

# Sustainable growth and financial markets in a natural resource rich country

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## *Abstract*

We study the optimal growth path of a natural resource rich country, which can borrow from international financial markets. More precisely, we explore to what extent international borrowing can overcome resource scarcity in a small open economy, in order to have sustainable growth. First, this paper presents a benchmark model with a constant interest rate. We then introduce technical progress to see if the economy's growth can be sustainable in the long-run. Secondly, we analyse the case of a debt elastic interest rate, with a constant price of natural resources and then with increasing prices. The main finding of this paper is that borrowing on international capital markets does not permit sustainable growth for a country with exhaustible natural resources, when the interest rate is constant. Nevertheless, when we endogenize the interest rate the consumption growth rate can be positive before declining.

**Key words:** Exhaustible natural resources, exogenous growth, financial markets

**JEL Classification:** E20, O40, Q32, E44

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# 1 Introduction

Surprisingly, oil or gaz producers, even though they have important revenues due to their natural resources, are usually facing high level of debt. For instance, Angola, which is one of the first oil producer of Africa has to deal with an external debt of 22,170 billion of US dollar, and Brazil had a debt of 440,478 billion of US dollar in 2012. In this paper, we shall investigate whether a natural resource dependent country with an external debt can achieve sustainable output and consumption growth rates, thanks to its access to international financial markets.

Since the pioneering work of Hotelling (1931), the natural resource literature flourished in the 1970s-1980s, in the aftermath of the 1973 and 1979 oil crises. A significant part of those studies were based on extraction models, where the producer maximizes its profits by lowering its exploration and production costs. For instance, Gilbert (1979), Heal (1979), and Loury (1978) examined optimal extraction paths under reserve uncertainty. Pindick (1978) added price volatility to this kind of models.

Our paper is at the crossroads of two literatures. The first stream of literature deals with sustainable growth with natural resources. The cornerstone of those studies is Stiglitz's (1974) neoclassical optimal growth model, where natural resources are essential for production. He points out that optimal growth and a sustainable per capita consumption is feasible with exhaustible natural resource in limited supply, if the ratio of the rate of technical change to the the rate of population growth is greater than or equal to the share of natural resources. Solow (1974) also uses a neoclassical growth model, where he applies the maxim principle to the intergenerational problem of optimal capital accumulation, in general and then in particular with scarce natural resources. Dasgupta and Heal (1974) have a different approach of technical change, they do not view it as a smooth gradual process, but as technological breakthrough. They thus introduce a "backstop technology", which represents a major discovery, a new substitute for the depletable resource. An example might be the substitution of fossil fuel energy by solar or wind energy. It should be emphasized that this date of discovering is completely unknown. Nevertheless, this substitution process is a way to overcome resource scarcity and avoid a falling level of per capita consumption. Dasgupta and Stiglitz (1976) and Heal (1978) follow this approach. More recently, Benckroun and Withagen (2011) show that in a Dasgupta-Heal-Solow-Stiglitz based model, the initial consumption under a utilitarian criterion starts below the maximin rate of consumption if and only the resource is abundant enough. They underline the fact that the present generation does not necessarily benefits most from a windfall of resources. In addition, some articles decided to introduce endogenous growth with non renewable natural resources in this literature, as Schou (1996). In particular, Barbier (1999) proposed a "Romer-Stiglitz" model, where endogenous technical change can overcome resource scarcity. Grimaud and Roug (2003) analyses the effect of intervention instruments on the rate of resource extraction and growth rate

in a Schumpeterian model. Furthermore, Groth and Schou(2002) show in a one-sector optimal endogenous growth model, growth per capital consumption is unstable unless there is population growth. They adopt a "semi-endogenous" framework to allow a stable long-run growth.

The second body of literature embrace extensions of the Ramsey model to an open economy. It is shown that the open economy version of the Ramsey model leads to a number of paradoxical conclusions. Indeed, Barro and Sala-i-Martin(2003) display that the speed of convergence for capital and output is infinite, ie the economy jumps into the steady state. Moreover, consumptions tends to zero, except for the most patient country. Some attempts have been made to improve those counterfactual resluts, by introducing a constraint on international credit or adjustment costs. The infinite speed of convergence for capital and output does not apply to countries that are effectively constrained to borrow, and consumption does not decrease to zero. With adjustment costs for investments, capital and output are less instantaneous, even if capital markets are perfect.

Our main contribution is to introduce international borrowing in an exogenous Ramsey growth model, with exhaustible natural resources, in absence and then in presence of technical change. As a matter of fact, very few studies examine optimal exogenous growth models with resource extraction in an open economy. There are some exceptions, as Poelhekke and Van der Ploeg (2008), Van der Ploeg and Venables (2011) who have introduced international finance in their models of resource-rich countries, which focus more on managing resource windfalls. This paper analyses whether a resource dependent economy can have a sustainable growth rate thanks to international borrowing or technical progress, while its main source of growth is bound to disappear. The resources here are essential for production, non renewable, and can apply to oil, gaz, shale gaz or to different minerals as copper, gold, diamonds. In other words, we explore whether indebtedness or technical change can help overcome resource scarcity.

In the next section, we set forth a general Ramsey model in a small open economy, with exhaustible natural resources. In section 3, we focus on a particular case where the interest rate  $r$  is constant. We then introduce a resource-augmenting technical progress that helps raise the efficiency of resource use. In section 4, we generalize our model with an interest rate which is increasing in the level of debt. We then try to see if our results are different when resource prices are increasing.

## 2 Theoretical framework

We consider a simple neoclassical growth model with natural resources. It is a small open economy with borrowing capacities.

Output is produced with a constant-returns-to-scale technology, using a Cobb-Douglas production function:

$$Y = F(K, R) = K^\alpha((1 - \gamma)R)^{1-\alpha}, 0 < \alpha < 1 \quad (1)$$

where  $K$  is the stock of man-made capital,  $R$  a nonrenewable natural resource, and  $\gamma$  the share of the country's natural resources that are exported abroad. Labor is supposed to be constant.

According to diminishing returns to the accumulation of capital,  $F'(K) = F_K > 0$  and  $F''(K) < 0$ .

The man-made capital depreciates at rate  $\delta$ :

$$\dot{K}(t) = I(t) - \delta K(t), \delta \in [0, 1] \quad (2)$$

with  $I(t)$  the investment.

### 2.1 Natural resources

Let  $S(t)$  be the stock of non-renewable resources available at time  $t$ , and  $R(t)$  the rate of extraction of this resource at time  $t$ . The economy has an initial stock of natural resources  $S(0)$ .

The stock decreases over time with the rate of extraction:

$$\dot{S}(t) = -R(t) \quad (3)$$

with  $R(t)$  a continuous function of time,  $K(0) > 0$  and  $S(0) > 0$ .

### 2.2 The government

We consider an economy with a government that has constant relative risk aversion preferences. The intertemporal utility function is:

$$\int_0^\infty e^{-\rho t} U(C(t)) dt \quad (4)$$

with  $U(C(t)) = \frac{C^{1-\eta}-1}{1-\eta}$  for  $\eta \neq 1$ ,  $\eta > 0$   
and  $U(C(t)) = \ln(C(t))$  for  $\eta = 1$

The variable  $C(t)$  denotes consumption at time  $t$ , and  $\eta$  is the coefficient of relative risk aversion. The discount rate  $\rho$  is assumed to be strictly positive.

The government has access to international capital markets, and can borrow or lend at an interest rate  $r$ , which is exogenous and which depends on the level of the country's debt  $B(t)$  at time  $t$ . One part of the country's resources  $\gamma R$  is exported abroad, and the rest  $(1 - \gamma)R$  is sold on the domestic market.

The government's dynamic budget constraint is:

$$\dot{B}(t) = C(t) + r(B(t))B(t) + I(t) - Y(t) - \gamma pR(t) \quad (5)$$

where  $p$  is the price of the natural resources sold abroad, and thus  $\gamma pR(t)$  is the natural resources revenue received by the government from exports.

The government maximises its lifetime utility on an infinite horizon, subject to the capital accumulation equation (2), to the natural resources stock constraint equation (3) and the budget constraint equation (5).

The problem to be solved is thus:

$$\max_{\{C, I, R\}} \int_0^{\infty} e^{-\rho t} U(C(t)) dt$$

s.t.

$$\dot{K}(t) = I(t) - \delta K(t)$$

$$\dot{S}(t) = -R(t)$$

$$\dot{B}(t) = C(t) + r(B(t))B(t) + I(t) - Y(t) - \gamma pR(t)$$

$$K(0) > 0, S(0) > 0$$

There are three state variables  $S(t)$ ,  $B(t)$ , and  $K(t)$ . Three control variables  $C(t)$ ,  $I(t)$  and  $R(t)$  to be chosen.

The current value Hamiltonian is :

$$H(C, R, I, \lambda_1, \lambda_2, \lambda_3, t) = U(C(t)) + \lambda_1(t)(I(t) - \delta K(t)) - \lambda_2(t)R(t) + \lambda_3(C(t) + r(B(t))B(t) + I(t) - Y(t) - \gamma p(t)R(t))$$

The co-states  $\lambda_1(t)$ ,  $\lambda_2(t)$ ,  $\lambda_3(t)$  represent respectively the shadow price of accumulated capital, of the resource stock and the shadow price of debt.

The optimality conditions are given by:

$$U'(C(t)) = -\lambda_3(t) \quad (6)$$

$$-\lambda_3(t)(F'_R + \gamma p) = \lambda_2(t) \quad (7)$$

$$-\lambda_3(t) = \lambda_1(t) \quad (8)$$

$$\dot{\lambda}_1(t) = \lambda_1(t)(\delta + \rho) + \lambda_3(t)F'_K \quad (9)$$

$$\dot{\lambda}_2(t) = \rho\lambda_2(t) \quad (10)$$

$$\dot{\lambda}_3(t) = \lambda_3(t)(\rho - r'(B(t))B(t) + r(B(t))) \quad (11)$$

At the optimum rate, the rate of return is the same at all points in time, being equal to the social discount rate (equation 10):  $\frac{\dot{\lambda}_2(t)}{\lambda_2(t)} = \rho$ . As the Hotelling rule goes, the shadow price of the resource in the ground, also called the scarcity rent, grows at the discount rate.

The transversality conditions are:

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_1(t) K(t) = 0$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_2(t) S(t) = 0$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_3(t) B(t) = 0$$

It is not possible to accumulate capital and debt indefinitely. As in the long run, the natural resource will be exhausted, the country cannot extract those resources indefinitely.

### 3 Model with a constant interest rate

The economy has unrestricted access to a perfect world capital market.

#### 3.1 Model without technical progress

We now assume that the interest rate equals a constant  $r$ . We also assume that  $r \leq \rho$  applies, because if not the economy would eventually accumulate enough assets to violate the small-country assumption that we made.

With an interest rate  $r$  constant, the government's dynamic budget constraint becomes:

$$\dot{B}(t) = C(t) + rB(t) + I(t) - Y(t) - \gamma pR(t) \quad (12)$$

The optimality conditions remain the same as in the general model, except for equation (11) that becomes:

$$\dot{\lambda}_3(t) = \lambda_3(t)(\rho - r) \quad (13)$$

Using (8) and (11) yields that the marginal productivity of capital is equal to the depreciation rate plus the interest rate:

$$F_K = \delta + r \quad (14)$$

Besides, using the production function and equation (14), the marginal productivity of capital can be reexpressed as:

$$F_K = \alpha \left( \frac{K}{(1-\gamma)R} \right)^{\alpha-1} = \delta + r$$

It leads to a ratio capital natural resource  $\frac{K}{(1-\gamma)R}$  which is constant, when the interest rate  $r$  is constant:

$$\frac{K}{(1-\gamma)R} = \left( \frac{\delta + r}{\alpha} \right)^{\frac{1}{\alpha-1}}$$

From (9) and (11), the marginal productivity of natural resources depends on the price of those resources and on the interest rate:

$$F_R + \gamma p = - \frac{\lambda_2(0)}{\lambda_3(0)} e^{rt} \quad (15)$$

As one can note, there is an incompatibility between equation (15), which grows exponentially, and the marginal productivity of the natural resources derived from the production function,  $F_R + \gamma p = (1-\alpha) \left( \frac{K}{(1-\gamma)R} \right)^\alpha + \gamma p$ , which is constant.

We have to distinguish two cases: - If the country does not export its natural resources, then  $\gamma = 0$ , the model does not work. Equation (15) becomes:

$$F_R = - \frac{\lambda_2(0)}{\lambda_3(0)} e^{rt},$$

which is not possible as  $F_R$  is constant.

- If the country exports its natural resources, equation (15) holds if and only if prices increase at a rate  $r$ . Therefore, prices can be reexpressed as:

$$p(t) = p(0)e^{rt}$$

Using equations (14) and (15) boils down to a new Solow-Stiglitz condition:

$$\frac{\dot{F}_R + \gamma \dot{p}}{F_R + \gamma p} = F_K - \delta = r \quad (16)$$

We can see that along the optimal path, the growth rate of the marginal productivity of the natural resources plus the growth rate of the prices have to be equal to the interest rate.

**Proposition:** *The optimal rate of consumption is given by*

$$g_C = \frac{\dot{C}}{C} = \frac{r - \rho}{\eta}, \eta > 1 \quad (17)$$

If  $\eta = 1$ ,  $\frac{\dot{C}}{C} = r - \rho$

**Proof:** This is straightforward from equations (6) and (11), which gives the Keynes-Ramsey rule.

As  $r \leq \rho$ , the rate of consumption is negative, therefore consumption decreases asymptotically towards zero. This result confirms what has been shown in the literature extending the Ramsey model to an open economy with international borrowing, where consumption also tended to zero.

We shall see now what are the implications of a constant ratio capital natural resource on output.

**Proposition:** *The optimal path of output and stock of capital approach zero.*

**Proof:** We know that the natural resources are exhaustible, so that the rate of extraction of those resources tends towards zero:  $\lim_{t \rightarrow +\infty} R = 0$

Since the ratio  $\frac{K}{R}$  is constant, this implies that the accumulation of capital also approaches zero

$$\lim_{t \rightarrow +\infty} K = 0$$

As K and R are falling to zero, therefore output also decreases towards zero:

$$\lim_{t \rightarrow +\infty} Y = 0$$

In other words, not just the growth rate but even the level of output will vanish.

In addition, from equation (2), we also have a level of investment approaching zero:

$$\lim_{t \rightarrow +\infty} I = 0$$

Those no-output and no-growth results appear to be counterfactual. Especially, the fact that production also falls towards zero leads to an economy on the way to extinction.

Adding exhaustible natural resources to an open economy version of the Ramsey model leads thus to different results from those models with no natural resources, where the speed of convergence for capital and output is infinite.

A balanced growth path (BGP) is defined as a path along which the quantities Y, C and K change at constant proportionate rates. Let  $g_x$  denote the



growth rate of the variable  $x > 0$ , that is  $g_x = \frac{\dot{x}}{x}$ .

**Proposition:** *Along a BGP,  $g_Y = g_K = g_I = g_C = g_R$*

**Proof:** Let reexpress the production function in growth rate, such as  $g_x = \frac{\dot{x}}{x}$ :

$$Y = \left( \frac{K}{(1-\gamma)R} \right)^\alpha R$$

Then,

$$g_Y = \alpha g_{\frac{K}{(1-\gamma)R}} + g_R$$

As the ratio  $\frac{K}{(1-\gamma)R}$  is constant, its growth rate equals zero.

Therefore,  $g_K = g_R$  and  $g_Y = g_R$

Then,  $g_Y = g_K = g_I = g_R = g_C = \frac{r-\rho}{\eta}$ ,  $\eta > 1$

Concerning the debt path, we need to consider the constraint budget:

$$\dot{B}(t) = C(t) + rB(t) + I(t) - Y(t) - \gamma pR(t):$$

$$B(t) = b_1 e^{rt} + b_2 e^{\frac{r-\rho}{\eta}t} + b_3 e^{(\frac{r-\rho}{\eta}+r)t}$$

Solving this, with the transversality condition  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_3(t) B(t) = 0$  and the corresponding solution  $\lambda_3(t) = \lambda_3(0) e^{(\rho-r)t}$  yields:

$$\lim_{t \rightarrow +\infty} \lambda_3(0) (b_1 e^{rt} + b_2 e^{\frac{r-\rho}{\eta}t} + b_3 e^{(\frac{r-\rho}{\eta}+r)t}) e^{-rt} = 0$$

$$\lambda_3(0) \lim_{t \rightarrow +\infty} (b_1 + b_2 e^{(\frac{r-\rho}{\eta}-r)t} + b_3 e^{\frac{r-\rho}{\eta}t}) = 0$$

From the transversality condition, we can see that  $b_1 = 0$  and as  $r \leq \rho$ ,  $\lim_{t \rightarrow +\infty} B(t) = 0$ .

This induces that

$$\max g_B = \frac{r-\rho}{\eta}$$

In conclusion, in a small open economy with exhaustible natural resources when there is no technical progress and a constant interest rate, all the variables output, capital, consumption, natural resources, the level of debt grow at the same rate along the BGP and decline asymptotically towards zero.

Therefore, as in the literature of sustainable growth with natural resources, such as Stiglitz (1974) or Dasgupta and Heal (1974), sustainability is not feasible without technical progress.

### 3.2 Model with technical progress

We now introduce a resource-augmenting technical progress in the model. Therefore, even though natural resources are still exhaustible, the rate of extraction does no longer fall asymptotically to zero as it used to do in the precedent sections. This assumption can be supported by the fact that technological change can lead to the discovery of new fields, like in Brazil, or allow the exploitation of resources previously thought not to be economically accessible, as offshore high pressure - high temperature wells for example.

The new Cobb-Douglas production function is:

$$Y = K^\alpha (A(1 - \gamma)R)^{1-\alpha}, 0 < \alpha < 1$$

with  $A$  the resource-saving technological change.  $A$  is growing at some constant exogenous rate  $\tau > 0$ , so that  $A = e^{\tau t}$

Prices and the interest rate are assumed to be constant.

We have the same optimality conditions as in the benchmark model, except that this time the marginal productivity of capital and the marginal productivity of the resource can be reexpressed as:

$$F_K = \alpha \left( \frac{K}{A(1-\gamma)R} \right)^{\alpha-1} \quad (18)$$

$$F_R = (1 - \alpha) \left( \frac{K}{A(1-\gamma)R} \right)^\alpha A \quad (19)$$

From equation (14), yields:

$$F_K = \delta + r$$

This implies that the ratio  $\frac{K}{A(1-\gamma)R}$  is constant:

$$\frac{K}{A(1-\gamma)R} = \left( \frac{\delta + r}{\alpha} \right)^{\frac{1}{\alpha-1}} \quad (20)$$

Moreover, the modified Solow-Stiglitz efficiency condition still holds:

$$\frac{F_R + \dot{\gamma} p}{F_R + \gamma p} = F_K - \delta = r$$

In other words, along the optimal path the the interest has to be equal to the growth rate of the marginal productivity of the resource plus the government's revenue of exported resources.

Using this last efficiency condition and equation (23) yields:

$$\frac{\dot{F}_R}{F_R} = \alpha \left( \frac{\dot{K}}{A(1-\gamma)R} \right) + \frac{\dot{A}}{A} \quad (21)$$

$$\frac{F_R \dot{+} \gamma p}{F_R + \gamma p} = \frac{(\alpha(\frac{\dot{K}}{A(1-\gamma)R}) + \frac{\dot{A}}{A})F_R}{F_R + \gamma p}$$

As the ratio  $\frac{K}{A(1-\gamma)R}$  is constant, therefore its growth rate  $g_{\frac{K}{AR}} = \frac{\frac{\dot{K}}{A(1-\gamma)R}}{\frac{K}{A(1-\gamma)R}}$  is equal to zero. Equation (25) can thus be reexpressed as:

$$\frac{F_R \dot{+} \gamma p}{F_R + \gamma p} = \frac{\tau F_R}{(1-\alpha)(\frac{K}{A(1-\gamma)R})^\alpha A + \gamma p}$$

**Proposition:** *The optimal growth rate of consumption is given by:*

$$g_C = \frac{\dot{C}}{C} = \frac{r - \rho}{\eta}$$

**Proof:** This is straightforward from equations (6) and (11), which gives the Keynes-Ramsey rule. Therefore, the consumption growth rate is negative, as  $r \leq \rho$ . Consumption thus cannot be sustainable in the long run.

Our result is thus different from Stiglitz (1974), where thanks to technical progress there is sustainability in a closed economy with exhaustible natural resources, if the ratio of the rate of technical change to population growth rate is higher or equal than the share of natural resource in production.

**Proposition:** *Along a BGP,  $g_Y = g_K = g_I = g_C$  and  $g_R < 0$*

**Proof:**

$$g_{A(1-\gamma)R} = g_A + g_{(1-\gamma)R} = g_A$$

We thus assume  $g_R < 0$

As the ratio  $\frac{K}{A(1-\gamma)R}$  is constant, by differentiating logarithmically we have:

$$g_{\frac{K}{A(1-\gamma)R}} = g_K - g_{A(1-\gamma)R} = 0$$

and  $g_{A(1-\gamma)R} = g_A + g_{(1-\gamma)R} = g_A + g_R$ , as  $\gamma$  constant

This implies that:

$$g_K = g_A + g_R$$

Furthermore, as  $g_K = g_C$ , we have:  $g_R = g_C - g_A$

Then,

$$g_R = \frac{r - \rho}{\eta} - \tau$$

As  $g_R < 0$ , therefore  $g_C = \frac{r-\rho}{\eta} < \tau$

Concerning the debt path, we need to consider the constraint budget, knowing that Y, I and R tend asymptotically towards zero, with constant prices:

$$B(t) = b_1 e^{rt} + b_2 e^{\frac{r-\rho}{\eta}t}$$

Solving this, with the transversality condition  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_3(t) B(t) = 0$  and the corresponding solution  $\lambda_3(t) = \lambda_3(0) e^{(\rho-r)t}$  yields:

$$\lim_{t \rightarrow +\infty} \lambda_3(0) (b_1 e^{rt} + b_2 e^{\frac{r-\rho}{\eta}t}) e^{-rt} = 0$$

$$\lambda_3(0) \lim_{t \rightarrow +\infty} (b_1 + b_2 e^{(\frac{r-\rho}{\eta}-r)t}) = 0$$

From the transversality condition, we can see that  $b_1 = 0$  and as  $r \leq \rho$ ,  $\lim_{t \rightarrow +\infty} B(t) = 0$ .

This implies that

$$max g_B = \frac{r-\rho}{\eta} - r$$

In conclusion, in a small open economy with exhaustible natural resources when there is technical progress and a constant interest rate, all the variables output, capital, consumption, natural resources, level of debt decline towards zero. Moreover, along the BGP those variables do not grow at the same rate.

Surprisingly, technical progress cannot overcome resource scarcity, as the interest rate is exogenous and constant. Therefore, contrary to Stiglitz (1974) positive growth rates for output and consumption cannot be sustained in the long run.

## 4 Model with a debt-elastic interest rate

The economy faces now limitations in its access to the world financial markets. It is indeed more realistic than a country that can borrow as much as it wants at a fixed rate.

### 4.1 Case with constant prices

The interest rate  $r(B)$  depends now on the level of debt. We thus assume that  $r(B)$  is increasing in the aggregate level of foreign debt.

**Proposition:** *The optimal level of debt decreases and output falls to zero in the long-run.*

**Proof:** Using the optimality conditions of the general model, especially equations (8), (9) and (11), we have the marginal productivity of the capital that depends on the interest rate:

$$F_K = \delta + r'(B).B + r(B) \quad (22)$$

As  $F_K = \alpha(\frac{K}{(1-\gamma)R})^{\alpha-1}$ , we can find the ratio  $\frac{K}{(1-\gamma)R}$ :

$$\frac{K}{(1-\gamma)R} = \left( \frac{\delta + r'(B).B + r(B)}{\alpha} \right)^{\frac{1}{\alpha-1}} \quad (23)$$

From equations (7), (10), (11) and (17), the marginal productivity of the natural resources is given by:

$$\frac{F_R + \dot{\gamma}p}{F_R + \gamma p} = F_K - \delta = r'(B).B + r(B) \quad (24)$$

This relation corresponds to the new Solow-Stiglitz efficiency condition in an open economy with a debt-elastic interest rate.

But  $F_R$  can be also written as:  $F_R = (1 - \alpha)(\frac{K}{(1-\gamma)R})^\alpha$

Therefore,

$$\frac{\dot{F}_R}{F_R} = \alpha \left( \frac{\dot{K}}{\frac{K}{(1-\gamma)R}} \right)$$

Using (18), yields:

$$\frac{F_R + \dot{\gamma}p}{F_R + \gamma p} = \frac{1}{\alpha} * G(B).r'(B) * \dot{B}$$

with

$$G(B) = \alpha \frac{\frac{1}{\alpha-1}(r'(B).B + r(B) + \delta)^{\frac{1}{\alpha-1}} - 1}{(r'(B).B + r(B) + \delta)^{\frac{1}{\alpha-1}} + \gamma p}$$

And thus with (20), we can have a function of B,  $r(B)$ ,  $r'(B)$ ,  $G(B) < 0$ :

$$\frac{1}{\alpha} * G(B).r'(B) * \dot{B} = r'(B).B + r(B)$$

We reexpress this condition as the following autonomous differential equation:

$$\dot{B} = H(B)$$

with  $H(B) = (r'(B).B + r(B)) * \frac{\alpha}{G(B)*r'(B)}$

As  $B > 0$ ,  $r(B) > 0$  and  $r'(B) > 0$ , and  $G(B) < 0$ ,  $H(B) = \dot{B}$  is thus negative.

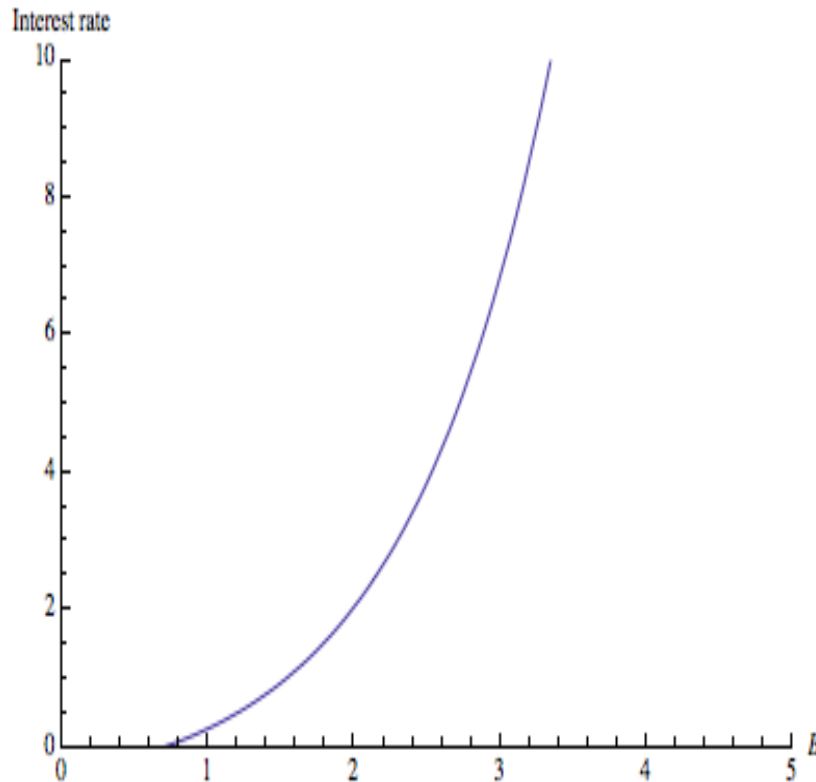
Therefore, the optimal level of debt  $B$  is decreasing with time. This is quite a paradoxical result, as the economic policy recommended here is to pay off its debt from the beginning and thus not to take any risks.

As in the literature it is more common to use an exponential interest rate, we will use this last formulation to see if our result is still robust.

We refer to Schmitt-Grohe and Uribe (2003) debt-elastic interest-rate premium. It can be expressed as:  $r(B) = r^* + \psi(e^{B-D} - 1)$

where  $\psi(e^{B-D} - 1)$  is the country-specific interest rate premium,  $r^*$  is the world interest rate,  $D$  is the steady-state level of foreign debt.  $r^*$ ,  $D$  and  $\psi$  are constant and positive parameters.

Figure 1: Exponential interest rate  $r(B)$  in function of the level of debt



**Proposition:** *When the interest rate is exponential, the optimal level of debt decreases and output falls to zero in the long-run.*

**Proof:** The formulation of the interest rate implies  $r(B) > 0$  and  $r'(B) = \psi e^{B-D} > 0$ .  $G(B)$  is still negative, thus  $\dot{B} < 0$ . Debt is thus still decreasing with time.

As the level of debt is falling to zero, the ratio  $\frac{K}{(1-\gamma)R}$  also approaches asymptotically a constant:

$$\frac{K}{(1-\gamma)R} = \left( \frac{\psi e^{B-D} \cdot B + r^* + \psi(e^{B-D} - 1) + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}$$

We provide a numerical example that illustrates the debt pattern in function of time when the interest rate is exponential (Figure 2). More precisely, we set  $\alpha = 0,32$ ,  $\delta = 0,1$ ,  $p = 1$ ,  $\gamma = 0,5$ ,  $r^* = 0,04$ ,  $D = 0,7442$  and  $\psi = 0,8$ , and we plot the optimal debt path in function of time.

In fact, despite the form of interest rate chosen, the level of debt  $B$  is still decreasing towards zero and the ratio  $\frac{K}{(1-\gamma)R}$  tends asymptotically towards a constant.

We thus find the same counterfactual results from the benchmark model with a constant interest rate  $r$  concerning the accumulation of capital, the level of investment, and output that decrease asymptotically towards zero.

**Proposition:** *The optimal rate of consumption is given by:*

$$\frac{\dot{C}}{C} = \frac{F_K - \delta - \rho}{\eta} = \frac{r'(B) \cdot B + r(B) - \rho}{\eta}, \eta > 1 \quad (25)$$

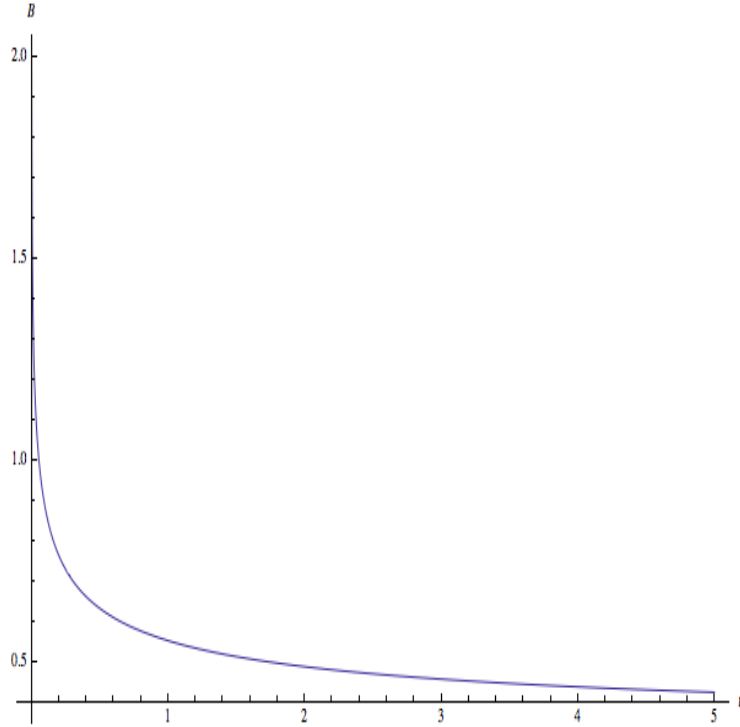
**Proof:** This is straightforward from equations (6) and (11). Consumption accumulates at a rate equal to the difference between the net marginal product of capital  $F_K - \delta$  and the discount rate  $\rho$ .

Nevertheless this time the consumption growth rate  $g_C$  is depending on the level of debt  $B_t$ , contrary to the benchmark model.

In fact during the transitional dynamics, the consumption growth rate is first positive: we have  $g_C \geq 0$  when  $r'(B) \cdot B + r(B) \geq \rho$

In the long-run, as the level of debt  $B$  tends asymptotically towards zero, this growth rate becomes negative, thus  $g_C \leq 0$ , and consumption decreases.

Figure 2: Debt path in function of time  
With an exponential interest rate  $r(B)$

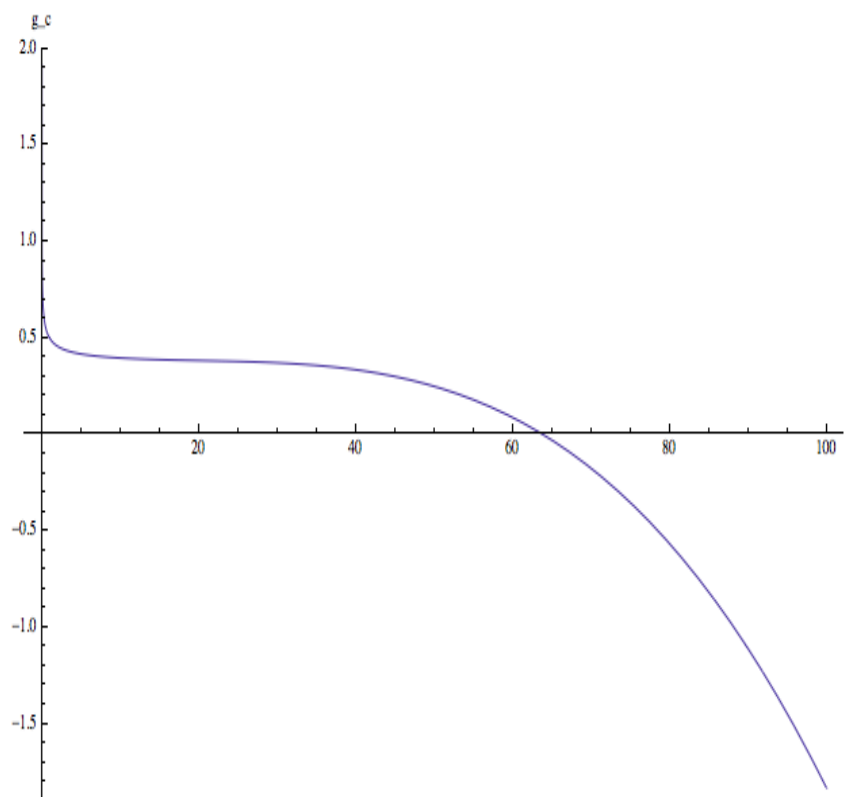


To illustrate the consumption growth rate in function of time when the interest rate is exponential (Figure 3), we set  $\alpha = 0,32$ ,  $\delta = 0,1$ ,  $p = 1$ ,  $\gamma = 0,5$ ,  $r^* = 0,04$ ,  $D = 0,7442$  and  $\psi = 0,8$ .

We can see that the consumption growth rate is first positive, and then declines in the long-run. Therefore, when we endogenize the interest rate, consumption can grow for a while before decreasing.



Figure 3: Consumption growth rate in function of time  
 With an exponential interest rate  $r(B)$



## 4.2 Case with increasing prices

From now on, prices have been taken as constant. We now assume prices to increase at a rate  $\theta$ , with:  $p(t) = p(0)e^{\theta t}$ . We shall see if increasing prices, like in 2002-2008 where the oil price went from 20\$ to 140\$, can help have a sustainable growth.

The equation (20) becomes:

$$\frac{\dot{F}_R + \gamma \dot{p}}{F_R + \gamma p} = \frac{G(B) \cdot r'(B) * \dot{B} + \gamma \dot{p}}{\alpha}$$

$$\frac{G(B) \cdot r'(B) * \dot{B} + \gamma \dot{p}}{\alpha} = r'(B) \cdot B + r(B)$$

We reexpress this condition as the following non autonomous differential equation, as  $p$  depends now on time:

$$\dot{B} = H_2(B)$$

with  $H_2(B) = \frac{\alpha(r'(B).B+r(B))-\gamma\dot{p}}{G(B)*r'(B)}$

As we know from below  $r(B) > 0$ ,  $r'(B) > 0$ , and  $G(B) < 0$ . Moreover,  $\gamma > 0$  and  $\dot{p} > 0$ . Therefore, when  $\alpha(r'(B).B + r(B)) < \gamma\dot{p}$ , then  $\dot{B} > 0$ . Let us apply this condition in the case where  $B(0)$  takes the value of zero and when the interest rate is exponential, this condition becomes:  $\alpha(r^* + \psi(e^{-D} - 1)) < \gamma\dot{p}(0)$ . As  $p(0)$  is assumed to be equal to one, then  $\dot{p}(0) = \theta$ . This inequality is thus verified when  $\psi(e^{-D} - 1) < \frac{\gamma\theta}{\alpha} - r^*$ . In that case, the country increases its level of debt. In the short-run, it is interesting to see that as debt is growing, consumption grows.

But when  $\alpha(r'(B).B + r(B)) > \gamma\dot{p}$ , then  $H_2(B) = \dot{B} < 0$ , the optimal level of debt is still decreasing, even though prices are increasing. Indeed when  $B(0)$  takes the value of zero and when the interest rate is exponential, this condition becomes:  $\psi(e^{-D} - 1) > \frac{\gamma\theta}{\alpha} - r^*$ .

This means that even in a period of relative prosperity, when the price of the natural resource is increasing, thus when the country earns more money than it used to do, the government ought to pay off its debt in priority. In the long-run, it confirms the results from the case with a debt-elastic interest rate and constant prices, which is quite similar to what we found in the benchmark model, where consumption declines.

## 5 Conclusion

In this paper, we have developed an exogenous optimal growth model of a resource-rich open economy, where natural resources are essential for production and non-renewable.

The main finding of this paper is that sustainability is not feasible when the interest rate is constant, even if there is technical progress, but when the economy faces a debt-elastic interest rate, consumption can grow for a while before decreasing in the long-run.

First, we show that when the interest rate is assumed constant, this economy is faced to decreasing consumption and production. This leads us to some counterfactual no-growth results, as all our variables approach asymptotically zero and grow at the same rate on a balanced growth path. When we add technical progress, results are quite similar to this first case except that natural resources and output, consumption do no longer grow at the same rate. Contrary to Stiglitz (1974), resource augmenting technical progress does not overcome resource scarcity. In the literature extending Ramsey models to an open economy, borrowing constraints with a collateral or adjustment costs are usually added in order to found less paradoxical results, but our approach is different. We introduce an interest rate that increases in the aggregate level of debt. It is a more realistic assumption, as countries usually face a risk premium. Second, we show that when we endogenize the interest rate, the optimal level of debt asymptotically declines, and so do the output, capital and investment. Therefore, the country ought to reduce its debt. Nevertheless, the growth rate of consumption is positive during the transitional dynamics before declining.

This paper can be considered as a contribution to the optimal growth models with exhaustible natural resources, that focused mainly on closed economies. One way to extend this paper would be to optimize the model according to  $\gamma$  the share of the country's natural resources exported abroad. Another extension would be to introduce a constraint on international credit, where natural resources could be used as collateral.

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