \geq 0$ is feasible given an initial capital stock $k > 0$ if there exists a capital stream ${}_0k = (k_0, k_1, k_2 \dots) \geq 0$ such that $k_0 = k$ and

$$c_t + k_t = k_{t-1} + f(k_{t-1}) \tag{1}$$

for all $t \in \mathbb{N}$.[1] Such a technology is referred to as a *Ramsey model*. Denote by

$$K(k) = \{{}_0k : k_0 = k \text{ and } 0 \leq k_t \leq k_{t-1} + f(k_{t-1}) \text{ for all } t \in \mathbb{N}\}$$

the set of feasible capital streams. Write $\mathcal{K} = \bigcup_{k\in\mathbb{R}_+} K(k)$.

Define

$$\mathbf{c}({}_0k) = (k_0 + f(k_0) - k_1, k_1 + f(k_1) - k_2, \dots, k_{t-1} + f(k_{t-1}) - k_t, \dots)$$

as the consumption stream that is associated with ${}_0k$, and denote by

$$C(k) = \{{}_1c : \text{there is } {}_0k \in K(k) \text{ s.t. } {}_1c = \mathbf{c}({}_0k)\}$$

the set of feasible consumption streams. Write $\mathcal{C} = \bigcup_{k\in\mathbb{R}_+} C(k)$. Say that ${}_1c \in C(k)$ is *efficient* if there is no ${}_1\tilde{c} \in C(k)$ such that ${}_1\tilde{c} > {}_1c$.

Map consumption c into utility by an increasing, strictly concave, and continuously differentiable utility function $u : \mathbb{R}_+ \to \mathbb{R}$, satisfying $u(0) = 0$ and $\lim_{c\to 0} u'(c) = \infty$.[2] Define $\mathbf{u}({}_1c) = (u(c_1), u(c_2), \dots, u(c_t) \dots)$ and denote by

$$U(k) = \{{}_1u : \text{there is } {}_1c \in C(k) \text{ s.t. } {}_1u = \mathbf{u}({}_1c)\}$$

the set of feasible consumption streams. Write $\mathcal{U} = \bigcup_{k\in\mathbb{R}_+} U(k)$.

Let labor be uniformly distributed over a continuum of dynasties i on the unit interval $[0,1]$. Let, at each time t, $\mathbf{k}_t : [0,1] \to \mathbb{R}_+$ be the profile of capital ownership across the dynasties and $\mathbf{c}_t : [0,1] \to \mathbb{R}_+$ the profile of consumption. Assume that $\mathbf{k}_t$ and $\mathbf{c}_t$ are Lebesque measurable and sum up to aggregate capital and consumption:

$$k_t = \int_0^1 \mathbf{k}_t(i)di\,, \quad c_t = \int_0^1 \mathbf{c}_t(i)di\,.$$

[1] For two vectors x and y, write $x \geq y$ if and only if $x_t \geq y_t$ for all t, $x > y$ if and only if $x \geq y$ and $x \neq y$, and $x \gg y$ if and only if $x_t > y_t$ for all t.

[2] The normalization $u(0) = 0$ will become important in Section 3 where the effect of atmospheric externalities enters multiplicatively.

At each time $t \in \mathbb{N}$, the budget constraint for each dynasty $i \in [0,1]$ is given by:

$$\mathbf{c}_t(i) + \mathbf{k}_t(i) = \mathbf{k}_{t-1}(i) + f\left(k_{t-1}\right) + f'\left(k_{t-1}\right)\left(\mathbf{k}_{t-1}(i) - k_{t-1}\right), \tag{2}$$

where the terms $f\left(k_{t-1}\right) + f'\left(k_{t-1}\right)\left(\mathbf{k}_{t-1}(i) - k_{t-1}\right)$ equal the value of the marginal products of the capital and labor held by dynasty i at time $t-1$.[3]

Assume that the present decision-maker of dynasty i cares for the descendants in the same dynasty, but not for descendants in other dynasties. In particular, the preferences of each dynasty is assumed to be represented by the *non-paternalistic altruistic* (NPA) *welfare function* $w_\alpha : \mathcal{U} \to \mathbb{R}$ defined by:

$$w_\alpha({}_1u) = (1-\alpha)\sum\nolimits_{t=0}^{\infty} \alpha^t u_{t+1}\,, \tag{3}$$

where $\alpha \in (0,1)$. It follows from the assumptions on f and u that w_α is well-defined on $\mathcal{U}$. The altruism is *non-paternalistic* in the terminology of Ray (1987), as the interests of generations t and $t+1$ are aligned for the stream beyond time $t+1$:

$$w_\alpha({}_tu) = (1-\alpha)u_t + \alpha w_\alpha({}_{t+1}u)\,.$$

Hence, we assume that the dynasties have the same utility function u and the same altruistic welfare function w_α. If, in addition, the profile of initial ownership to capital is assumed to be uniform, so that $\mathbf{k}_0(i) = k$ for all $i \in [0,1]$, the dynasties will behave in the same manner, implying that it holds for each $t \in \mathbb{N}$ that $\mathbf{k}_t(i) = k_t$ for all $i \in [0,1]$ so that eq. (2) simplifies to eq. (1). Since u is strictly concave, we may perform the analysis by considering a representative dynasty maximizing $w_\alpha({}_1u)$ over all ${}_1u \in U(k)$.

The NPA *welfare function* $v_\alpha : \mathcal{K} \to \mathbb{R}$ defined over capital streams is given by:

$$v_\alpha({}_0k) = w_\alpha(\mathbf{u}(\mathbf{c}({}_0k))) = (1-\alpha)\sum\nolimits_{t=0}^{\infty} \alpha^t u(k_t + f(k_t) - k_{t+1})\,,$$

with $\alpha \in (0,1)$. Say that ${}_0k \in K(k)$ is a NPA *optimum* if

$$v_\alpha({}_0k) \geq v_\alpha({}_0\tilde{k}) \quad \text{for all } {}_0\tilde{k} \in K(k)\,.$$

Before characterizing the optimum, we need to define the *modified golden rule*. Define $k_\infty : (0,1) \to \mathbb{R}_+$ by, for all $\alpha \in (0,1)$,

$$\alpha\left(1 + f'(k_\infty(\alpha))\right) = 1\,.$$

[3] To derive eq. (2), note that, for dynasty i, the value of the marginal products at time t equals:

$$F_1(k_t,1)\cdot \mathbf{k}_t(i) + F_2(k_t,l)\cdot 1 = F(k_t,1) + F_1(k_t,1)(\mathbf{k}_t(i) - k_t)$$

by using the property that F is homogeneous of degree 1. We obtain $f\left(k_t\right) + f'\left(k_t\right)\left(\mathbf{k}_t(i) - k_t\right)$ by substituting $f(k_t)$ for $F(k_t,1)$ and $f'(k_t)$ for $F_1(k_t,1)$.

It follows from the properties of f that k_∞ is well-defined, continuous, and increasing, with $\lim_{\alpha\to 0} k_\infty(\alpha) = 0$ and $\lim_{\alpha\to 1} k_\infty(\alpha) = \infty$. For given $\alpha \in (0,1)$, $k_\infty(\alpha)$ is the capital stock corresponding to the modified golden rule.

By Beals and Koopmans (1969), we have the following result:

Proposition 1 *There is a unique NPA optimum,* $\hat{k}(k)$*, with associated NPA optimal consumption stream* $\hat{c}(k) = \mathbf{c}(\hat{k}(k))$*. Furthermore,* $\hat{k}(k)$ *is monotone, with* $\lim_{t\to\infty} \hat{k}_t(k) = k_\infty(\alpha)$*, and* $\hat{c}(k)$ *is efficient, with* $\lim_{t\to\infty} \hat{c}_t(k) = f(k_\infty(\alpha))$*.*

The result that also $\hat{c}(k)$ is monotone is left out of Proposition 1 to make the statement parallel to that of Proposition 2, where the consumption stream need not be monotone. The following is a straight-forward implication of Proposition 1:

Corollary 1 *Let* $\alpha^* = 1/\left(1 + f'(k)\right)$*.*

(a) If $\alpha \in (\alpha^*, 1)$*, then the NPA optimum,* $\hat{k}(k)$*, is increasing.*

(b) If $\alpha = \alpha^*$*, then the NPA optimum,* $\hat{k}(k)$*, is constant.*

(c) If $\alpha \in (0, \alpha^*)$*, then the NPA optimum,* $\hat{k}(k)$*, is decreasing.*

The central question of this paper is whether non-paternalistic intergenerational altruism leads a normatively attractive outcome. Normative criteria – derived from axiomatic analysis and designed to balance the interests of present and future generations from an impartial perspective – can be used to evaluate the normative attractiveness of outcomes. In Appendix A, several such criteria are considered.

An fundamental criterion is *Suppes-Sen maximality* (Suppes, 1966; Sen, 1970), which combines *efficiency* in the sense of being sensitive to the interests of each generation (the Strong Pareto axiom) with *equal treatment* in the sense of being insensitive to all finite re-orderings of the consumption stream (the Finite Anonymity axiom).[4]

The *Calvo criterion* (Calvo, 1978) maximizes the infimum of the altruistic welfare of all generations. Calvo (1978) shows that there exists a unique and time-consistent Calvo optimum in the Ramsey model.

Appendix A also presents the *Sustainable discounted utilitarian* (SDU) criterion (Asheim and Mitra, 2010) and the *Rank-discounted utilitarian* (RDU) criterion (Zuber

[4]The Suppes-Sen criterion is incomplete as it compares only pairs that dominate each other after some finite re-ordering. It follows from the Lauwers–Zame (Lauwers, 2010; Zame, 2007) impossibility result (see Asheim, 2010, Section 3.2) that no explicitly describable complete criterion can satisfy both the Strong Pareto and Finite Anonymity axioms.

and Asheim, 2012). In these criteria, utility u_t as an indicator of wellbeing for each generation t is turned into generalized utility $v(u_t)$ by a continuous and increasing function $v : \mathbb{R}_+ \to \mathbb{R}$. Both SDU and RDU discount future generalized utility by a discount factor $\beta \in (0, 1)$, as long as the future is better off than the present, thereby trading-off current sacrifice and future gain. In this case, future's higher consumption is discounted because,

- at a higher level, added consumption contributes less to utility if the composite function $v \circ u$ is strictly concave, and
- being better off, its utility is assigned less weight since $\beta < 1$.

Hence, if consumption is perfectly correlated with time, these criteria work as the ordinary *Time-discounted utilitarian* (TDU) criterion which economists usually promote. The important difference is that, in the criteria of SDU and RDU, the future is discounted because priority is given to the worse off earlier generations.

However, if the present is better off than the future, then priority shifts to the future. In this case, future utility is not discounted, implying that zero relative weight is assigned to present consumption. The criteria of SDU and RDU can therefore capture the intuition that we should seek to assist future generations if they are worse off than us, while not having an unlimited obligation to save for their benefit if they turn out to be better off. Also, RDU is compatible with equal treatment of generations by being insensitive to all re-orderings of the consumption stream (formally, RDU satisfies the Strong Anonymity axiom).

The coming result, which is proven in Appendix A, makes the following main point: The NPA optimum – interpreted as the outcome implemented by maximizing dynasties with non-paternalistic altruism – is an outcome that can be endorsed by the Suppes-Sen, Calvo, SDU and RDU criteria if and only if there is sufficient altruism.

Theorem 1 *Let $\alpha^* = 1/(1 + f'(k))$, and assume that all generations are non-paternalistically altruistic according to eq. (3). Then the implemented capital stream, $\hat{k}(k)$, with associated consumption stream, $\hat{c}(k) = \mathbf{c}(\hat{k}(k))$, is*

- *Suppes-Sen maximal,*
- *Calvo optimal, and*
- *SDU optimal and RDU optimal with $\beta = \alpha$ and $v(u) = u$,*

if and only if $\alpha \geq \alpha^*$. *Furthermore, long-term consumption,* $\lim_{t\to\infty} \hat{c}_t(k) = f(k_\infty(\alpha))$, *is increasing as a function of* α *for* $\alpha > \alpha^*$.

Hence, in the context of the Ramsey model, sufficient non-paternalistic altruism leads to streams that can be justified by criteria that balance the interests of present and future generations from an impartial perspective (and, in the case of the RDU, treat generations equally).

3 Atmospheric externalities

Assume now that there are two kinds of capital, *brown* and *green* capital, where brown capital is more productive than green capital, but where the aggregate stock of brown capital produces negative externalities that reduce utility for everyone by the same multiplicative factor. We refer to such negative effects as *atmospheric externalities.*

In this alternative version of the Ramsey model, a consumption stream ${}_1c = (c_1, c_2, \dots) \geq 0$ is feasible given a pair of initial capital stocks $(b, g) > 0$ if there exist streams of brown capital ${}_0b = (b_0, b_1, b_2 \dots) \geq 0$ and green capital ${}_0g = (g_0, g_1, g_2 \dots) \geq 0$ such that $(b_0, g_0) = (b, g)$ and

$$c_t + b_t + g_t = b_{t-1} + g_{t-1} + f(b_{t-1} + (1-\gamma)g_{t-1}) \tag{4}$$

for all $t \in \mathbb{N}$, where $\gamma \in (0,1)$ measures to what extent green capital is less productive.[5]

Let, as before, labor be uniformly distributed over a continuum of dynasties i on the unit interval $[0,1]$. Let, at each time t, $\mathbf{b}_t : [0,1] \to \mathbb{R}_+$ and $\mathbf{g}_t : [0,1] \to \mathbb{R}_+$ be the profiles of capital ownership across the dynasties and $\mathbf{c}_t : [0,1] \to \mathbb{R}_+$ the profile of consumption. Assume that $\mathbf{b}_t$, $\mathbf{g}_t$ and $\mathbf{c}_t$ are Lebesque measurable and sum up to aggregate amounts of the two types of capital and aggregate consumption:

$$b_t = \int_0^1 \mathbf{b}_t(i)di\,, \quad g_t = \int_0^1 \mathbf{g}_t(i)di\,, \quad c_t = \int_0^1 \mathbf{c}_t(i)di\,.$$

At each time $t \in \mathbb{N}$, the budget constraint for each dynasty $i \in [0,1]$ is given by:

$$\begin{aligned} \mathbf{c}_t(i) + \mathbf{b}_t(i) + \mathbf{g}_t(i) = {} & \mathbf{b}_{t-1}(i) + \mathbf{g}_{t-1}(i) + f\left(b_{t-1} + (1-\gamma)g_{t-1}\right) \\ & + f'\left(b_{t-1} + (1-\gamma)g_{t-1}\right)\left(\mathbf{b}_{t-1}(i) - b_{t-1} + (1-\gamma)(\mathbf{g}_{t-1}(i) - g_{t-1})\right)\,, \end{aligned} \tag{5}$$

[5] By assuming that γ is constant we abstract from directed technological progress. Our conclusions are robust to assuming learning-by-doing in the accumulation of green capital, as dynasties will chose to accumulate brown capital only and no learning-by-doing will occur.

where the terms

$$f(b_{t-1} + (1-\gamma)g_{t-1}) + f'(b_{t-1} + (1-\gamma)g_{t-1})\,(\mathbf{b}_{t-1}(i) - b_{t-1} + (1-\gamma)(\mathbf{g}_{t-1}(i) - g_{t-1}))$$

equal the value of the marginal products of the capital and labor held by dynasty i at time $t-1$.

Assume from now on that the map from consumption to utility for dynasty i in generation t depends on the aggregate amount of brown capital accumulated by generation $t-1$:

$$a(b_{t-1})u(\mathbf{c}_t(i))\,,$$

where the continuous and decreasing function $a : \mathbb{R}_+ \to \mathbb{R}$, satisfying $a(0) = 1$ and $\lim_{b\to\infty} a(b) = 0$, captures the effect of the atmospheric externalities caused by brown capital. Refer to $a(b_{t-1})u(\mathbf{c}_t(i))$ as *adjusted utility*.

This is of course a simplified and crude representation of the effects of economic activity on climate change. In reality, greenhouse gas emissions are dependent on the use of brown capital; it is not that the stock of greenhouse gases is a function of the stock of brown capital. Under such alternative modeling, the concentration of greenhouse gases in the atmosphere may increase even if the stock of brown capital remains constant. One can therefore claim that our modeling of climate change is conservative in the sense of understating the long-term effects of brown capital accumulation.

In this alternative model each dynasty can invest either in brown $\mathbf{b}_t(i)$ or green $\mathbf{g}_t(i)$ capital. It is individually rational for each dynasty to invest in brown capital only, as this relaxes its budget constraint, while not influencing the aggregate stock of brown capital that adjusts its utility. However, efficient policies at a societal level requires that the superior productiveness of brown capital is weighted against the atmospheric externalities that such capital leads to. In this section we model and analyze decentralized decision-making, while turning to the opportunities for superior aggregate policies in the subsequent Section 4.

As in the previous section we assume that the dynasties have the same utility function u defined on consumption, and have same altruistic welfare function w_α defined by (3) on the set of feasible streams of adjusted utility. In particular, as before, we assume that the present decision-maker of dynasty i cares for the descendants in the same dynasty, but not for descendants in other dynasties. Moreover, we assume that the dynasties are affected by the atmospheric externalities caused by brown capital in the same manner, as captured by the function a.

If, in addition, the profile of initial ownership to capital is assumed to be uniform, so that $\mathbf{b}_0(i) = b$ and $\mathbf{g}_0(i) = g$ for all $i \in [0,1]$, the dynasties will behave in the same manner. This implies that it holds for each $t \in \mathbb{N}$ that $\mathbf{b}_t(i) = b_t$ and $\mathbf{g}_t(i) = 0$ for all $i \in [0,1]$ so that eq. (5) simplifies to eq. (4). Since u is strictly concave, we may perform the analysis by considering a representative dynasty.

The analysis is simplified by considering the case where $b > 0$ and $g = 0$, so that in each period, also period 0, there is a positive stock of brown capital only.[6] Under this assumption and taking into account that dynasties will choose to accumulate only brown capital, the set of (brown) capital streams and consumption streams as a function of the initial stock are the same as in the original Ramsey model analyzed in Section 2: $K(b)$ and $C(b)$, respectively.

The representative dynasty maximizes the NPA *welfare function* over all consumption streams ${}_1c \in C(b)$ while taking the atmospheric externalities caused by the stream of brown capital, ${}_0b \in K(b)$, as given. Let

$$\mathbf{u}({}_1c,\, {}_0b) = (a(b_0)u(c_1), a(b_1)u(c_2), \ldots, a(b_{t-1})u(c_t), \ldots)$$

denote the stream of adjusted utility. The NPA *welfare function* $v_\alpha : \mathcal{K} \times \mathcal{K}$ defined over capital streams in the model with atmospheric externalities is given by:

$$v_\alpha({}_0k,\, {}_0b) = w_\alpha(\mathbf{u}(\mathbf{c}({}_0k),\, {}_0b)) = (1-\alpha)\sum\nolimits_{t=0}^{\infty} \alpha^t a(b_t) u(k_t + f(k_t) - k_{t+1})\,,$$

with $\alpha \in (0,1)$, where $k_0 = b$ and k_t is brown capital held by the representative dynasty for $t \in \mathbb{N}$. Note that the representative dynasty takes ${}_0b$ as given when maximizing $v_\alpha({}_0k,\, {}_0b)$ over all ${}_0k \in K(b)$. However, in equilibrium, ${}_0k = {}_0b$, leading to the following definition: Say that ${}_0b \in K(b)$ is a NPA *equilibrium* if

$$v_\alpha({}_0b,\, {}_0b) \geq v_\alpha({}_0\tilde{k},\, {}_0b) \quad \text{for all } {}_0\tilde{k} \in K(b)\,.$$

The following result is established in Appendix B:

Proposition 2 *Assume $b > 0$ and $g = 0$. Then there is a unique NPA equilibrium, $b^*(b)$, with associated NPA equilibrium consumption stream $c^*(b) = \mathbf{c}(k^*(b))$. Furthermore, $b^*(b)$ is strictly monotone, with $\lim_{t\to\infty} b^*_t(b) = k_\infty(\alpha)$, and $c^*(b)$ is efficient, with $\lim_{t\to\infty} c^*_t(b) = f(k_\infty(\alpha))$.*

[6] The generalization of the analysis to the case where $g > 0$ is straightforward, but has a notational cost. Furthermore, in this case the result in Propostion 2 on the monotonicity of the stream of brown capital holds only beyond period 1, not between period 0 and 1.

As for Proposition 1, we obtain the following corollary:

Corollary 2 *Assume $b > 0$ and $g = 0$. Let $\alpha^* = 1/(1 + f'(b))$.*

(a) If $\alpha \in (\alpha^*, 1)$, then the NPA equilibrium, $b^*(b)$, is increasing.

(b) If $\alpha = \alpha^*$, then the NPA equilibrium, $b^*(b)$, is constant.

(c) If $\alpha \in (0, \alpha^*)$, then the NPA equilibrium, $b^*(b)$, is decreasing.

Proposition 2 sets the stage for the following comparative statics result, which is also proven in Appendix B:

Theorem 2 *Assume $b > 0$ and $g = 0$, and that all generations of each dynasty i are non-paternalistically altruistic according to (3) in the model with atmospheric externalities. Then long-term utility adjusted for atmospheric externalities,*

$$\lim_{t\to\infty} a(b_t^*(b))u(c_t^*(b)) = a(k_\infty(\alpha))u(f(k_\infty(\alpha))) ,$$

approaches 0 as $\alpha \to 1$.

Hence, with competitive altruism, caring more for future generations may undermine the future wellbeing.[7] In the next section we show that the resulting outcomes cannot be endorsed by any of the criteria considered in Appendix A, provided that altruism is sufficiently strong.

4 Efficient streams

It is a straightforward observation that the stream of adjusted utilities implemented by the dynasties through the NPA equilibrium is not efficient in the alternative model with atmospheric externalities if there is excessive competitive altruism. In particular, if the altruism parameter α exceeds a critical level $\tilde{\alpha}$, defined by

$$\tilde{\alpha} = \sup\{\alpha \in (0,1) : a(k_\infty(\alpha))u(f(k_\infty(\alpha))) \geq u(f((1-\gamma)k_\infty(\alpha)))\} , \qquad (6)$$

then there exists $\tau \in \mathbb{N}$ such that adjusted utility will be increased for all $t > \tau$ by switching from brown to green capital for all periods beyond τ, while keeping the stream

[7] Problems discussed in this section cannot be solved by instead assuming that the present representative of dynasty i cares not only for the descendants in the same dynasty, but for all descendants. Imposing this assumption gives rise to another and more serious public good problem.

unchanged for all $t \leq \tau$. This follows from the observation that the streams, $b_t^*(b)$ and $c_t^*(b)$, under decentralized decision-making approaches $k_\infty(\alpha)$ and $f(k_\infty(\alpha))$ as time goes to infinity. By the properties of a and k_∞, we have that the set in the brackets is non-empty and, furthermore, $\tilde{\alpha} \in (0, 1)$.

Since efficiency is a necessary condition for Suppes-Sen maximality, it follows that the NPA equilibrium is not Suppes-Sen maximal if the altruism parameter α exceeds the critical level $\tilde{\alpha}$, defined by (6).

It is not a trivial matter to analyze streams that are optimal according to the criteria considered in Appendix A, in the alternative model with atmospheric externalities. Instead, we establish the existence of an efficient maximin stream. We use this in turn to show how excessive competitive altruism leads to streams that are not optimal according to the Calvo, SDU and RDU criteria.

A maximin stream in the model with atmospheric externalities maximizes the infimum, taken over all time periods, of adjusted utility. If a stream with constant adjusted utility is efficient, so that adjusted utility cannot be increased in one period without being decreased in another period, then clearly the infimum of adjusted utility cannot be increased. Such an efficient stream with constant adjusted utility is an *efficient maximin stream*.

Efficient policies at a societal level require that the superior productiveness of brown capital is weighted against the atmospheric externalities that such capital leads to. In each period $t \in \mathbb{N}$, the mix of capital must be chosen so that the gain in production obtained by accumulating brown capital is weighted against the loss in adjusted utility, for given consumption, that such accumulation leads to.

To be specific, let k be the total capital that is accumulated in period t, to be used in period $t+1$ and let k' be the total capital that is accumulated in period $t+1$, to be used in period $t+2$. In period t, the asset management problem for given k and k' is to split k into brown capital, μk, and green capital, $(1-\mu)k$, so that

$$a(\mu k)u(k + f(\mu k + (1-\gamma)(1-\mu)k) - k')$$

is maximized over all $\mu \in [0, 1]$, where $k' \in [0, k + f(k))$ so that more than k' in period $t+1$ is attainable given k in period t with use of the more productive brown capital only. Obtaining such short-run efficiency is a necessary condition for an efficient stream of adjusted utility, because otherwise adjusted utility can be increased in period $t+1$ without being decreased in any other period.

Since the functions a, u and f are all continuous and $[0, 1]$ is a compact set, it follows that the expression is maximized for some μ and we may define:

$$\hat{u}(k,k') = \max_{\mu\in[0,1]} a(\mu k)u(k + f(\mu k + (1-\gamma)(1-\mu)k) - k')\,. \tag{7}$$

Let $b \geq 0$ and $g \geq 0$ be the initial stocks of brown and green capital, with $b+g > 0$. In Appendix C we show that the unique maximin path is found by choosing total capital $\bar{k}$ from period 1 and on such that

$$a(b)u(b + g + f(b + (1-\gamma)g) - \bar{k}) = \hat{u}(\bar{k},\bar{k})\,, \tag{8}$$

leading to following constant stream of adjusted utilities:

$$(a(b)u(b + g + f(b + (1-\gamma)g) - \bar{k}), \hat{u}(\bar{k},\bar{k}), \dots, \hat{u}(\bar{k},\bar{k}), \dots)\,.$$

As stated in the following proposition this is indeed an efficient maximin stream:

Proposition 3 *Assume $b \geq 0$, $g \geq 0$ and $b+g > 0$. There is a unique maximin stream in the model with atmospheric externalities. This stream is efficient and has constant and positive adjusted utility.*

Note that when this efficient maximin stream is followed, the NPA altruistic welfare is constant and equal to

$$w_\alpha(\hat{u}(\bar{k},\bar{k}), \hat{u}(\bar{k},\bar{k}), \dots) = (1-\alpha)\sum\nolimits_{t=0}^{\infty} \alpha^t \hat{u}(\bar{k},\bar{k}) = \hat{u}(\bar{k},\bar{k})\,.$$

Hence, when this efficient maximin stream is valued by the Calvo criterion, it yields $\hat{u}(\bar{k},\bar{k})$, while it cannot exceed $\lim_{t\to\infty} a(b_t^*(b))u(c_t^*(b)) = a(k_\infty(\alpha))u(f(k_\infty(\alpha)))$ in the case of decentralized decision-making.

When this efficient regular stream is valued by the SDU and RDU criteria, it yields $\upsilon(\hat{u}(\bar{k},\bar{k}))$ (where $\upsilon : \mathbb{R}_+ \to \mathbb{R}$ is the increasing and continuous function that turns adjusted utility into generalized utility), while it cannot exceed $\lim_{t\to\infty} \upsilon(a(b_t^*(b))u(c_t^*(b)))$ $= \upsilon(a(k_\infty(\alpha))u(f(k_\infty(\alpha))))$ in the case of decentralized decision-making.

Hence, it follows from Theorem 2 that the outcome of decentralized decision-making cannot be endorsed by any of these criteria, if the altruism parameter exceeds a critical level $\bar{\alpha}$, defined by

$$\bar{\alpha} = \sup\{\alpha \in (0,1) : a(k_\infty(\alpha))u(f(k_\infty(\alpha))) \geq \hat{u}(\bar{k},\bar{k})\}\,, \tag{9}$$

if the set in the brackets is non-empty and $\bar{\alpha} = 0$ otherwise. It follows from the properties of a and k_∞ that $\bar{\alpha} \in [0,1)$.

Theorem 3 *Assume $b > 0$ and $g = 0$, and that all generations of each dynasty i are non-paternalistically altruistic according to (3) in the model with atmospheric externalities. Then the implemented pair of capital streams, $b^*(b)$ and ${}_0g = (0, 0, \dots, 0, \dots)$, with associated consumption stream, $c^*(b) = \mathbf{c}(b^*(b))$, is*

- *not Suppes-Sen maximal*

if $\alpha > \tilde{\alpha}$, so that long-term utility, $\lim_{t\to\infty} a(b_t^*(b))u(c_t^*(b))$, adjusted for atmospheric externalities, would be increased by shifting from brown to green capital, and

- *not Calvo optimal and*
- *not SDU optimal nor RDU optimal for any $\beta \in (0, 1)$ and any increasing and continuous υ function,*

if $\alpha > \bar{\alpha}$, so that long-term utility, $\lim_{t\to\infty} a(b_t^*(b))u(c_t^*(b))$, adjusted for atmospheric externalities, is smaller than the maximin level, $\hat{u}(\bar{k}, \bar{k})$.

5 Collective action

A feasible policy is to require the dynasties to accumulate green capital only. This can be done by direct command, or by imposing a prohibitive tax on brown capital. If such policies are enforced, then the dynasties will accumulate green capital only and the atmospheric externalities will be avoided. Under the assumption that $b = 0$ and $g > 0$, so that also in period 0 there is a positive stock of green capital only, and replacing $f(g)$ with $\tilde{f}(g) = f((1 - \gamma)g)$ in the original Ramsey model of Section 2, the analysis becomes straight-forward.

Denote by

$$G(g) = \{{}_0g : g_0 = g \text{ and } 0 \leq g_t \leq g_{t-1} + \tilde{f}(g_{t-1}) \text{ for all } t \in \mathbb{N}\ \}$$

the set of feasible green capital streams subject to ${}_0b = (0, 0, \dots, 0, \dots)$, as a function of the initial stock of green capital g, and define

$$\tilde{\mathbf{c}}({}_0g) = (g_0 + \tilde{f}(g_0) - g_1, g_1 + \tilde{f}(g_1) - g_2, \dots, g_{t-1} + \tilde{f}(g_{k-1}) - g_t, \dots)$$

as the constrained consumption stream that is associated with ${}_0g$.

Say that ${}_0g \in G(g)$ is a constrained NPA *optimum* if

$$v_\alpha({}_0g) \geq v_\alpha({}_0\tilde{g}) \quad \text{for all } {}_0\tilde{g} \in G(g)\,.$$

Before characterizing the constrained optimum, we need to redefine the *modified golden rule*. Define $g_\infty : (0,1) \to \mathbb{R}_+$ by, for all $\alpha \in (0,1)$,

$$\alpha \Big(1 + (1-\gamma)\tilde{f}'(g_\infty(\alpha))\Big) = 1\,.$$

It follows from the properties of $\tilde{f}$ that g_∞ is well-defined, continuous, and increasing, with $\lim_{\alpha\to 0} g_\infty(\alpha) = 0$ and $\lim_{\alpha\to 1} g_\infty(\alpha) = \infty$. For given $\alpha \in (0,1)$, $g_\infty(\alpha)$ is the green capital stock corresponding to the modified golden rule.

As in Section 2, we have the following result by Beals and Koopmans (1969):

Proposition 4 *Assume $b = 0$ and $g > 0$ and impose the constraint that $b_t = 0$ for all $t \in \mathbb{N}$. Then there is a unique constrained NPA optimum, $\hat{g}(g)$, with associated constrained NPA optimal consumption stream $\hat{c}(g) = \tilde{\mathbf{c}}(\hat{g}(g))$. Furthermore, $\hat{g}(g)$ is monotone, with $\lim_{t\to\infty} \hat{g}_t(g) = g_\infty(\alpha)$, and $\hat{c}(g)$ is constrained efficient, with $\lim_{t\to\infty} \hat{c}_t(g) = \tilde{f}(g_\infty(\alpha))$.*

Furthermore, the constrained NPA optimum, $\hat{g}(g)$, is increasing if $\alpha \in (\alpha^*, 1)$, constant if $\alpha = \alpha^*$, and decreasing if $\alpha \in (0, \alpha^*)$, where $\alpha^* = 1/\Big(1 + (1-\gamma)\tilde{f}'(g)\Big)$.

Such policies need not be efficient in terms of adjusted utility if short-run efficiency (as defined by problem (7)) requires a mixture of brown and green capital with a positive amount of brown capital. This will be the case if γ is large, so that brown capital is much more productive than green capital, and a is only moderately decreasing, so that the effects of the atmospheric externalities are small.

In the opposite case, where γ is small and a decreases steeply with the amount of brown capital, then the efficiently loss of such policies may be small or even non-existent. Indeed, the following result, proven in Appendix D, shows under what condition the efficiency loss in terms of adjusted utility is non-existent in the special case where the function capturing the effects of the atmospheric externalities is given by $a(\mu k) = e^{-\rho\mu k}$, with $\rho > 0$. The theorem demonstrates that, for any $\rho > 0$, there is a $\gamma > 0$ small enough to ensure that the efficiently loss in terms of adjusted utility associated with accumulating green capital only is zero.

Theorem 4 *Assume $b = 0$ and $g > 0$. Also, assume $a(\mu k) = e^{-\rho\mu k}$, with $\rho > 0$. Then the constrained NPA optimum, $\hat{g}(g)$, with associated constrained NPA optimal consumption stream $\hat{c}(g) = \tilde{\mathbf{c}}(\hat{g}(g))$, is short-run efficient in all periods if and only if $\gamma \in (0,\ \gamma(\rho)]$, where $\gamma(\rho) > 0$ is an increasing function of ρ, with $\lim_{\rho\to 0} \gamma(\rho) = 0$.*

Furthermore, with a small γ, the model with such policies will lead to behavior and outcomes that are close to those obtained in the ordinary Ramsey model. As we have shown in Sections 3 and 4, the behavior and outcomes will be very different and adverse without such collective action, even in the case where γ is small.

6 Concluding remarks

In this paper we have shown how competitive intergenerational altruism may seriously undermine long-term wellbeing in a setting that is designed to model the intergenerational game that will be played by dynasties trying to get ahead in a world threatened by climate change.

Its main insight is that stronger concern for future generations need not be an effective substitute for collective climate action. If the helping hand that present members of each dynasty lend to its future members accelerates climate change, then increased uncoordinated transfers towards the future might aggravate the conditions under which future generations will live.

Appendix A

Definitions of Suppes-Sen maximality, Calvo optimality, TDU optimality, SDU optimality, and RDU optimality

Let U be a set of feasible utility streams for a given initial condition (k in the model of Section 2 and (b, g) in the model of Section 3). That is, the utility stream ${}_1u = (u_1, u_2, \dots)$ is feasible if and only if ${}_1u \in U$. Let $\mathcal{U}$ be the union of U when varying the initial conditions in both models.

Suppes-Sen maximality (see Asheim, Buchholz and Tungodden, 2001, for a presentation and analysis of this criterion).

Say that the utility stream ${}_1u \in U$ is *Suppes-Sen maximal* given the set of feasible utility streams U if there do not exist ${}_1\tilde{u} \in U$ and a finite permutation ${}_1u'$ of ${}_1u$ such that ${}_1\tilde{u} > {}_1u'$.

The *Calvo criterion* (see Calvo, 1978, for a presentation and analysis of this criterion) evaluates streams according to the infimum of the non-paternalistic altruistic welfare. Hence, the Calvo welfare function is defined on $\mathcal{U}$ as follows:

$$\inf_{t\geq 1} w_\alpha({}_t u)\,.$$

Say that ${}_1u \in U$ is a *Calvo optimum* if

$$\inf_{t\geq 1} w_\alpha({}_t u) \geq \inf_{t\geq 1} w_\alpha({}_t \tilde{u}) \quad \text{for all } {}_1\tilde{u} \in U\,.$$

Let $\upsilon : \mathbb{R}_+ \to \mathbb{R}$ be a continuous and increasing function that maps utility into *generalized utility.*

Time-discounted generalized utilitarianism (TDU). Define the TDU welfare function $W_\beta^T : \mathcal{U} \to \mathbb{R}$ for $\beta \in (0,1)$ as follows:

$$W_\beta^T({}_1u) = (1-\beta)\sum\nolimits_{t=0}^{\infty} \beta^t \upsilon(u_{t+1})\,.$$

Say that ${}_1u \in U$ is a *time-discounted generalized utilitarian* (TDU) *optimum* given the set of feasible utility streams U if

$$W_\beta^T({}_1u) \geq W_\beta^T({}_1\tilde{u}) \quad \text{for all } {}_1\tilde{u} \in U\,.$$

Sustainable discounted generalized utilitarianism (SDU) (see Asheim and Mitra, 2010, for a presentation and analysis of this criterion, including an axiomatization). Under SDU, the future is discounted if and only if the future is better off than the present. Define the SDU welfare function $W_\beta^S : \mathcal{U} \to \mathbb{R}$ for $\beta \in (0,1)$ as follows: $W_\beta^S({}_1u) = \lim_{\tau\to\infty} z(1,\tau)$, where $z(1,\tau)$ is constructed as follows:

$$\begin{aligned}
&z(\tau,\tau) = W_\beta^T({}_\tau u)\\
&z(\tau-1,\tau) = \min\{(1-\beta)\upsilon(u_{\tau-1}) + \beta z(\tau,\tau),\ z(\tau,\tau)\}\\
&\ldots\\
&z(1,\tau) = \min\{(1-\beta)\upsilon(u_1) + \beta z(2,\tau),\ z(2,\tau)\}\,.
\end{aligned}$$

Say that ${}_1u \in U$ is a *sustainable generalized discounted utilitarian* (SDU) *optimum* given the set of feasible utility streams U if

$$W_\beta^S({}_1u) \geq W_\beta^S({}_1\tilde{u}) \quad \text{for all } {}_1\tilde{u} \in U\,.$$[8]

[8]Asheim and Mitra (2010, Section 2) use the construction presented here to establish the existence of a SDU welfare function, while using their requirements (W.1)–(W.4) as the primitive definition.

Rank-discounted generalized utilitarianism (RDU) (see Zuber and Asheim, 2012, for a presentation and analysis of this criterion, including an axiomatization). Under RDU, streams are first reordered into a non-decreasing stream, so that discounting becomes according to rank, not according to time. The definition takes into account that streams like $(1, 0, 0, 0, \dots)$, with elements of infinite rank, cannot be reordered into a non-decreasing stream. Therefore, let $\ell({}_1u)$ denote lim inf of ${}_1u$ if it exists (set $\ell({}_1u) = \infty$ otherwise), and let $L({}_1u) := \{t \in \mathbb{N} \mid u_t < \ell({}_1u)\}$. If $|L({}_1u)| = \infty$, then let ${}_{[1]}u = (u_{[1]}, u_{[2]}, \dots)$ be a non-decreasing reordering of all elements u_t with $t \in L({}_1u)$ (so that $u_{[r]} \leq u_{[r+1]}$ for all ranks $r \in \mathbb{N}$). If $|L({}_1u)| < \infty$, then let $(u_{[1]}, u_{[2]}, \dots, u_{[|L({}_1u)|]})$ be a non-decreasing reordering of all elements u_t with $t \in L({}_1u)$ (so that $u_{[r]} \leq u_{[r+1]}$ for all ranks $r \in \{1, \dots, |L({}_1u)|\}$), and set $u_{[r]} = \ell({}_1u)$ for all $r > |L({}_1u)|$.

Define the RDU welfare function $W^R_\beta : \mathcal{U} \to \mathbb{R}$ for $\beta \in (0, 1)$ as follows:

$$W^R_\beta({}_1u) = W^T_\beta({}_{[1]}u)\,.$$

Say that ${}_1u \in U$ is a *rank-discounted generalized utilitarian* (RDU) *optimum* given the set of feasible utility streams U if

$$W^R_\beta({}_1u) \geq W^R_\beta({}_1\tilde{u}) \quad \text{for all } {}_1\tilde{u} \in U\,.$$

It follows by the assumptions made in Sections 2 and 3 that the generalized utilitarian welfare functions W^T_β, W^S_β and W^R_β are well-defined on $\mathcal{U}$ if the function υ that turns utility into generalized utility is (weakly) concave.

Proof of Theorem 1

The result on Suppes-Sen maximality follows from Asheim, Buchholz and Tungodden (2001, Propositions 5 & 6 and Example 2). The result on Calvo optimality follows from Calvo (1978, Proposition 2). The result on SDU optimality follows from Asheim and Mitra (2010, Theorem 2). The result on RDU optimality follows from Zuber and Asheim (2012, Proposition 10).

Appendix B

Proof of Proposition 2

Throughout this proof, we assume that ${}_0g = 0$ so that the set of brown capital streams for the representative dynasty is given by $K(b)$. In particular, the transformation between the stock of brown capital at time $t-1$, denoted k_{t-1}, and the stock of brown capital at time t, denoted k_t, provided that c_t is consumed at time t, is given by eq. (1).

A necessary condition for the maximization by ${}_0k \in K(b)$ of $v_\alpha({}_0\tilde{k}, {}_0b)$ over all ${}_0\tilde{k} \in K(b)$ is short-run optimality in the following sense: For all $t \in \mathbb{N}$, $k_{t-1} > 0$ and $k_{t+1} \in [0,\, k_{t-1} + f(k_{t-1}) + f(k_{t-1} + f(k_{t-1}))]$, k_t maximizes

$$a(b_{t-1})u(k_{t-1} + f(k_{t-1}) - \tilde{k}_t) + \alpha a(b_t)u(\tilde{k}_t + f(\tilde{k}_t) - k_{t+1})$$

over all $\tilde{k}_t \le k_{t-1} + f(k_{t-1})$ satisfying $\tilde{k}_t + f(\tilde{k}_t) \ge k_{t+1}$. Because otherwise, a reallocation of consumption between periods t and $t+1$ could have increased $v_\alpha({}_0k, {}_0b)$ without affecting consumption in any other period. This yields the first-order condition

$$a(b_{t-1})u'(c_t) = \alpha a(b_t)u'(c_{t+1})(1 + f'(k_t))\,, \tag{B1}$$

where $c_t = k_{t-1} + f(k_{t-1}) - k_t > 0$ and $c_{t+1} = k_t + f(k_t) - k_{t+1} > 0$ by the properties of u. Eq. (B1) is the key to proving the following useful result.

Lemma 1 *Assume that ${}_0k \in K(b)$ maximizes $v_\alpha({}_0\tilde{k}, {}_0b)$ over all ${}_0\tilde{k} \in K(b)$ and ${}_0k' \in K(b')$ maximizes $v_\alpha({}_0\tilde{k}, {}_0b')$ over all ${}_0\tilde{k} \in K(b')$. If, for some $t \in \mathbb{N}$, $k_t = b_t \le k'_t = b'_t$ and $a(b_{t-1})u'(c_t) < a(b'_{t-1})u'(c'_t)$, then $a(b_t)u'(c_{t+1}) < a(b'_t)u'(c'_{t+1})$, $c_{t+1} > c'_{t+1}$ and $k_{t+1} < k'_{t+1}$.*

Proof. It follows from (B1), $k_t = b_t \le k'_t = b'_t$, $a(b_{t-1})u'(c_t) < a(b'_{t-1})u'(c'_t)$, the monotonicity of a and the strict concavity of f that

$$\begin{aligned} a(b_t)u'(c_{t+1})(1 + f'(k_t)) &< a(b'_t)u'(c'_{t+1})(1 + f'(k'_t)) \\ &\le a(b'_t)u'(c'_{t+1})(1 + f'(k_t)) \le a(b_t)u'(c'_{t+1})(1 + f'(k_t))\,. \end{aligned}$$

Hence, $a(b_t)u'(c_{t+1}) < a(b'_t)u'(c'_{t+1})$ and $u'(c_{t+1}) < u'(c'_{t+1})$. By the strict concavity of u, we have that $c_{t+1} > c'_{t+1}$. Since $k_t \le k'_t$, it follows from eq. (1) and the monotonicity of f that $c_{t+1} + k_{t+1} = k_t + f(k_t) \le k'_t + f(k'_t) = c'_{t+1} + k'_{t+1}$, which combined with $c_{t+1} > c'_{t+1}$ implies that $k_{t+1} < k'_{t+1}$. ■

The proof of Proposition 2 modifies Beals and Koopmans's (1969) proof of their Theorem 1, and will be divided into nine steps.

Step 1: For every $b \in \mathbb{R}_+$, the set of streams $K(b)$ is compact. Set $k^*_0 = b$ and define ${}_1k^*$ recursively by $k^*_t = k^*_{t-1} + f(k^*_{t-1})$ for $t \in \mathbb{N}$. So ${}_0k^*$ is the stream with no consumption. $K(b)$ is contained in the product of the closed intervals $[0, k^*_t]$. The latter set is compact and $K(b)$ is a closed subset.

Step 2: For given ${}_0b$, $v_\alpha({}_0k,\, {}_0b)$ *is continuous and strictly quasi-concave as a function of* ${}_0k$. This follows from the properties of u.

Step 3: For given ${}_0b$, *there exists a unique NPA optimal stream* $\hat{k}(b,\, {}_0b)$ *and the associated consumption stream* $\hat{c}(b,\, {}_0b) = \mathbf{c}(\hat{k}(b,\, {}_0b))$ *is efficient.* Existence follows from Weierstrass' extreme value theorem and uniqueness from the strict quasi-concavity of $v_\alpha({}_0k,\, {}_0b)$ as a function of ${}_0k$. The optimality of $\hat{k}(b,\, {}_0b)$ implies that $\hat{c}(b,\, {}_0b)$ is efficient.

Step 4: If $b > 0$, *then* $\hat{k}(b,\, {}_0b) \gg 0$ *and* $\hat{c}(b,\, {}_0b) \gg 0$. If $k_{t-1} > 0$, then short-optimality (cf. (B1)) implies that $c_t = k_{t-1}+f(k_{t-1})-k_t > 0$, $c_{t+1} = k_t+f(k_t)-k_{t+1} > 0$ and $k_t > 0$. Since $k_0 = b > 0$, the result follow by induction.

Step 5: $\hat{k}(b,\, {}_0b)$ *is continuous in* ${}_0b$. Write $\hat{v}_\alpha(b,\, {}_0b) = v_\alpha(\hat{k}(b,\, {}_0b),\, {}_0b)$.

We first establish that $\hat{v}_\alpha(b,\, {}_0b)$ is strictly decreasing in b_t and continuous from the right in b_t. Let $\tilde{b} > b_t$ and let ${}_0b'$ be given by $b'_t = \tilde{b}$ and $b'_\tau = b_\tau$ for $\tau \neq t$. Then

$$\hat{v}_\alpha(b,\, {}_0b) \geq v_\alpha(\hat{k}(b,\, {}_0b'),\, {}_0b) > v_\alpha(\hat{k}(b,\, {}_0b'),\, {}_0b') = \hat{v}_\alpha(b,\, {}_0b')$$

since going from ${}_0b$ to ${}_0b'$ does not influence $K(b)$ but decreases adjusted utility derived from $\hat{k}(b,\, {}_0b')$ at time $t+1$ from $a(b_t)u'(\hat{c}_{t+1}(b,\, {}_0b'))$ to $a(b'_t)u'(\hat{c}_{t+1}(b,\, {}_0b'))$. Moreover,

$$\hat{v}_\alpha(b,\, {}_0b) > \hat{v}_\alpha(b,\, {}_0b') \geq v_\alpha(\hat{k}(b,\, {}_0b),\, {}_0b')\,,$$

using the property that $\hat{v}_\alpha(b,\, {}_0b)$ is strictly decreasing in b_t. Continuity from the right follows since v_α is continuous in b_t.

Suppose ${}_0b^{(n)} \to {}_0b$. Since $K(b)$ is compact, $\hat{k}(b,\, {}_0b^{(n)})$ has at least one convergent subsequence. It suffices to show that any convergent subsequence must converge to $\hat{k}(b,\, {}_0b)$. Dropping terms and renumbering, we may assume that $\hat{k}(b,\, {}_0b^{(n)})$ itself converges to ${}_0k$. By continuity of v_α, we have that

$$v_\alpha({}_0k,\, {}_0b) = \lim_{n\to\infty} v_\alpha(\hat{k}(b,\, {}_0b^{(n)}),\, {}_0b^{(n)}) = \lim_{n\to\infty} \hat{v}_\alpha(b,\, {}_0b^{(n)}) \geq \hat{v}_\alpha(b,\, {}_0b) \geq v_\alpha({}_0k,\, {}_0b)\,,$$

using fact that $\hat{v}_\alpha(b,\, {}_0b)$ is strictly decreasing in b_t and continuous from the right in b_t. Therefore, $v_\alpha({}_0k,\, {}_0b) = \hat{v}_\alpha(b,\, {}_0b) = v_\alpha(\hat{k}(b,\, {}_0b),\, {}_0b)$, and by the uniqueness of $\hat{k}(b,\, {}_0b)$, ${}_0k = \hat{k}(b,\, {}_0b)$. This proves that $\hat{k}(b,\, {}_0b)$ is continuous in ${}_0b$.

Step 6: For given ${}_0b$, *there exists a unique NPA equilibrium stream* $b^*(b)$ *where associated consumption stream* $c^*(b) = \mathbf{c}(b^*(b))$ *is efficient.* By steps 1 and 5, Brouwer's fixed point theorem implies that there exists ${}_0\hat{b} \in K(b)$ such that ${}_0\hat{b} = \hat{k}(b,\, {}_0\hat{b})$.

To show that ${}_0\hat{b}$ is unique, let ${}_0\hat{b}$, ${}_0\hat{b}' \in K(b)$ satisfy ${}_0\hat{b} = \hat{k}(b, {}_0\hat{b})$ and ${}_0\hat{b}' = \hat{k}(b, {}_0\hat{b}')$. Note that $\hat{b}_0 = \hat{b}'_0 = b$. Suppose that there is a first time $t \in \mathbb{N}$ such that $\hat{b}_t < \hat{b}'_t$. Then, by eq. (1), $\hat{c}_t(b, {}_0\hat{b}) > \hat{c}_t(b, {}_0\hat{b}')$ and $a(\hat{b}_{t-1})u'(\hat{c}_t(b, {}_0\hat{b})) < a(\hat{b}'_{t-1})u'(\hat{c}_t(b, {}_0\hat{b}'))$, keeping in mind that $\hat{b}_{t-1} = \hat{b}'_{t-1}$. By repeated use of Lemma 1, $\hat{c}_\tau(b, {}_0\hat{b}) > \hat{c}_\tau(b, {}_0\hat{b}')$ for all $\tau \geq t$. Since $\hat{c}_\tau(b, {}_0\hat{b}) = \hat{c}_\tau(b, {}_0\hat{b}')$ for all $\tau = 1, \ldots, t-1$ if $t > 1$, it follows that $\hat{c}_t(b, {}_0\hat{b}')$ is inefficient. By Step 3, this leads to a contradiction.

The efficiency of $c^*(b)$ follows from Step 3.

Step 7: If $b > b' > 0$, then $b^*(b) \gg b^*(b')$. Let $b > b' > 0$, and suppose that $b^*_1(b) \leq b^*_1(b')$. Then, by eq. (1) and $b^*_0(b) = b > b' = b^*_0(b')$, it follows that $c^*_1(b) > c^*_1(b')$ and $a(b^*_0(b))u'(c^*_1(b)) < a(b^*_0(b'))u'(c^*_1(b'))$ by the properties of a and u. By repeated use of Lemma 1, $c^*_t(b) > c^*_t(b')$ and $b^*_t(b) < b^*_t(b')$ for all $t \geq 2$.

If $b^*_1(b) = b^*_1(b') = b_1$, then this would contradict that $b^*(b_1)$ is unique.

If $b^*_1(b) < b^*_1(b')$, then the conclusion that $c^*(b^*_1(b))$ strictly dominates $c^*(b^*_1(b'))$ contradicts the efficiency of $c^*(b^*_1(b'))$.

Therefore, $b^*_1(b) > b^*_1(b')$. Repeating the argument implies that $b^*(b) \gg b^*(b')$.

Step 8: $b^*(b)$ is strictly monotone in time. Assume $b = b^*_0(b) < b^*_1(b)$. Then, by Step 7, $b^*_{t-1}(b) < b^*_t(b)$ for all $t \in \mathbb{N}$. Likewise for $b = b^*_0(b) = b^*_1(b)$ and $b = b^*_0(b) > b^*_1(b)$.

Step 9: $b^*(b)$ converges to $k_\infty(\alpha)$. By Step 8, $b^*(b)$ is strictly monotone in time and there are three possible limits for $b^*_t(b)$ as $t \to \infty$:

(i) $b^*(b)$ is decreasing with $\lim_{t\to\infty} b^*_t(b) = 0$,

(ii) $b^*(b)$ is increasing with $\lim_{t\to\infty} b^*_t(b) = \infty$,

(iii) $b^*(b)$ is strictly monotone with $\lim_{t\to\infty} b^*_t(b) \in (0, \infty)$.

In case (i), the properties of f implies that there is $t \in \mathbb{N}$ such that $\alpha(1+f'(b^*_t(b))) > 1$. It follows from (B1) that, for $\tau > t$, $a(b^*_{\tau-1}(b))u'(c^*_\tau(b))$ is decreasing and $a(b^*_{\tau-1}(b))$ is increasing. Hence, $u'(c^*_\tau(b))$ is decreasing and, by the strict concavity of u, $c^*_\tau(b)$ is increasing. This is infeasible if $\lim_{t\to\infty} b^*_t(b) = 0$ by eq. (1) and the properties of f.

In case (ii), the properties of f implies that there is $t \in \mathbb{N}$ such that $\alpha(1+f'(b^*_t(b))) < 1$. It follows from (B1) that, for $\tau > t$, $a(b^*_{\tau-1}(b))u'(c^*_\tau(b))$ is increasing and $a(b^*_{\tau-1}(b))$ is decreasing. Hence, $u'(c^*_\tau(b))$ is increasing and, by the strict concavity of u, $c^*_\tau(b)$ is decreasing. Consider the feasible stream in $K(b^*_t(b))$ with $k_\tau = b^*_t(b)$ for all $\tau \geq t$. It follows from eq. (1) that $c_\tau = f(b^*_t(b)) > c^*_{t+1}(b) \geq c^*_\tau(b)$ for $\tau > t$, implying that $c^*(b)$ is inefficient. This contradicts Step 3.

Therefore, only case (iii) remains, in which case $\lim_{t\to\infty} b_t^*(b) = k_\infty \in (0,\infty)$ and $\lim_{t\to\infty} c_t^*(b) = f(k_\infty)$. By eq. (B1), and the properties of a, u and f,

$$a(k_\infty)u'(f(k_\infty)) = \alpha a(k_\infty)u'(f(k_\infty))(1+f'(k_\infty))\,.$$

This implies that $\alpha(1+f(k_\infty)) = 1$ and $k_\infty = k_\infty(\alpha)$.

Proof of Theorem 2

By Proposition 2, $\lim_{t\to\infty} b_t^*(b) = k_\infty(\alpha)$ and

$$\lim_{t\to\infty} a(b_t^*(b))u(c_t^*(b)) = a(k_\infty(\alpha))u(f(k_\infty(\alpha)))\,.$$

As $\alpha \to 1$, it follows from the properties a and f that $k_\infty(\alpha) \to \infty$, $a(k_\infty(\alpha)) \to 0$, and $u(f(k_\infty(\alpha))) \to u(\bar{c})$. Hence, as $\alpha \to 1$, $a(k_\infty(\alpha))u(f(k_\infty(\alpha))) \to 0 \cdot u(\bar{c}) = 0$.

Appendix C

Proof of Proposition 3

It follows from envelope theorem that the partial derivative of $\hat{u}(k,k')$ with respect to the first variable satisfies:

$$\hat{u}_1(k,k') \geq a(\mu k)u'(c)\left(1+f'(\mu k+(1-\gamma)(1-\mu)k)(1-\gamma)\right) > 0\,,$$

and with equality if $\mu < 1$, where $c = k + f(\mu k + (1-\gamma)(1-\mu)k) - k'$. The partial derivative with respect to the second variable is given by:

$$\hat{u}_2(k,k') = -a(\mu k)u'(c) < 0\,.$$

Furthermore, if total capital is constant between the two periods, so that $k = k'$, then

$$\tfrac{d}{dk}\hat{u}(k,k) \geq a(\mu k)u'(f(\mu k+(1-\gamma)(1-\mu)k))f'(\mu k+(1-\gamma)(1-\mu)k)(1-\gamma) > 0\,.$$

This means that $\hat{u}(k,k)$ is an increasing function of k with $\hat{u}(0,0) = 0$. The function

$$\tilde{u}(k) = a(b)u(b+g+f(b+(1-\gamma)g)-k)$$

is a decreasing function of k with $\tilde{u}(k) > 0$ if $k = 0$ and $\tilde{u}(k) = 0$ if $k = b+g+f(b+(1-\gamma)g)$. It follows that $\bar{k}$ defined by eq. (8) — that is, $\tilde{u}(\bar{k}) = \hat{u}(\bar{k},\bar{k})$ — is uniquely determined and in the interval $(0,\, b+g+f(b+(1-\gamma)g))$.

The proof is concluded by showing that the stream

$$(\tilde{u}(\bar{k}), \hat{u}(\bar{k}, \bar{k}), \ldots, \hat{u}(\bar{k}, \bar{k}), \ldots) \tag{C1}$$

is a *regular maximin* stream in the sense of Dixit et al. (1980). For this purpose, define present-value prices of adjusted utility and capital as follows: Set $\lambda_1 = 1$ and define p_1 by $p_1 = -\lambda_1 \tilde{u}'(k)$. Furthermore, for all $t \in \mathbb{N}$, define λ_{t+1} and p_{t+1} inductively by:

$$\lambda_{t+1} \hat{u}_1(\bar{k}, \bar{k}) = p_t \text{ and } p_{t+1} = -\lambda_{t+1} \hat{u}_2(\bar{k}, \bar{k})\,.$$

It follows from these definitions that

$$\lambda_1 \tilde{u}(\bar{k}) + p_1 \bar{k} \geq \lambda_1 \tilde{u}(k) + p_1 k \tag{C2}$$

for all $k \geq 0$. Moreover, for $t \in \mathbb{N}$:

$$\lambda_{t+1} \hat{u}(\bar{k}, \bar{k}) + p_{t+1} \bar{k} - p_t \bar{k} \geq \lambda_{t+1} \hat{u}(k, k') + p_{t+1} k' - p_t k \tag{C3}$$

for all $(k, k') \in \mathbb{R}^2_+$. Also, for $t \geq 2$:

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{p_t}{p_{t+1}} = -\frac{\hat{u}_1(\bar{k}, \bar{k})}{\hat{u}_2(\bar{k}, \bar{k})} \geq 1 + f'(\bar{\mu}\bar{k} + (1-\gamma)(1-\bar{\mu})\bar{k})(1-\gamma) > 1\,,$$

where $\bar{\mu} = \operatorname{argmax}_{\mu \in [0,1]} a(\mu \bar{k}) u(f(\mu \bar{k} + (1-\gamma)(1-\mu)\bar{k}))$. Hence, $\lim_{t\to\infty} \lambda_t = 0$ and $\lim_{t\to\infty} p_t = 0$ and $\sum_{t=1}^{T} \lambda_{t+1}$ is convergent. Therefore,

$$\lambda_1 \left(\tilde{u}(\bar{k}) - \tilde{u}(\tilde{k}_1)\right) + \sum_{t=1}^{T} \lambda_{t+1} \left(u(\bar{k}, \bar{k}) - u(\tilde{k}_t, \tilde{k}_{t+1})\right) \geq p_{T+1}(\tilde{k}_{T+1} - \bar{k})$$

by (C2) and (C3) for any alternative stream ${}_1\tilde{k} = (\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_t, \ldots)$ of total capital. Since $p_t > 0$ and $\tilde{k}_t \geq 0$ for all $t \in \mathbb{N}$ and $\lim_{t\to\infty} p_t = 0$, we obtain:

$$\liminf_{T\to\infty} \left[\lambda_1 \left(\tilde{u}(\bar{k}) - \tilde{u}(\tilde{k})\right) + \sum_{t=1}^{T} \lambda_{t+1} \left(u(\bar{k}, \bar{k}) - u(\tilde{k}_t, \tilde{k}_{t+1})\right)\right] \geq 0$$

Since $\lambda_t > 0$ for all $t \in \mathbb{N}$, this means that it is not possible to increase adjusted utility at one time without decreasing adjusted utility at some other time. The constant adjusted utility stream defined by (C1) is therefore efficient and thus a maximin stream.

Appendix D

Proof of Theorem 4

Consider the period t asset management problem (7). Recall that, for given total capital accumulated in period t, k, and total capital accumulated in period $t+1$, k', the problem is to split k into brown capital, μk, and green capital, $(1-\mu)k$, so as to achieve short-run efficiency. Assume that $a(\mu k) = e^{-\rho\mu k}$, with $\rho > 0$, and that $k' \in [0, k + f(k))$ so that more than k' in period $t+1$ is attainable given k in period t with use of the more productive brown capital only.

With $a(\mu k) = e^{-\rho\mu k}$, the period t asset management problem (7) becomes:

$$\max_{\mu} e^{-\rho\mu k}\ u(k + f(\mu k + (1-\gamma)(1-\mu)k) - k') \text{ subject to } \begin{cases} -\mu \leq 0. \\ \mu \leq 1. \end{cases} \tag{D1}$$

The corresponding Lagrangian is:

$$L(\mu) = e^{-\rho\mu k}\ u(c) - \lambda_1(-\mu) - \lambda_2(\mu - 1),$$

where $c = k + f(\mu k + (1-\gamma)(1-\mu)k) - k'$.

The optimal value of μ, $\hat{\mu}$, satisfies the first-order condition:

$$\underbrace{-\rho k\ e^{-\rho\hat{\mu}k}\ u(c) + e^{-\rho\hat{\mu}k}\ u'(c)\ f'(\hat{\mu}k + (1-\gamma)(1-\hat{\mu})k)\ \gamma k}_{= s(\gamma|\hat{\mu})} + \lambda_1 - \lambda_2 = 0\,, \tag{D2}$$

as well as the following complementary slackness conditions:

$$\lambda_1 \geq 0, \text{ with } \lambda_1[-\hat{\mu}] = 0.$$

$$\lambda_2 \geq 0, \text{ with } \lambda_2[\hat{\mu} - 1] = 0.$$

It follows from the properties of u and f that λ_1 is a continuous function of γ.

Figure 1 shows the function $s(\gamma|\hat{\mu})$. Recall that we, for any $\rho > 0$, seek the highest possible $\gamma(\rho)$ ensuring that problem (D1) is solved for the boundary solution $\hat{\mu} = 0$ rather than an interior solution or the boundary solution $\hat{\mu} = 1$. Therefore $\lambda_2 = 0$.

Set $\hat{\mu} = 0$ and $\lambda_2 = 0$ in eq. (D2), and let λ_1 approach 0:

$$\begin{aligned} &\lim_{\lambda_1 \to 0} \left[-\rho k\ u(c) + u'(c)\ f'((1-\gamma)k)\ \gamma k + \lambda_1\right] \\ &\qquad = -\rho k\ u(c) + u'(c)\ f'((1-\gamma)k)\ \gamma k = 0. \end{aligned}$$

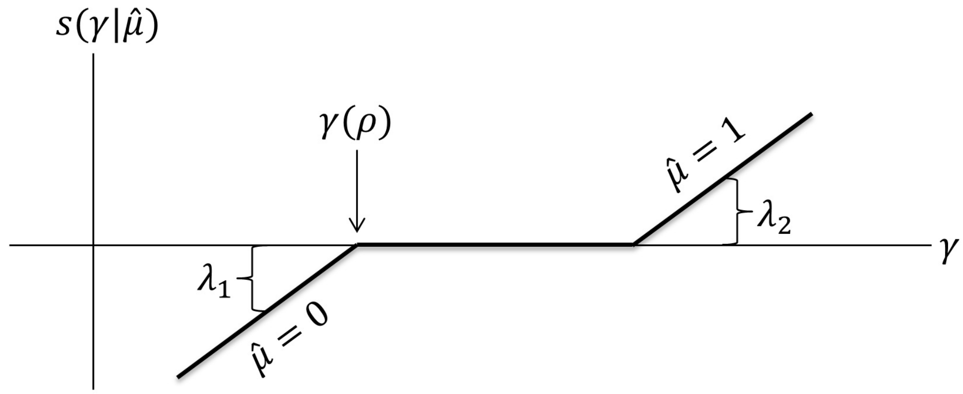

Figure 1: $s(\gamma|\hat{\mu})$, with $\hat{\mu}$ indicating optimal μ.

Rearranging terms, we can define $\gamma(\rho)$, for any $\rho > 0$, by

$$\rho \, \frac{u(c)}{u'(c) \; f'((1-\gamma(\rho))k) \; \gamma(\rho)} \;=\; 1.$$

It follows from the properties of u and f and the assumption that $k' \in [0, k + f(k))$ that $\gamma(\rho)$ is well-defined, continuous, and increasing, with $\lim_{\rho \to 0} \gamma(\rho) = 0$. For any $\rho > 0$, $\gamma(\rho)$ is the highest possible value of γ consistent with short-run efficiency in all periods in the case where only green capital is accumulated.

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