Sustainability of an economy relying on two reproducible assets

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Abstract

We study the sustainability of an economy relying on two reproducible assets, which can be renewable resources or manufactured capital stocks. Within the maximin framework, we define the utility level which can be sustained under different assumptions about the substitutability of the two assets. The degree of substitutability has strong effects on the regularity or non-regularity of the maximin problem, and thus on the shadow-values of the capital stocks. This has important consequences for the computation of durable savings and the sustainability prospects of future generations.

Key words: sustainable development, maximin, sustainability indicator.

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1 Introduction

The growing utilization of the environment in human activity, related to growth in population and in material consumption, has increased concern for the sustainability of society, of the production of goods and of the natural world. Sustainability is a complex issue, and there is not even a universally accepted definition of it.

To an economist, sustainability means the ability to continue to support a standard of living of human beings (Solow, 1974, 1993). The standard of living is usually represented as a mathematical aggregate of all components of human well being, including the consumption of manufactured goods, the flow of services from the environment and so on, both now and in the future.

The "contributions" of the components of well being can be considered abstractly to be "produced", and factors that produce or contribute and that last over periods of time can be considered to be capital. The concepts are abstract, but it is easy to write them mathematically: the levels of the components at any time can be written as a vector $c = (c_1, \ldots, c_n)$, for a very large n, and the aggregate as $u(c)$. The lasting factors that produce c can be written as a vector $K = (K_1, \ldots, K_m)$, again for a very large m. The elements of K at any time can be manufactured capital goods such as machines, buildings and vehicles, environmental goods such as forests, oil reserves and fish stocks, some measure(s) of the technology available for production, some measure(s) of human knowledge and skills, etc. These elements of capital can be added to or drawn down over time in order to provide a dynamic realization of $u(c)$ through time, from the present to kingdom come, that can be given a value $V(K)$.

To get beyond mathematical triteness, one must assume that one has a comprehensive list of the elements of c and K, as well as expressions for $u(c)$ and $V(K)$. This assumption is heroic, but making it helps to focus attention on the properties of sustainabililty, if it can be defined precisely. Each component of well being, c_i, even the purity of air being breathed, can be considered to have a marginal contribution given by $\pi_i = \partial u / \partial c_i$. Each component of capital K_j can be considered to have a marginal contribution $v_j = \partial V / \partial K_j$. These marginal contributions are called *shadow prices.*

A common way of trying to represent sustainability is to postulate that value is the flow of utility discounted at a constant rate ρ:

$$V(K_t) = \int_t^\infty u(c_s)\, e^{-\rho(s-t)} \mathrm{d}s.$$

This representation has obvious similarities to the discounting of future money values at a money rate of interest. Theoretically, the two are linked. Some economists consider the society to be sustained at time t if $\mathrm{d}V(K_t)/\mathrm{d}t = 0$. This condition implies that $\sum_{j=1}^{m} (\partial V/\partial K_j)(\mathrm{d}K_j/\mathrm{d}t) = \sum_{j=1}^{m} v_j (\mathrm{d}K_j/\mathrm{d}t) = 0$, so that net investment, evaluated at the shadow prices v_j, is zero. This definition, however, may apply only at date t, and it may well be that at some future date s, $\mathrm{d}V(K_s)/\mathrm{d}s < 0$, and indeed that such an inequality can hold forever afterward. If $\sum_{j=1}^{m} v_j (\mathrm{d}K_j/\mathrm{d}t) = 0$, value can be said to be maintained at time t but not to be sustained forever. Such a possibility is by no means ruled out because the "power of compound interest" is such that the well being of the consumer at time s is heavily discounted by the factor $e^{-\rho(s-t)}$. A striking example in a simple model is given by Asheim (1994).

To make sure that well being is sustained forever another approach, unfortunately no less abstract, can be taken. Sustained well being $\sigma(K)$ can be defined as the minimal level of current well being attained at any future date s: $\sigma(K) = \min_{s\geq t} u(c_s)$. The reason for making this provision is that it is not possible to say that a level of well being is sustained if some generation in the future does not enjoy it. The capital stocks, $(K_1, \ldots, K_m)$, are the same because the components of well being and the rest of the society are the same. Suppose that the minimal level is maximized by the choices of the society. The maximized function, the so-called *maximin* level of well being, can be written $m(K) = \max \sigma(K) = \max \min_{s\geq t} u(c_s)$. It can be differentiated to find shadow prices by which the capital stocks can be evaluated, $p_j = \partial m(K)/\partial K_j$. In many interesting cases, the value $m(K)$ remains constant through time, even if the stocks change. In this case, investment evaluated at the prices p_j is zero: $\sum_{j=1}^{m} p_j \mathrm{d}K_j/\mathrm{d}t = 0$. This condition is the true form of what is known as *Hartwick's rule* for net investment, since Hartwick (1977) derived his formula explicitly for a sustainable (constant-consumption in his model) economy.

There are, then, two approaches by which a similar condition of zero current net investment is obtained. For prices v_j that apply to discounted utility, $\sum_{j=1}^{m} v_j \mathrm{d}K_j/\mathrm{d}t = 0$.

For prices p_j that apply to the maximin value, $\sum_{j=1}^{m} p_j \mathrm{d}K_j/\mathrm{d}t = 0$. The programs are different, however, and give rise to different values of $\mathrm{d}K_j/\mathrm{d}t$ and of the corresponding prices: $p_j \neq v_j$.

Both programs are highly idealized. Neither can use market prices. For maximin, what is being investigated is whether well being is sustained or is sustainable. If the maintained hypothesis is that the society is not sustained, it can hardly be held that the prices that direct the decisions in the economy are the correct, maximin prices p_j. If one admits that the world is not a Pareto-efficient economy for the discounted-utilitarian formulation, it is by the same token not possible to hold that the prices v_j are the correct, optimizing prices, even approximately or when adjusted by a cost-benefit analysis. Even in a Pareto-efficient economy, the analyst cannot "back out" the value function $V(K)$ from observed, market realizations, because the society may be at an efficient point but not an optimum for $V(K)$. Rather, correct shadow prices are obtained going forward from the primitive $V(K)$, and these are, with probability one, not the prices given by even a Paretian economy, let alone an inefficient economy as exists in reality. One has only to realize that there are an uncountably infinite number of values of ρ, the utility-discount rate, each giving its own sets $(v_1, \ldots, v_m)$, $c = (c_1, \ldots, c_n)$ and $K = (K_1, \ldots, K_m)$ for each future time s.

Each problem is formidable. It is no easier to solve a discounted-utilitarian problem than a maximin one. It is prudent, then, in studying and eventually trying to provide a measure of sustainability, to drop the discounted-utilitarian framework in favor of the maximin one. To get the prices p_j one must solve the maximin problem. The only way to do it seems to be to build up from simpler problems and to try to gain a greater understanding of the economic issues involved.

There are only a few solved maximin problems. Among them is the maximin problem for the so-called Dasgupta-Heal-Solow model (Dasgupta and Heal, 1974; Solow, 1974), which has two sectors, one for production of a composite consumption–capital good and one for use of a nonrenewable resource. The solution to the problem was found by Solow (1974), but the solution to the discounted-utilitarian problem in the same model economy required exotic mathematics and even at that eluded solution for nearly forty years (Benchekroun and Withagen, 2011). The maximin solution has since been made more transparent by several authors, as well as generalized by Stollery (1998), d'Autume et al. (2010) and Asheim and Zuber (2013).

Another solved maximin problem is that of the simple fishery, with a single capital input and a single component of well being. It exhibits a form of *non-regularity*, a subtle mathematical property that makes it inefficient to assure each generation the same level of well being. The maximin value is still defined and is constant. Another form of non-regularity arises in a simple representation of Easter Island (Cairns and Tian, 2010), which tries to examine a sustainable path for this notoriously unsustained, isolated economy. In this case, as the definition implies, maximin value is not decreasing. The non-regular part of the program appears after a period of regularity and before another.

Non-regularity has been studied in a good deal of generality by Doyen and Martinet (2012) and by Cairns and Martinet (2014). It appears to be more than just an isolated anomaly in a few special cases but to be frequent. The present paper seeks to gain a greater understanding of the relationship between regularity and non-regularity in more complicated model economies. It is necessary to begin small, with a model suggested by the simple fishery but with two renewable assets such as two physically separate fisheries. Solving this problem entails expressing the paths of the shadow prices of the two assets. Ultimately, an aim is to find conditions applicable to the maximin shadow prices in an economy with several renewable assets.

When there is more than one natural asset, the question of substitutability arises. This question is controversial in examinations of sustainability. In economic analyses, so-called weak sustainability postulates that there is substitutability among all capital stocks, while so-called strong sustainability postulates that there are natural assets that have no substitutes in well being. The study of two natural assets in the present paper considers perfect complementarity, perfect substitutability, and the more neoclassical smooth, non-linear substitutability between the stocks. If substitutability can be represented by a single parameter in the function $u(c)$, then as substitutability decreases the set of initial stocks for which there are regular solutions extends.

2 The single-resource case

The simple fishery is a canonical model to study a single renewable resource economy.[1] It has one state variable, a renewable resource stock X, and one control variable, the catches $C \geq 0$. Natural growth of the stock is represented by a strictly concave, differentiable function $F(X)$ that takes positive values and reaches a maximum at X^{MSY} for $X \in]0, X_{sup}[$, with $F(0) = 0 = F(X_{sup})$. Utility is derived from the catch, and is represented by a function $U(C) = C$ for simplicity. The evolution of the stock is given by $\dot{X} = F(X) - C$. The highest sustainable level of the catch, the "maximum sustainable yield" is $MSY \equiv F(X^{MSY})$.

Formally, the maximin value of a given fish stock X is defined as

$$\begin{aligned} m(X) \quad &\equiv \quad \max \underline{U} \\ \text{s.t.} \quad & X(t) = X \text{ and} \\ & \forall s \geq t, \dot{X}(s) = F(X(s)) - C(s) \text{ and } C(s) \geq \underline{U} \, . \end{aligned} \tag{1}$$

This value is the highest level of utility that can be sustained over all feasible paths starting from state X.

If $X \leq X^{MSY}$, the maximin criterion (1) prescribes a catch level $C = F(X)$. If $X > X^{MSY}$, the highest sustainable catch is Y. The maximin value is given by

$$m(X) = \begin{cases} F\left(X^{MSY}\right) = Y & \text{if } X \geq X^{MSY}, \\ F(X) & \text{if } X \leq X^{MSY}. \end{cases}$$

The shadow value is $p(X) = \frac{\mathrm{d}m(X)}{\mathrm{d}X}$, with

$$p(X) = \begin{cases} 0 & \text{if } X \geq X^{MSY}, \\ F'(X) & \text{if } X \leq X^{MSY}. \end{cases}$$

[1]Another canonical model would be the one-sector economy *à la Solow*, with capital depreciation. Results would not change. Maximun Sustainable Yield is replaced by the golden rule consumption.

Figure 1 plots the maximin consumption (in bold) and the evolution of the stock according to the region the harvest takes place.

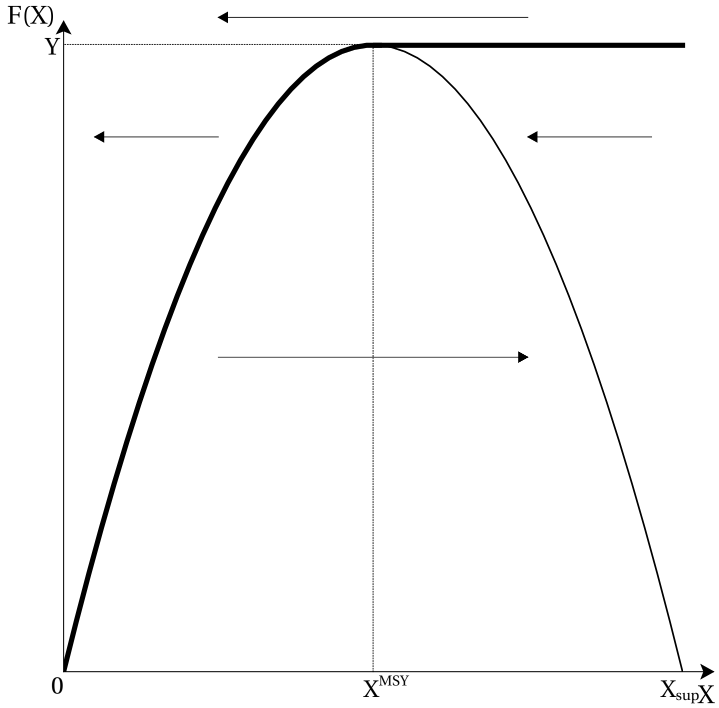

Figure 1: The maximin consumption with one resource

3 Two resources

3.1 Economic model

We assume that the economy is composed of infinitely many generations of identical consumers each living for an instant in continuous time. Utility is derived from the consumption of two goods C_i, with $i = 1, 2$. The instantaneous utility function $U(C_1, C_2)$ is assumed to be twice differentiable and strictly quasi-concave unless otherwise stated.[2] Further, $\lim_{C_i \to 0} U'_{C_i}(C_1, C_2) = \infty, \ i = 1, 2$.

[2]As we are considering time autonomous problems, we will omit the argument t in the functions.

The stocks interact only through utility.[3] Each good i is produced using a reproducible asset X_i, which is either a natural-resource or an manufactured-capital asset. The growth of the asset $F_i(X_i)$ depends only on the stock X_i and is assumed to be continuous, concave and differentiable. Stock dynamics are described by the differential equations

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} \equiv \dot{X}_i = F_i\left(X_i(t)\right) - C_i(t)\ . \tag{2}$$

Marginal productivity is assumed to be positive for low stock levels but we allow it to turn negative at some point, for example, if the asset is a natural renewable resource stock with carrying capacity or a neoclassical manufactured capital stock with decreasing marginal gross productivity and constant decay rate. When this is the case, we denote by X_i^{MSY} the stock level for which $F_i'(X_i^{MSY}) = 0$.[4] Such a sector is a source of non-regularity in a maximin problem, as in the single-resource case described above (see Asako, 1980).

3.2 Maximin value function and shadow value of the stocks

A maximin problem aims at defining, among all feasible paths starting from an initial state, the development trajectory with the highest minimal utility level over time. The maximin value of a state, $m(X_1, X_2)$, is the highest level of utility that can be sustained forever from that state. Formally, the maximin value of state (X_1, X_2) is defined as follows:

$$\begin{aligned} m(X_1, X_2) &= \max \underline{U}, && (3)\\ \text{s.t. } (X_1(0), X_2(0)) &= (X_1, X_2) \\ \dot{X}_i &= F_i(X_i(t)) - C_i(t), i = 1, 2 \text{ and} \\ U(C_1(t), C_2(t)) &\geq \underline{U} \text{ for all } t \geq 0. && (4) \end{aligned}$$

The marginal or shadow value of a stock in a maximin problem is $p_i(X_1, X_2) = \frac{\partial m(X_1, X_2)}{\partial X_i}$.

Differentiation of the maximin value in respect to time yields a direct link with net

[3]The case of interactions between sectors (e.g., an extraction sector providing an input to the other sector or ecological models of predator and prey) is left for future work. One can refer to Solow (1974) for the case of a nonrenewable resource being extracted and used as an input for a manufacturing sector.

[4]This is the stock which yields the Maximum Sustainable Yield or the Golden Rule production level.

investments (Cairns and Martinet, 2014, Lemma 1):

$$\dot{m}(X_1, X_2) = \frac{\partial m}{\partial X_1}\dot{X}_1 + \frac{\partial m}{\partial X_2}\dot{X}_2 = p_1\dot{X}_1 + p_2\dot{X}_2 \ . \tag{5}$$

The following lemmata are established in the Appendix.

Lemma 1. *For any state (X_1, X_2), the maximin value is at least equal to the utility derived from consumption at the corresponding stationary state:*

$$m(X_1, X_2) \geq U\left(F_1(X_1), F_2(X_2)\right) \ .$$

Lemma 2. *If the maximin value of a state (X_1, X_2) is greater than the utility from keeping the state stationary, i.e., if $m(X_1, X_2) > U\left(F_1(X_1), F_2(X_2)\right)$, then (i) the consumption of at least one good is greater than the production of the corresponding stock and (ii) that stock decreases.*

4 Regularity and Non-Regularity

Solow (1993, pp. 167-168) has a vision that transforms sustainability from being an undisciplined, popular, modern expression of the conservationism that has arisen from time to time in advanced countries, into an economic prescription of durable decisions for the current generation: "If sustainability means anything more than a vague emotional commitment, it must require that something be conserved for the very long run. It is very important to understand what that thing is: I think it has to be a generalized capacity to produce economic well-being."

This notion had earlier been given an economic meaning by Solow (1974). In the midst of a severe global economic situation, the energy crisis of the 1970s led many to re-interpret the social objective in terms of conservation of scarce petroleum. He adapted the maximin formulation of equity enunciated by Rawls (1971) to a dynamic problem, the intergenerationally equitable use of an exhaustible, petroleum resource. Maximin insists on raising the level of the least well-off, to the extent possible. As reasoned in Sections 2 and 3 for the simple fishery or for a society with two complementary fish stocks, the redistribution can be performed by continually shifting consumption or utility from the best-off generation to the poorest, to the extent possible. Solow's (1974) examination of intergenerational equality in the Dasgupta-Heal-Solow model is an instance in which such a policy can be followed to achieve intergenerational equality. In others, however, even though a maximin solution exists, equity does not mean equality. Therefore,

- a redistribution to achieve equality is not always possible and
- if it is possible, intergenerational equality is the outcome of the intergenerational-equity problem and not the objective.

The maximin solution determines the properties of a path of the economy through time (eq. 3). That it may not be possible to attain intergenerational equality in the solution to this problem is important to our understanding of sustainability or intergenerational equity. In a regular problem (the term used by Burmeister and Hammond (1977)) it is possible to redistribute the wealth of the economy equally to each generation. If it is not possible, however, the problem is called non-regular. There are simple examples of both regular and non-regular problems. One way to define regularity and non-regularity is by the attainment or non-attainment of equality. It is also possible to dig deeper into the mathematical properties of the problem and the roles of the stocks at the initial time in order to understand why regularity is or is not possible.

A direct approach to solve the maximin problem A direct approach to the maximin problem was provided by Cairns and Long (2006), using a mechanical application of optimal-control theory to problem (3).

Taking the sustained utility level $\underline{U}$ as a control parameter and denoting the adjoint variables of the stocks by p_i, one can write the Hamiltonian associated with the problem (3), subject to the equations of motion (2), as

$$\mathcal{H}(X, C, p) = p_1 \dot{X}_1 + p_2 \dot{X}_2 = p_1 \left(F_1(X_1) - C_1 \right) + p_2 \left(F_2(X_2) - C_2 \right) \ . \tag{6}$$

Let ω be the multiplier associated with the constraint (4), so as to write the Lagrangian as

$$\mathcal{L}(X, C, p, \underline{U}, \omega) = \mathcal{H}(X, C, p) + \omega \left(U(C_1, C_2) - \underline{U} \right) \ . \tag{7}$$

The necessary conditions are[5], for $i = 1, 2$ and for any time t, as follows.

$$\frac{\partial \mathcal{L}}{\partial C_i} = 0 \, , \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial X_i} = -\dot{p}_i \, , \tag{9}$$

$$\int_0^\infty \left(-\frac{\partial \mathcal{L}}{\partial \underline{U}} \right) \mathrm{d}s = \int_0^\infty \omega(s) \, \mathrm{d}s = 1; \tag{10}$$

$$\lim_{t \to \infty} p_i X_i = 0 \, , \tag{11}$$

$$\lim_{t \to \infty} \mathcal{H}(X, C, p) = 0 \, , \tag{12}$$

$$(p_1, p_2, \omega) \neq (0, 0, 0) \, , \tag{13}$$

along with the usual complementary slackness condition

$$\omega \geq 0, \; \omega \left(U(C_1, C_2) - \underline{U} \right) = 0. \tag{14}$$

All shadow-values (i.e., stocks shadow values $p(t)$ and the equity shadow value $w(t)$) are evaluated at time 0 and are non-negative.

Definitions of regularity All of the optimality conditions hold and can be given economic meaning in a regular problem. The simple fishery is an instructive example of a non-regular problem. With the stock $X > X^{MSY}$, the marginal unit of stock does not contribute at all to the maximin value, which is $F\left(X^{MSY}\right)$. Its shadow price is zero. Thus, it is possible on many legitimate maximin paths that $U\left(C_1, C_2\right) - \underline{U} = U\left(C_1, C_2\right) - m\left(X_1, X_2\right) > 0$, so that some generations may enjoy a level of well-being that is strictly higher than the maximin level to which generations are held once the stock has declined to X^{MSY}. When $U\left(C_1, C_2\right) - m\left(X_1, X_2\right) > 0$, $w = 0$. Finally, Cairns and Long (2006) show that the simple fishery fails a sufficient condition called the constraint qualification.

Therefore, defining a non-regular problem is mathematically subtle. One can propose a number of definitions for a regular problem:

1. Utility is constant, $\forall t$

[5] They are necessary only for a regular maximin path with positive shadow values of the stocks.

2. $U(C_1, C_2) = m(X_1, X_2)$, $\forall t$ (*true for the second type of non-regularity, different from 1*)

3. $w > 0$, $\forall t$

4. $p > 0$, $\forall t$

5. All of the necessary conditions hold, $\forall t$;

6. The constraint qualification holds, $\forall t$.

The first is the motivating characteristic, and the others are mathematical conditions. Condition 6 is a sufficient, not a necessary like conditions 2, 3, 4 and 5.

The upshot is that a non-regular problem is characterized by the failure of conditions that typically apply in optimal-control problems. Mechanical maximization is not the way to solve a non-regular problem. Another method of solution is required. The failure of these conditions is the mathematical reason for the failure of intergenerational equality (constant utility through time). Hartwick's (1977) net-investment rule that $H = 0$ is trivial on a non-regular path, since $p = 0$ on an interval: it is not a condition for choice of the changes of the components of X.

Asako (1980) first noticed the possibility of non-regularity in a maximin program. Two types of non-regularity have been studied by Doyen and Martinet (2012) and Cairns and Martinet (2014). The models of Sections 2 and 3 are examples of the first. They are examples of redundant stocks—of stocks that are beyond what is known in growth theory as the golden-rule level. The second type is not examined here. It arises when there is a constraint on smoothing utility. Cairns and Tian (2010) studied one example, in which a labor force could not grow fast enough in the context of the economy. Another, heuristic example is that with some sorts of technical progress, some generations may become successively richer without being able to redistribute utility backward in time.

5 Perfect complementarity

An issue raised by environmentalists and ecologists is the possibility that some natural assets do not have substitutes. Accordingly, we begin with the case of perfect complementarity. It allows us to emphasize the way considering two reproducible assets modifies the results for a single asset, in particular in terms of non-regularity.

Suppose that the two resources are perfectly complementary in utility. The utility function is written as

$$U(C_1, C_2) = \min\{a_1 C_1, a_2 C_2\}, \quad a_1, a_2 > 0, \tag{15}$$

where a_1 and a_2 are fixed proportions in which the resources are combined to provide utility.

Let $C_i^\star(X_i(0))$ denote the maximal sustainable level of consumption for each stock given its initial state $X_i(0)$.[6] This quantity corresponds either to the production $F_i(X_i(0))$ if $X_i(0) < X_i^{MSY}$ or to Y_i if $X_i(0) \geq X_i^{MSY}$. The stock with the lowest value $a_i C_i^\star$ is called the limiting sustainable utility. We can thus define the maximin value as follows:

$$m(X_1, X_2) = \min\{a_1 C_1^\star(X_1), a_2 C_2^\star(X_2)\} \ . \tag{16}$$

Usually, the maximin value is determined by the "limiting resource". This notion of "limiting resource" is subtle because of the dynamic dimension of sustainability encompassed in the $C_i^\star$.

There are two ways for a resource to be limiting in our problem. (1) The resource can be scarce relative to the other resource given its current stock and impose a limit on current utility. We use the terminology "locally limiting" then. (2) The resource can be relatively less productive in the long-run, notwithstanding short-term scarcity, and impose a limit on long-run utility. We use the terminology "globally limiting" then.

Globally limiting resource To illustrate the influence of a "globally limiting resource", consider large resource stocks (e.g., virginal natural resource stocks) characterized by $(X_1(0), X_2(0)) > \left(X_1^{MSY}, X_2^{MSY}\right)$. Clearly, neither resource limits current utility, as under our assumptions the extraction at initial time is not bounded for either stock. In this case, $C_i^\star(X_i(0)) = Y_i$ for both resources. The maximin value for such an initial state is $m(X_1(0), X_2(0)) = \min\{a_1 Y_1, a_2 Y_2\}$. This level is the upper bound for the maximin

[6]This quantity corresponds to the maximin value for a single resource problem.

value over the whole state domain. It is a generalization of the Maximum Sustainable Yield in the single-resource case. We thus have a non-regularity of the redundant stock type (Asako, 1980; Doyen and Martinet, 2012). This non-regularity affects the maximin shadow values of the stocks as well. When $a_i Y_i < a_j Y_j$, stock i puts a tighter limit on sustainable utility than stock j.

Also let $X_j^\star < X_j^{MSY}$ such that $a_j F_j(X_j^\star) = a_i F(X_i^{MSY}) = a_i Y_i$. This is the smallest stock for resource j such that Y_i is limiting.[7] Increasing stock j above level $X_j^\star$ does not increase the maximin value. Therefore, stock j's maximin value is zero. More generally, any state $(X_i(0), X_j(0)) \geq (X_i^{MSY}, X_j^\star)$ has a maximin value $m(X_i(0), X_j(0)) = a_i Y_i$, and both resource stocks above these levels are redundant and have nil maximin shadow values.

Locally limiting resource If both stocks are not redundant, i.e., if $(X_i(0), X_j(0)) < (X_i^{MSY}, X_j^\star)$, then $m(X_i(0), X_j(0)) = \min\{a_i F_i(X_i(0)), a_j F_j(X_j(0))\}$. One resource is "locally limiting." Let $a_k F_k(X_k(0)) < a_l F_l(X_l(0))$, i.e., stock k is limiting current utility. The current utility and the maximin value are given by the production of the "locally limiting" resource, which may not be the globally limiting one. We can define a stock $\tilde{X}_l(X_k(0)) < X_l(0)$ such that $a_l F_l(\tilde{X}_l) = a_k F_k(X_k(0))$. This is the smallest stock for resource l such that resource k is limiting. One has $m(X_k(0), X_l(0)) = m(X_k(0), \tilde{X}_l(X_k(0)))$. The non-limiting resource has no marginal maximin value above stock $\tilde{X}_l(X_k(0))$. Its stock is redundant, even though it is below its MSY value.

Graphical representation Figure 2 is a four-quadrant graph, in which the north-east (NE) quadrant represents production $F_1(X_1)$ as a function of stock X_1 and the south-west (SW) quadrant, production $F_2(X_2)$ as a function of stock X_2. The north-west (NW) quadrant plots utility as a function of the consumption levels (C_1, C_2). The south-east (SE) quadrant is the state map (X_1, X_2) in which state trajectories can be drawn as well as level curves for the maximin value.

In the case illustrated, resource 2 is globally limiting ($a_1 Y_1 > a_2 Y_2$). All the representations are derived from the results above with $i = 2$ and $j = 1$. Any stock

[7] If stock X_j decreases below level $X_j^\star$, resource j becomes locally limiting in place of the Y_i. See the discussion of a "locally limiting" resource below.

$(X_1, X_2) \geq (X_1^\star, X_2^{MSY})$ has a maximin value equal to upper bound $a_2 Y_2$. The hatched area corresponds redundant stocks.

For any pair $(X_1, X_2) < (X_1^\star, X_2^{MSY})$, the maximin value is given by the locally limiting resource. The curve AB, from $(0, 0)$ to $(X_1^\star, X_2^{MSY})$, represents the states for which $a_1 F_1(X_1) = a_2 F_2(X_2)$. For these states, both resources are limit utility; there is no redundant stock along AB. The maximin entails staying at the original point (X_1, X_2). The corresponding sustained level of utility (the maximin value) is given by $U(C_1, C_2)$, for consumption levels in (NW) corresponding to (X_1, X_2). The maximin value increases along AB, from 0 at $(0, 0)$ to $a_2 Y_2$ at $(X_1^\star, X_2^{MSY})$. Both stocks have nil maximin shadow values because increasing one stock alone does not increase the maximin value. The "cross marginal value" m''_{X_1, X_2}, however, is positive.

For all states East of the curve, resource 2 is locally limiting. A part of stock X_1 is redundant. Maximin value is given by the point on the curve directly west. Stock X_2 has a positive marginal maximin value. For states South of the curve, resource 1 is limiting, with a similar interpretation. Iso-maximin-value curves are given by perpendicular lines starting from any state on AB.

Maximin value, shadow values and non-regularity Having two complementary renewable resources thus increases the redundancy of the stocks, either globally (one resource stock may be redundant due to the level of the other resource's Maximum Sustainable Yield) or locally (one stock is redundant because the other's current production is low).

The maximin value is non-decreasing function in (X_i, X_j) with an upper bound $a_i Y_i$ for stocks above $(X_i^{MSY}, X_j^\star)$. When both stocks are strictly above $(X_i^{MSY}, X_j^\star)$, the maximin value is equal to the upper bound $a_i Y_i$. From any state in this area of double redundancy, one can momentarily increase the utility and the maximin path may be non-regular in the sense that utility can be larger than the maximin value. Both stock's maximin shadow values are nil in this area.

When the current utility is limited by one of the resource stocks, for the non-limiting stock j, $p_j = 0$ but for the limiting stock i, $p_i > 0$. In this case, utility cannot be increased above the maximin value even if one stock is redundant. The redundant stock cannot be overconsumed to increase utility.

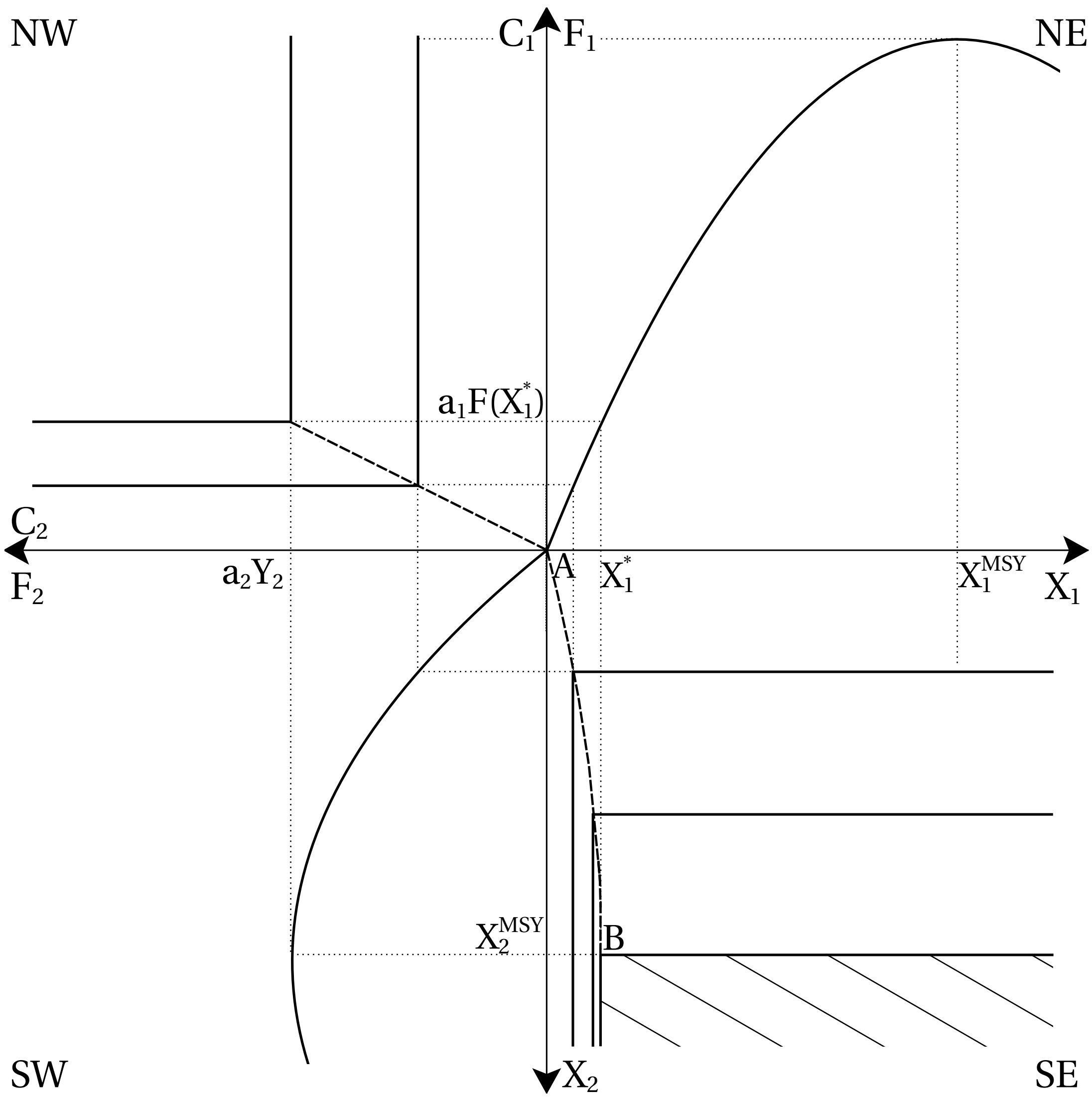

Figure 2: Perfect complements

6 Diminishing marginal substitutability

Consider a positively valued utility function $U(C_1, C_2)$ such that $U'_{C_i}(0, C_j) = +\infty$ (both goods are essential in consumption), $U'_{C_i} > 0$ and $U''_{C_i,C_i} < 0$ (both goods contribute positively to consumption but at a decreasing marginal rate) and $U''_{C_i,C_j} > 0$.[8] Without loss of generality, let $U(0,0) = 0$; in the worst case utility is equal to nil. The following makes sense only when both stocks are positive, i.e., $X_i(0) > 0$ for $i = 1, 2$. If this is not the case, one is back to the single resource problem.

Under these conditions, and given Lemma 1, one can say that $C_i > 0$ on a maximin path. The problem cannot be solved by general reasoning and we use the direct approach presented in Section 4.

Optimality conditions From (8), we get the expression of the shadow values of the stocks in utility terms

$$p_i = wU'_{C_i}, \quad i = 1, 2. \tag{17}$$

At the optimum, the shadow value of an extra unit of stock i (the value of investment in stock i) is equal to the marginal utility of consumption weighted by the shadow value of the equity constraint (14). As any unit of consumption reduces investment (see 2), one has to be indifferent between consuming an extra unit or investing it.

Combining this result for the two stocks, and defining the relative price $\pi = p_1/p_2$, one gets $\pi = \frac{U'_{C_1}}{U'_{C_2}}$. Along an optimal path, relative prices are equal to the marginal rates of substitution in consumption.

From Conditions (9), one has

$$\dot{p}_i = -p_i F'_i(X_i) \quad \Leftrightarrow \quad \frac{\dot{p}_i}{p_i} = -F'_i(X_i), \quad i = 1, 2. \tag{18}$$

Each stock's shadow price decreases at a rate equal to its marginal productivity.

Shadow value of equity and discounting Taking the time derivative of eq. (17) and using eq. (18), we obtain the following conditions:

$$F'_i(X_i) = -\frac{\dot{w}}{w} - \frac{\dot{U}_{C_i}}{U'_{C_i}}, \quad i = 1, 2 \tag{19}$$

---

[8] If, in addition, the utility function is homothetic, substitutability can be measured by Hicksian 1932-elasticity $\chi = \frac{U'_{C_1}U'_{C_2}}{U''_{C_1,C_2}}$. Goods are complements in the limit where $\chi \to 0$ and perfect substitutes in the limit where $\chi \to +\infty$. The Cobb-Douglas case corresponds to $\chi = 1$.

or

$$\delta = -\frac{\dot{w}}{w} = F_i'(X_i) + \frac{\dot{U}_{C_i}}{U_{C_i}} = F_i'(X_i) + \frac{\dot{p}_i}{p_i}.$$

The LHS of this equation is the utility-discount rate, δ. This rate is equalized across the products $i = 1, 2$, by arbitrage of the values of the stocks. The first term on the RHS, $F_i'(X_i)$, can be interpreted as a dividend taken from the stock, at the margin, on the optimal path. The rate of change in the price is a capital gain (or loss) on the stock. At the margin the rate of return from each stock, in terms of utility as numeraire, is equalized at δ. In utility units, the rate of discount that applies to stock i is δ. The two stocks pay different dividends, which are grossed up to δ (as a no-arbitrage condition), by the changes in price. If accounting is done in terms of C_1 as numeraire then its price v_1 does not vary (there is no capital gain on the numeraire: $\dot{v}_1 = 0$). Its return is $F_1'(X_1) = r_1$, the rate of interest for C_1 as numeraire, and by arbitrage $r_1 = F_2'(X_2) + \dot{v}_2/v_2$.

Steady state A steady state ($\dot{C}_1 = \dot{C}_2 = 0$ and $\dot{X}_1 = \dot{X}_2 = 0$) is characterized by an equality of the marginal productivity of the stocks (we denote steady states values with asterisks):[9]

$$F_i'(X_i^*) = F_j'(X_j^*). \tag{20}$$

Furthermore, at a steady state at time t, we have

$$m(X_1^*, X_2^*) = U(F_1(X_1^*), F_2(X_2^*)); \tag{21}$$

$$w(s) = \mathrm{e}^{-F_i'(X_i^*)(s-t)}, \quad p_i(s) = U'_{C_i^*}\mathrm{e}^{-F_i'(X_i^*)(s-t)}, \quad i = 1, 2. \tag{22}$$

Transition path When the economy is not at a steady state, it follows a dynamic path with the depletion of one of the stocks (Lemma 2). Cairns and Long (2006, Proposition 1) show that as long as the necessary conditions hold, the Hamiltonian is nil over time.

$$\mathcal{H}(X, C, p) = 0 \quad \Leftrightarrow \quad p_1\dot{X}_1 + p_2\dot{X}_2 = 0. \tag{23}$$

[9]It is worth noting here that since conditions (17)-(19) remain valid for n stocks, this condition is a general one.

Equation (23) is known as the converse of Hartwick's rule (Dixit et al., 1980; Withagen and Asheim, 1998). It states that, if utility remains constant at the maximin level, i.e. $U(C_1, C_2) = m(X_1, X_2)$, net investments are nil over time. This implies that one stock increases while the other decreases, i.e., $C_i < F(X_i)$ and $C_j > F(X_j)$. It is of interest to determine which stock is used up to build the other one.

Since the 'equity' shadow value ω is the same for all stocks, one has from (19)

$$F_i' - F_j' = \dot{C}_i \left(\frac{U''_{C_j C_i}}{U'_{C_j}} - \frac{U''_{C_i C_i}}{U'_{C_i}} \right) - \dot{C}_j \left(\frac{U''_{C_i C_j}}{U'_{C_i}} - \frac{U''_{C_j C_j}}{U'_{C_j}} \right) \tag{24}$$

Under the properties of the utility function assumed above (increasing, concave, and $U''_{C_i C_j} > 0$), the expressions in both parenthesis are positive. If $F_i' > F_j'$, then a simple rearrangement yields

$$\dot{C}_i > \dot{C}_j \left(\frac{U''_{C_i C_j}}{U'_{C_i}} - \frac{U''_{C_j C_j}}{U'_{C_j}} \right) / \left(\frac{U''_{C_j C_i}}{U'_{C_j}} - \frac{U''_{C_i C_i}}{U'_{C_i}} \right)$$

IF $\dot{C}_j > 0$ then $\dot{C}_i > 0$. Both levels of consumption cannot increase along the maximin path, where utility is constant over time. However, having $\dot{C}_j < 0$ is compatible with having $\dot{C}_i > 0$. Eventually, either $F_i' = F_j'$, or else $C_i < F_i(X_i)$. (If $C_i > F_i(X_i)$ always, the stock is exhausted in finite time.) Hence, if $F_i' > F_j'$, stock i, which is more productive at the margin, is built up and resource stock j is depleted.

On regular parts, either stocks are at a steady state (X_1^*, X_2^*) such that marginal productivities are equal $(F_1'(X_1^*) = F_2'(X_2^*))$ or the stock with the highest productivity is built up by setting $C_i < F(X_i)$ (*under-consumption*) and by compensating the utility loss by an *over-consumption* of the less productive stock $(C_j > F(X_j))$ until the economy reaches a steady state. Under and over-consumptions decreases as the gap between marginal productivity decreases and the state gets closer to the equilibrium.

Maximin value, maximin prices and durable savings The partial derivatives of the maximin value give the shadow price of the stocks, i.e., $\frac{\partial m(X_1, X_2)}{\partial X_i} = p_i(X_1, X_2)$. From condition (17), one gets $\frac{m'_{X_1}}{m'_{X_2}} = \frac{U'_{C_1}}{U'_{C_2}}$. At the optimum, the marginal rate of transformation equals the marginal rate of substitution.[10]

Durable investment reads $\frac{\mathrm{d}m(X_1,X_2)}{\mathrm{d}t} = \frac{\partial m}{\partial X_1}\dot{X}_1 + \frac{\partial m}{\partial X_2}\dot{X}_2 = p_1\dot{X}_1 + p_2\dot{X}_2$. According to Hartwick's rule, $p_1\dot{X}_1 + p_2\dot{X}_2 = 0$. Using eq. (17), this implies

$$\frac{\dot{X}_1}{\dot{X}_2} = -\frac{p_2}{p_1} = -\frac{U'_{C_2}}{U'_{C_1}} \ . \tag{25}$$

Along a regular maximin path, consumption is constant. One has $\frac{\mathrm{d}U(C_1,C_2)}{\mathrm{d}t} = \dot{C}_1 U'_{C_1} + \dot{C}_2 U'_{C_2} = 0$. This gives us

$$\frac{\dot{C}_1}{\dot{C}_2} = -\frac{U'_{C_2}}{U'_{C_1}} \ . \tag{26}$$

Along an optimal path, stocks are either at their steady state or vary in opposite directions according to the marginal utility ratio. Combining conditions (25) and (26), one gets

$$\frac{\dot{X}_1}{\dot{X}_2} = \frac{\dot{C}_1}{\dot{C}_2} \quad \Leftrightarrow \quad \frac{\mathrm{d}X_1}{\mathrm{d}X_2} = \frac{\mathrm{d}C_1}{\mathrm{d}C_2} \ . \tag{27}$$

That gives us information on the shape of the iso-maximin values in the sates space for regular paths.

Regarding the prices, we can state the following. Here again, we assume that resource X_i is relatively less abundant than resource X_j in the sense $F'_i(X_i) > F'_j(X_j)$. Differentiating the price ratio $\pi = \frac{p_i}{p_j}$ with respect to time and using conditions (18), we get

$$\frac{\dot{\pi}}{\pi} = \frac{\dot{p}_i}{p_i} - \frac{\dot{p}_j}{p_j} = -F'_i(X_i) + F'_j(X_j) < 0. \tag{28}$$

The previous equation tells us that relative prices have to decrease, so, from the condition (27), the tangents (in absolute value) to the paths in the states space (X_i, X_j) have to decrease as well. One moves down along iso-maximin curves, which are convex in the state space, to equilibrium at which the marginal productivity of the two resources equalizes.

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[10] This is also the case in a discounted-utilitarian formulation, see for example Weitzman (1976), Arrow et al. (2012, p. 652).

Graphical representation Figure 3 is a plot of the solution, with quadratic curves used to represent production levels. Other conventions are the same as for Figure 2.

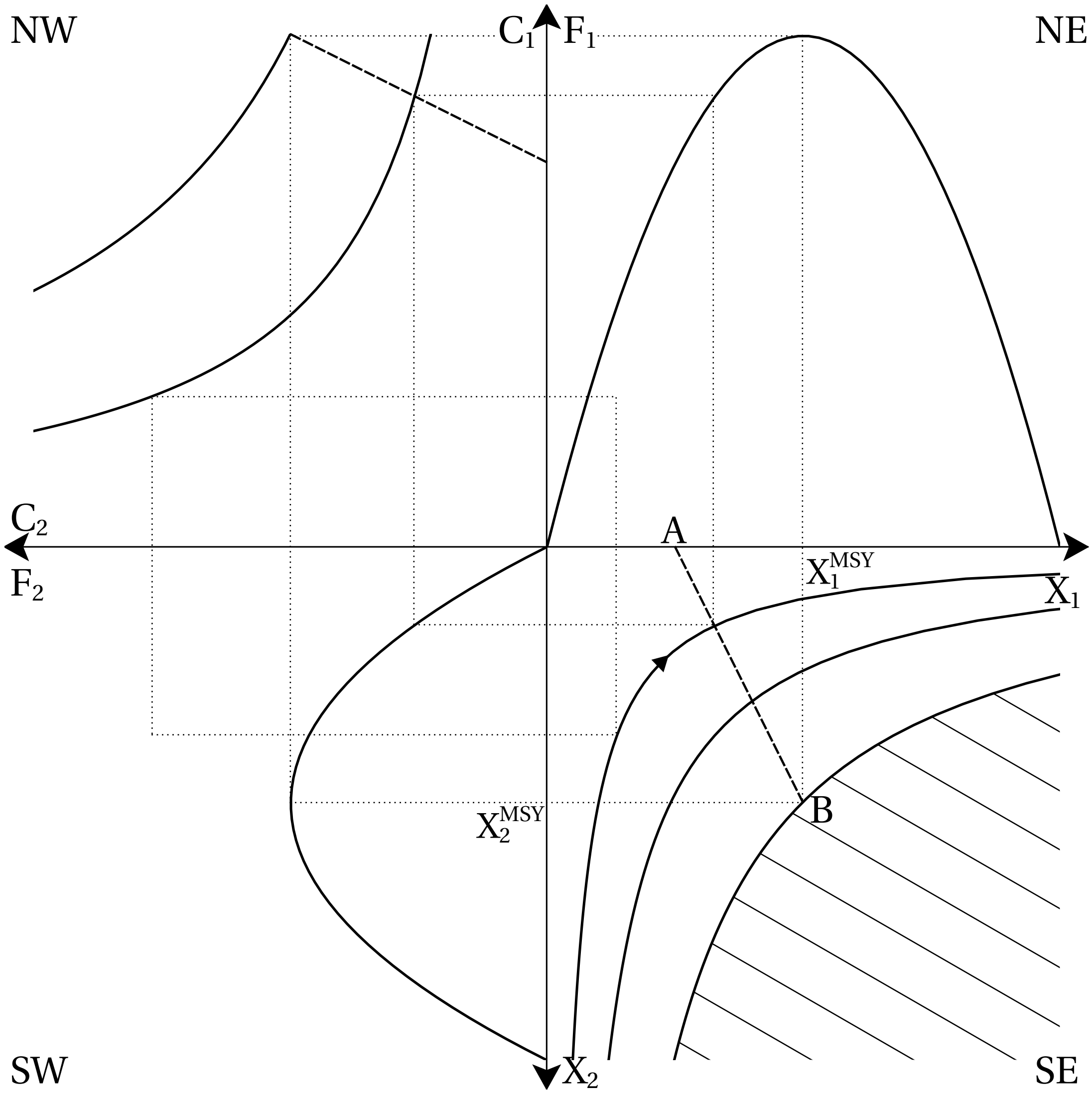

Figure 3: Imperfect substitutes

The straight line[11] AB in the state space (SE) corresponds to the states for which $F_i'(X_i) = F_j'(X_j)$, and thus to equilibria for the maximin problem. The corresponding

[11]This is a straight line because we assumed quadratic growth functions.

maximin value is given by the utility at equilibrium in (NW).

South-West of the equilibrium line, $F_1'(X_1) > F_2'(X_2)$. Stock 2 is less productive at the margin. Consumption of stock 2 exceeds its production, while consumption of stock 1 is lower than its production. The trajectory goes North-East toward the equilibrium line. Along that trajectory, utility is constant at the maximin value. The two consumption levels converge toward an equilibrium value as the state converges to an equilibrium with equal marginal productivities. The same pattern occurs (*mutatis mutandis*) north-east of the equilibrium line.

From any state in the hatched area, maximin value is equal to that of the state (X_1^{MSY}, X_2^{MSY}). Stocks are redundant and have maximin values of zero. For any state North-West of the hatched area, maximin prices are positive.

About non-regularity Here again, the hatched area represents a plateau in the maximin value function; all paths starting from a state in this area are non-regular in the sense that one can reach a utility larger than the maximin value for some time, until the state reaches the equilibrium (X_1^{MSY}, X_2^{MSY}) and the utility equals the maximin value. Along such paths (including the limiting case at the edge of the hatched area), prices are nil.

The interesting feature with two resources substitutable in utility is that the non-regular part is "smaller" than in the single-resource case. If one stock is below its MSY level and the other one above, a regular path may be obtained. The less productive is used in order to let the other one grow.

A remark on the equilibrium condition It seems anomalous that the equilibrium condition for a steady state is that

$$F_1'(X_1) = F_2'(X_2)\,,$$

and that there is no role for the utility function in the condition. In general, on the optimal path, equations (25) and (27) state that

$$\frac{\mathrm{d}C_1}{\mathrm{d}C_2} = \frac{\mathrm{d}X_1}{\mathrm{d}X_2} = \frac{\dot{X}_1}{\dot{X}_2} = -\frac{U'_{C_2}}{U'_{C_1}}\,.$$

The last equality indicates that the marginal utilities govern the slopes $\frac{dC_1}{dC_2}$ and $\frac{dX_1}{dX_2}$ at any time in dynamic equilibrium (as well as in the steady state).

In the steady state, $\dot{X}_i = 0, i = 1, 2$, so that

$$F_i(X_i) = C_i, \text{ or } F_i'(X_i)\,dX_i = dC_i. \tag{29}$$

Dividing through both sides of equation (29) yields that

$$\frac{F_1'(X_1)\,dX_1}{F_2'(X_2)\,dX_2} = \frac{dC_1}{dC_2},$$

so that eq. (29) implies that in the steady state,

$$F_1'(X_1) = F_2'(X_2).$$

The marginal productivities do help to determine the steady-state equilibrium in that they govern the relative values of the infinitesimals dC_i or dX_i. The opportunity cost of a unit of consumption of i in terms of the stock is

$$\frac{dC_i}{dX_i} = F_i'(X_i) \ .$$

and these opportunity costs are equalized as in eq. (27).

7 Perfect substitutes

We now assume that the two resources are perfectly substitutable, as, for example, two groundwater stocks with different qualities or two fish stocks with different nutritive values. This case also covers the more general problem of sustaining a small economy with trade, as long as the terms of trade are not changing.[12]

[12] An economy exporting two goods at given prices has an utility monotonically increasing with the value of the production, whatever the actual preference for consumption goods (even complementarity). It thus aims at sustaining the export values. This follows from the separating hyperplane theorem.

The utility function is assumed to be linear in the consumption of each resource:

$$U(C_1, C_2) = a_1 C_1 + a_2 C_2. \tag{30}$$

In this case, we cannot exclude nil consumptions.

Optimality conditions The necessary conditions (8) become, for $i = 1, 2$:

$$\frac{\partial \mathcal{L}}{\partial C_i} = -p_i + a_i w \leq 0 \ , C_i \geq 0 \ , C_i \frac{\partial \mathcal{L}}{\partial C_i} = 0 \ , \tag{31}$$

the other conditions being unchanged. The complementary slackness reads

$$w \geq 0, \ w \left(a_1 C_1 + a_2 C_2 - \underline{U}\right) = 0. \tag{32}$$

From conditions (9), we get

$$-\frac{\partial \mathcal{L}}{\partial X_i} = -p_i F_i' = \dot{p}_i, \quad \Leftrightarrow \quad \frac{\dot{p}_i}{p_i} = -F_i' \tag{33}$$

By Lemma 1, $C_1 = C_2 = 0$ cannot be solution of the problem.

Steady state Suppose that both consumption levels are strictly positive, i.e., $C_i > 0, i = 1, 2$. Eq (31) then imply that $\frac{\partial \mathcal{L}}{\partial C_i} = 0$ and thus that $w = \frac{p_1}{a_1} = \frac{p_2}{a_2}$. Since $(p_1, p_2, w) \neq (0, 0, 0)$, $w > 0$ and thus $U(C_1, C_2) = \underline{U}$. The maximin path is regular in this case.

Taking the time derivative of w, we get $\frac{\dot{w}}{w} = \frac{\dot{p}_1}{p_1} = \frac{\dot{p}_2}{p_2}$. Moreover, from Eq. (33), we get the conditions $\frac{\dot{p}_i}{p_i} = -F_i'(X_i)$. Combining these two conditions, we can state that a solution with $C_1 > 0$ and $C_2 > 0$ is possible only for states (X_1, X_2) such that $F_1'(X_1) = F_2'(X_2)$. This is dynamically possible only for a stationary state. We thus get the maximin value $m(X_1, X_2) = U(F_1(X_1^*), F_2(X_2^*))$.

Transition paths For any initial state (X_1, X_2) such that $F_1'(X_1) \neq F_2'(X_2)$, the maximin solution cannot correspond to positive consumption of both stocks. The following Proposition states that the asset with the lower marginal productivity is used up while the more productive asset builds up.

Proposition 1. *If $F_i'(X_i) < F_j'(X_j)$ and $F_j'(X_j) > 0$, the maximin path is characterized by $C_i \geq F_i(X_i) > 0$ and $C_j = 0$. Moreover, the gap between marginal productivities is reduced until either $F_i'(X_i) = F_j'(X_j)$ or $X_i = 0$.*

Proposition 2. *Assume that $F_j'(0) > F_i'(0)$, and denote by $X_j^0 > 0$ the stock level such that $F_j'(X_j^0) = F_i'(0)$. The maximin path starting from any state (X_i, X_j) such that $F_i'(X_i) < F_j'(X_j)$ and $m(X_i, X_j) \leq a_j F_j(X_j^0)$ exhausts asset X_i.*

Maximin shadow prices From the necessary conditions, we have $w = \frac{p_i}{a_i}$ and $\frac{\dot{w}}{w} = \frac{\dot{p}_i}{p_i} = -F_i'(X_i)$. One also has $-p_j + a_j w \leq 0 \Leftrightarrow p_j \geq \frac{a_j}{a_i} p_i \Leftrightarrow \frac{p_j}{p_i} \geq \frac{a_j}{a_i}$. Given that $p_i = m'_{X_i}$ and $a_i = U'_{C_i}$, one gets that the marginal rate of transformation of maximin value is greater than the marginal rate of substitution of consumption.

Let us define the relative price $\pi = \frac{p_i}{p_j} (\leq \frac{a_i}{a_j})$ and examine how this relative price evolves over time. Given conditions (33), we obtain

$$\frac{\dot{\pi}}{\pi} = \frac{\dot{p}_i}{p_i} - \frac{\dot{p}_j}{p_j} = -F_i'(X_i) + F_j'(X_j) > 0 \ . \tag{34}$$

The relative price of stock i rises. Since one harvests only the stock i, $F_i'(X_i)$ increases and $F_j'(X_j)$ decreases until the marginal levels of growth are equal.[13] In other words, if one begins with a resource relatively abundant and with a low relative (shadow) price, one has to harvest only that resource at a level that keeps utility constant. Meanwhile the other one grows until the steady state at which both resources have the same marginal productivity.

From condition (27), the optimal path is characterized by

$$-\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = \frac{p_1}{p_2}, \tag{35}$$

but contrary to the general case, here the utility function and the maximin value do not have the same shape.

[13]One cannot pass from $\frac{p_i}{p_j} < \frac{a_i}{a_j}$ to $\frac{p_i}{p_j} > \frac{a_i}{a_j}$ since the equality is a steady state of the system.

Graphical representation A solution is plotted in the four-quadrant plot in Figure 4. The interpretation is the same as that of Figure 3, except that one has corner solutions for the consumption. A path exhausting resource 1 illustrates Proposition 2.

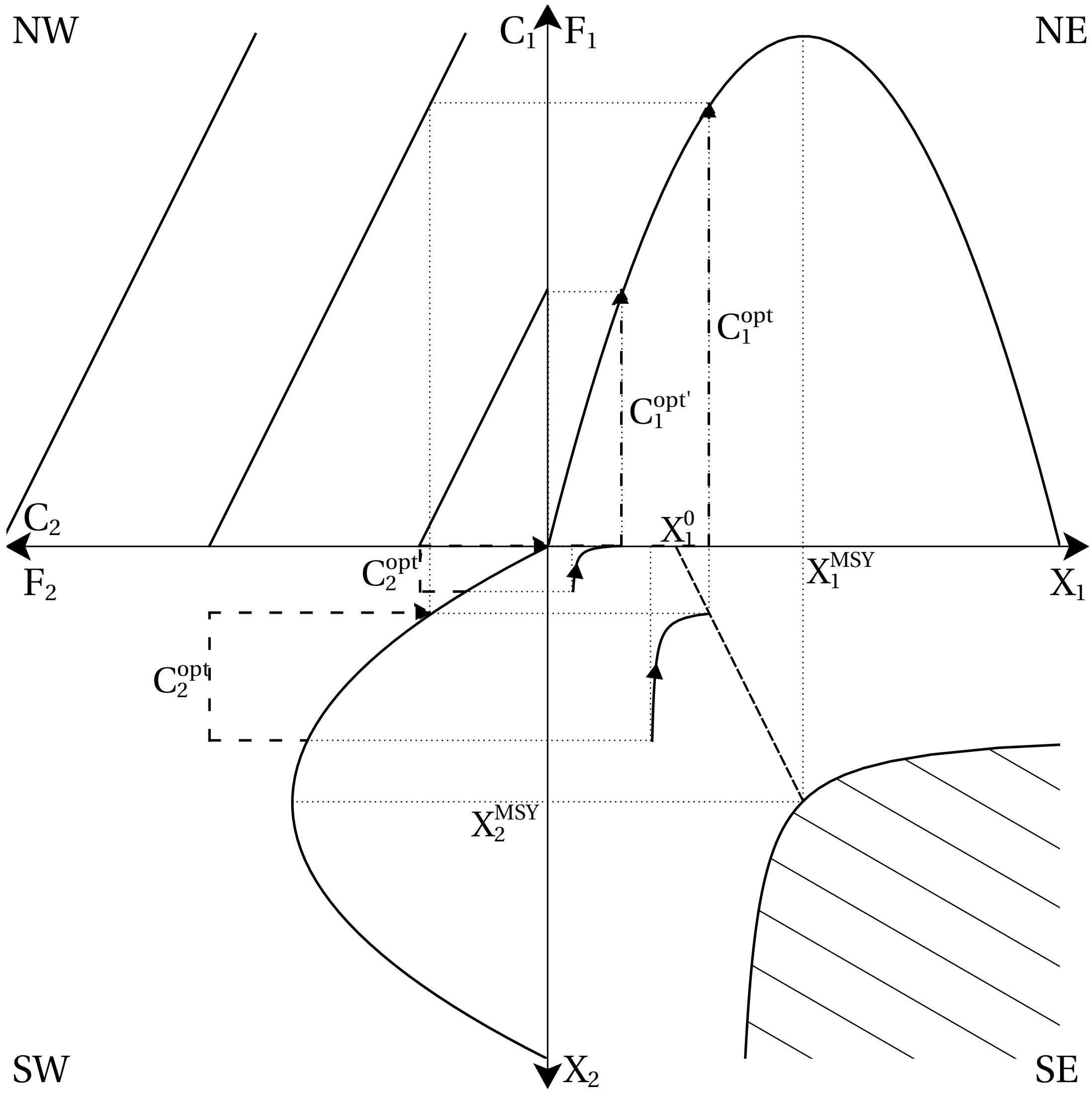

Figure 4: Perfect substitutes

8 Discussion

The idea of sustainability has a resonance with many people. "Progress" and not decline, has been sought by ambitious generations, but some have questioned whether continuing progress might some day give way to a maintenance objective, often one that is within natural constraints. These are economic notions and can be addressed using economic tools.

Two important questions have been addressed in this paper, namely,

- to understand more fully the implications of non-regularity in more realistic problems;
- to characterize more fully the properties of maximin prices with the aim of eventually being able to estimate them for decentralization to economic decision-makers.

Non-regularity is a persistent phenomenon in the study of maximin. The investigation in the present paper appears to imply that non-regularity can arise where there is complementary between different natural assets. Environmentalists who insist that produced capital is not substitutable for natural capital may be chagrined to learn that, (a) where prices cannot be defined and decentralized, it is because there is a non-regularity in which some actions on quantities are non-unique and difficult to characterize, and that (b) seeking to attain maximum sustainable yields of renewable resources may not be even close to socially desirable and that (c) if indifference curves meet axes, extinction may be optimal. Neoclassical economists may be unhappy that perfect substitutability gives rise to a form of non-regularity. To guarantee regularity in a given problem, it may be that it is necessary to postulate that indifference curves are asymptotic to axes. If the elasticity of marginal utility can be defined by a single parameter, then increasing substitutability decreases the range of initial stocks for which there are regular, sustainable paths.

A Proofs

Proof of Lemma 1 The dynamic path $\dot{X}_i = 0$ driven by decisions $C_i = F(X_i)$ is feasible and yields the constant utility $U(F_1(X_1), F_2(X_2))$. This provides a lower bound for the maximin value.

Proof of Lemma 2 This is a direct result from Lemma 1 and the dynamics.

Proof of Proposition 1 Consider a state $(X_i, X_j) > (0,0)$ such that $F_i'(X_i) < F_j'(X_j)$ and $F_j'(X_j) > 0$. This last condition ensures that the "less abundant resource" is below its MSY level X_j^{MSY}.[14] We shall demonstrate that, under these conditions, stock X_i is consumed alone while stock X_j builds up as long as the previous inequality holds by proving that the opposite is not possible.

Assume that $C_j > 0$ and $C_i = 0$. Let us denote the maximin value by m. Along the maximin path, on has $U(C_i, C_j) = a_j C_j = m$. By Lemma 2, we get $C_j \geq F_j(X_j)$. Stock X_j decreases while stock X_i increases, which implies that F_j' increases and F_i' decreases. The condition $F_i'(X_i) < F_j'(X_j)$ holds all over the path, and thus the described consumption pattern goes on. Stock X_j is exhausted in a finite time τ after which utility is derived only from the sustained consumption of stock X_i. At exhaustion time, stock X_i has increased to a level $X_i(\tau) \equiv X_i^*$ such that $F_i(X_i^*) = m/a_i$, allowing consumption C_i to sustain exactly the maximin utility. The equilibrium state is thus $(X_i^*, 0)$.

Let us make a step backward and examine the states through which the path goes just prior to exhaustion. The dynamics before exhaustion is

$$\begin{aligned} \dot{X}_i = F_i(X_i) \quad &\Leftrightarrow \quad \mathrm{d}X_i = F_i(X_i)\mathrm{d}t \\ \dot{X}_j = F_j(X_j) - C_j \quad &\Leftrightarrow \quad \mathrm{d}X_j = \left(F(X_j) - m/a_j\right)\mathrm{d}t \end{aligned}$$

Consider an infinitesimal time laps $\mathrm{d}t$. At time $\tau - \mathrm{d}t$, stock X_i is equal to $\tilde{X}_i = X_i^* - \mathrm{d}X_i = X_i^* - \frac{m}{a_i}\mathrm{d}t$. Stock X_j is equal to $\tilde{X}_j = dX_j = \frac{m}{a_j}\mathrm{d}t$.

By Lemma 1, we know that the maximin value at time $\tau - \mathrm{d}t$ is greater than or equal to the equilibrium utility of state $(\tilde{X}_i, \tilde{X}_j)$. Let us denote this utility level by $\tilde{U} = U(\tilde{X}_i, \tilde{X}_j) = a_i F_i(X_i^* - \frac{m}{a_i}\mathrm{d}t) + a_j F_j(\frac{m}{a_j}\mathrm{d}t)$. We have $m(\tau - \mathrm{d}t) \geq \tilde{U}$. By substracting

[14] If both marginal productivities are negative, both stocks are above their MSY value X^{MSY} and we are in a non-regular case in which the two stocks decrease and converge to a steady state at (X_i^{MSY}, X_j^{MSY}). Proposition 1 is relevant only if one stock is below its MSY level.

$m(\tau)$ to both sides of the equation, we obtain the following.

$$\begin{aligned}
m(\tau - \mathrm{d}t) - m(\tau) &\geq \tilde{U} - m(\tau) \\
&\geq a_i F_i\left(X_i^* - \frac{m}{a_i}\mathrm{d}t\right) + a_j F_j\left(\frac{m}{a_j}\mathrm{d}t\right) - a_i F_i(X_i^*) \\
&\geq a_i\left(F_i\left(X_i^* - \frac{m}{a_i}\mathrm{d}t\right) - F_i(X_i^*)\right) + a_j\left(F_j\left(0 + \frac{m}{a_j}\mathrm{d}t\right) - F_j(0)\right) \\
&\geq -m\mathrm{d}t\frac{F_i\left(X_i^* - \frac{m}{a_i}\mathrm{d}t\right) - F_i(X_i^*)}{-\frac{m}{a_i}\mathrm{d}t} + m\mathrm{d}t\frac{F_j\left(0 + \frac{m}{a_j}\mathrm{d}t\right) - F_j(0)}{\frac{m}{a_j}\mathrm{d}t}
\end{aligned}$$

Let us write $\epsilon_i = -\frac{m}{a_i}\mathrm{d}t$ and $\epsilon_j = \frac{m}{a_j}\mathrm{d}t$, as well as $\tilde{\tau} = \tau - \mathrm{d}t$ (and thus $\tau = \tilde{\tau} + \mathrm{d}t$). We get

$$\begin{aligned}
m(\tilde{\tau}) - m(\tilde{\tau} + \mathrm{d}t) &\geq -m\mathrm{d}t\frac{F_i\left(X_i^* + \epsilon_i\right) - F_i(X_i^*)}{\epsilon_i} + m\mathrm{d}t\frac{F_j\left(0 + \epsilon_j\right) - F_j(0)}{\epsilon_j} \\
\Leftrightarrow \quad \frac{1}{m}\left(\frac{m(\tilde{\tau}+\mathrm{d}t) - m(\tilde{\tau})}{\mathrm{d}t}\right) &\leq \frac{F_i\left(X_i^* + \epsilon_i\right) - F_i(X_i^*)}{\epsilon_i} - \frac{F_j\left(0 + \epsilon_j\right) - F_j(0)}{\epsilon_j}
\end{aligned}$$

By taking the limits $\epsilon_i, \epsilon_j, \mathrm{d}t \to 0$, we obtain

$$\frac{\dot{m}}{m} \leq \left(F_i' - F_j'\right) < 0 \ .$$

As the maximin value cannot decrease along a maximin path, we get a contradiction.

We thus can say that if $F_i'(X_i) < F_j'(X_j)$, $C_i > 0$ and $C_j = 0$. Moreover, by Lemma 1, one has $a_i C_i \geq a_i F(X_i) + a_j F(X_j) \Rightarrow C_i \geq F(X_i)$ as $F(X_j) > 0$. Stock X_i decreases (and thus F_i' increases) while stock X_j increases (and thus F_j' decreases). The gap between the marginal productivities decreases as long as the inequality holds.

Also, $pi/pj < ai/aj$. Relative prices rise until it equalizing the ratio of marginal utilities. As $-\frac{\mathrm{d}X_2}{\mathrm{d}X_1} = \frac{p_1}{p_2}$, tangents to iso-maximin curves rise in absolute values. One converge to a steady state satisfying $F_j'(X_j) = F_i'(X_i)$.

Proof of Proposition 2 From Proposition 1, if $F_i'(X_i) < F_j'(X_j)$, one consumes only stock C_i as long as the marginal productivities are not equalized. For any $X_j < X_j^0$, one has $F'_j(X_j) > F'_i(0)$. For such X_j, there is no equilibrium satisfying $F'_i(X_i) = F'_j(X_j)$. Stock X_i is exhausted at some time τ with $X_j^* = X_j(\tau) < X_j^0$. Maximin value is $a_j F_j(X_j^*) < a_j F_j(X_j^0)$. Using the reasoning as in the proof of Proposition 1, we can show that $\tilde{U} < m$ at exhaustion. Exhaustion is optimal.

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