Uniform Price Auctions with Asymmetric Entry Costs*

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October 2014

Abstract

Uniform price auctions admit a continuum of inefficient equilibria due to bidders' market power. In this paper, using a framework where bidders cannot anticipate the seller's strategy, I show that the vulnerability of these inefficient equilibria to strategic rationing is increased when bidders have asymmetric entry costs. It is therefore valuable for the seller to attract high cost bidders to the auction because this induces low cost bidders to bid more aggressively in order to eliminate high cost ones. Moreover, there does not exist ex-post optimal equilibria of the auction with non-increasing demand schedules except when the support of the uncertainty about the seller's strategy is infinite.

Keywords: Uniform Price Auctions, Entry Costs, EU ETS, Treasury Auctions.

JEL Classification: D44

1 Introduction

Uniform price auctions are often criticized because they give rise to inefficient equilibria due to bidders' market power. This paper analyzes the existence of such equilibria when there is uncertainty about the seller's strategy and asymmetric entry costs for bidders. I show that this uncertainty induces bidders to bid more aggressively, mitigating their market power and that this effect is enhanced when bidders have asymmetric entry costs.

To reduce carbon dioxide and other greenhouse gas emissions, The European Union has designed in 2003 its emissions trading scheme (ETS). From 2013, the EU will use a uniform price auction to allocate almost 50% of the allowances. The uniform price auction format is also

*Thanks for helpful comments to Bruno Biais, Mauricio Bermudez, Roberto Burguet, Antoni Calvó-Armengol, Jacques Crémer, Denis Gromb, Paul Klemperer, Didier Laussel, Inés Macho-Stadler, Diego Moreno, David Pérez-Castrillo, Jean-Charles Rochet, Jean Tirole and especially Bruno Jullien. I have benefited from comments by participants at the ASSET 2004 conference, the XXIX Simposio de Análisis Económico, the 2nd Europlace Institut of Finance Forum, the Workshop on Auctions and Public Service Procurement, the RES 2013, the EEA 2014, the Lunch Seminar on Game Theory and Social Choice at the Universitat Autonoma de Barcelona, the CCP seminar at University of East Anglia and the Lunch Seminar at the University of Toulouse. All remaining errors are mine.

commonly used in financial and electricity markets to sell a divisible good. UK treasury also sells index-linked bonds using the uniform price auction format. Uniform price auctions are also used in California to buy electricity in the power exchange. Open IPO, a web-based underwriter proposes to sell shares using a uniform price auction.[1]

In a uniform price auction, bidders strategically submit demand schedules for a divisible good and the price is set to equate supply and demand. Within this framework, Wilson (1979)[2] shows that there exists arbitrarily low equilibrium prices that may be sustained by the bidders. The intuition for this result is the following. As a bidder is only concerned with his demand schedule at the market clearing price, the slope is not determined by the equilibrium condition. This indetermination allows them to submit a rather inelastic demand schedule which offsets other bidders' incentives to bid more aggressively. They cannot a get a larger share with only a modest price increase. Consequently, underpricing equilibria emerge.

Wilson (1979) was the first to point that indivisibility is a critical assumption in the revenue equivalence theorem for multi-unit auctions. Multi-unit auctions for an indivisible and homogenous good have the same nice properties than single-unit auctions while share auctions (multi-unit auctions for a perfectly divisible good) have a multiplicity of equilibria, some of them being very inefficient.

In practice, the rules of the uniform price auction are often modified. In the RedEnvelope, Inc. Open IPO, the pro-rata percentage for the entire offering was approximately 56%. Keloharju, Nyborg and Rydqvist (2004) document that the Finnish Treasury never chooses the supply that maximizes revenue given the bids in an auction. They show that the quantity demanded but not awarded at the offering price averages to 38.5% of the quantity awarded.

This rationing strategy may be explained by the fact that the seller's goal is not only to maximize proceeds from the current sale but to have a higher number of winning bidders. Indeed, the seller may want to keep control over its firm after the IPO in order to avoid the possibility of a hostile takeover from a large shareholder (see Brennan and Franks, 1997). Amihud, Hauser and Kirsh (2003) also support the hypothesis that rationing is a way to increase ownership dispersion and thus, serves the interest of owners who derive private benefits of control. Booth and Chua (1996) argue that the issuer's demand for ownership dispersion creates an incentive to underprice.[3]

Rationing strategic bidders may also improve the liquidity in the secondary market, which is one of the main seller's objectives in IPOs, Treasury auctions or tradeable carbon permit auctions. Indeed, this may give them incentives to buy the quantity they didn't get during the

[1] Google IPO is a recent example of a uniform price aution IPO underwrited by Open IPO, see Berg, Neumann and Rietz (2009).

[2] The Wilson's (1979) result has been generalized by Back and Zender (1993), Biais and Faugeron-Crouzet (2002) and Wang and Zender (2002).

[3] Another justification to include a preference towards dispersion in allocation in the objective of the auctioneer can be found in the procurement literature which states that second sourcing can be used as a strategic device in order to maintain future competition in a given supplier base. See Farrell and Gallini (1988) or Lyon (2006) among others.

auction in the secondary market. In such markets, illiquidity is costly for the auctioneer because it increases the risk of some bidders may be able to exercise their market power.[4] Boehmer and Fishe (2001) show, in an IPO context, that the underwriter benefits from high trading activity in the secondary market due to trading profits and brokerage commissions. Thus, in order to raise secondary market trading, the underwriter rations bidders in the auction to attract investors with lower valuations. Booth and Chua (1996) state that rationing ensures oversubscription and results in a greater dispersion of ownership which may improve liquidity in the secondary market. Rationing reduces the firm's cost of capital and increases the market value of the firm in expanding the firm's investor base. As the seller often disposes of additional shares after the IPO, this induces her to ration bidders. Castellanos and Oviedo (2008) and Roseboro (2002) also document that an objective of Governments in Treasury auctions is to improve treasury securities liquidity in the secondary market in order to encourage more aggressive bidding in the primary market. Cramton and Kerr (1999) argue that illiquidity of the secondary market is costly for the seller. They also note that it reduces efficiency in tradeable carbon permit auctions, because it increases the ability of bidders to exercise their market power.

I therefore modify the uniform price auction rules in allowing the seller to increase the number of winning bidders in favoring non-strategic or non-competitive bidders.[5] She may allocate them some quantity of the good even though she has to discount the offer price to do so. Consequently, the seller strategically rations competitive bidders in allocating them only a part of the quantity demanded. The degree of rationing is chosen by the seller in order to maximize a weighted sum of the auction profits and the number of winning bidders.

Moreover, the seller's objective is assumed to be uncertain for the bidders. This may be explained by the market conditions which are better known by the seller than by the bidders. Another explanation may be that the seller's objective is perfectly known by the seller, but not by the bidders. These asymmetries of information between the seller and the bidders could explain the presence of uncertainty in the seller's objective.[6]

In this uniform price auction framework, I analyze the effects of introducing asymmetric entry costs supported by the bidders. Those asymmetric entry costs can be thought as the costs of hiring an intermediary in order to participate in the auction, or the costs to unfreeze funds in order to finance the acquisition of the good, or the costs of holding large positions and therefore having a too risky portfolio, or also, the costs of gathering information about the current value of the good that depends on the history of the bidder in this market.

The main contribution of this paper is that, when these entry costs are low enough, the vulnerability of inefficient equilibria is increased. The intuition for this result is the following.

[4]See Klingenfeld (2007) for Carbon auctions.

[5]For instance, all interested parties as non-polluting entities or financial institutions may be favored in the context of carbon auctions. In the U.S. Treasury auctions, non-competitive bidders have priority over the competitive bids, see Back and Zender (1993) or Wang and Zender (2002).

[6]For instance, there is no explicit rule for discounting in the prospectus of IPOs underwrited by OpenIPO or within the Mise en Vente process in France. Keloharju, Nyborg and Rydqvist (2004) also note that the Finnish Treasury did not have an explicit policy regarding the choice of quantity and stop-out price.

As in Bourjade (2009) paper, in which bidders do not have any entry costs, rationing prevents bidders from submitting too flat demand schedules. Indeed, due to the possibility of rationing, bidders are facing an uncertain residual supply. As they do not know the objective of the seller with certainty, they are not only concerned with their demand at the market clearing price but also with any price that could result from an ex-post rationing. This prevents them from submitting demand functions as flat as they want. The bidders' ability to inhibit competition from their rivals is reduced. They cannot manipulate bidding as they made it in the Wilson (1979) model.

However, due to the presence of entry costs, a second effect arises. From some price level, it is no longer optimal for high cost bidders to participate in the auction and they stop to bid. At this level, low cost bidders can get a significant increase in his allocation with only a minor price increase in forcing high cost bidders to quit the auction. This makes price competition more aggressive from low cost bidders. The gain in profits due to the quantity increase may compensate the loss due to a higher price. This more aggressive price competition therefore restricts the ability of bidders to manipulate the uniform price auction. It is therefore valuable for the seller to attract high cost bidders to the auction. An interesting feature of this result is that attracting high cost bidders is not only in the seller's interest because this increases the number of bidders. Indeed, this also induces low cost bidders to bid more aggressively close to the pivotal price at which high cost bidders quit the auction in order to eliminate them. Consequently, this prevents low cost bidders to exert their market power and eliminates inefficient equilibria.

The seminal contribution by Klemperer and Meyer (1989) is one of the first papers to select among the continuum of equilibria of the uniform price auction. Klemperer and Meyer (1989) introduce uncertainty about the supplied quantity and search for ex-post optimal supply function equilibria. They prove that the set of equilibria is significantly reduced when uncertainty is unbounded. The intuition is that the bidders' demand schedules has to be defined not only at the market clearing price, but also around it.[7]

This paper is also related to a growing literature on the ways to mitigate the bidders' market power in uniform price auctions. For instance, McAdams (2007a), and Back and Zender (2001) show that, when sellers have the possibility to adjust the quantity put for sale ex-post, bidders cannot sustain all equilibrium prices as before and then, underpricing is significantly reduced. The economic sense of this result is that when the seller can adjust the supply ex-post, bidders are not only concerned with the market clearing price that would occur without possibility of adjustment, but also with any price that could result from an ex-post adjustment. LiCalzi and Pavan (2005) study uniform price auctions when the seller is strategic and can precommit to an increasing supply schedule. They prove that even if underpricing is still present, it is significantly reduced. In this setting of increasing supply, with a little price increase the residual

[7] Despite this promising result, Back and Zender (1993) and Wang and Zender (2002) in the risk averse case, show that arbitrarily low equilibrium prices always exist in the case of uniform price treasury auctions when supply is perfectly inelastic but subject to a horizontal shock.

supply increases a lot and bidders get a bigger quantity. This reduces each bidder's market power. Consequently, bidders bid more aggressively and low price equilibria are eliminated. Allowing the seller to ration bidders when there is uncertainty about the seller's objective allows us to share this result. However, none of those papers analyzes the asymmetric bidders' case.

Vives (2010) and (2011) allows bidders of a uniform price auction to have private information about their own costs. The main result of the paper is that, when the bidders' costs functions are positively correlated, which is the most relevant case empirically, private information increases bidders' ability to exert their market power. The equilibrium price therefore tends to the collusive level. We also consider asymmetric bidders even though this asymmetry is about their fixed costs and not their marginal costs as in Vives. However, while Vives only consider symmetric linear Bayesian demand function equilibria, we allow bidders to submit non-linear demand functions. McAdams (2007b) also provide examples of uniform price auction's frameworks in which all equilibria are non-monotone when either the independence or risk-neutrality assumptions are relaxed. In this paper, all equilibria are also non monotone. Nevertheless, this is due to the bidders' asymmetric costs of participating in the auction.

But, adjusting the supply schedule is not the only way to boost competition in uniform price auctions. Kremer and Nyborg (2004a) and (2004b) show that allowing a finite number of bids is also a good way to eliminate bad equilibria.[8] However, their results are proved in a framework in which bidders are allowed to submit discrete bids while I consider a share auction.

Rationing in IPOs has also been studied by Parlour and Rajan (2005) in a model of book building. Following Sherman (2005), they model the Book Building process as a multi-unit common value auction and use the symmetric equilibrium characterized by Milgrom (1981). They show that rationing mitigate the winner's curse, i.e., bidders bid more aggressively.[9] They also determine the optimal degree of rationing in the class of credible mechanism.[10] However, a critical assumption in their model is that the bidders have unit demand. Instead, I consider bidders who have multi-unit demand, an assumption which is necessary in most of the real-life applications of uniform price auction.

The paper is organized as follows. Section 2 describes the model. In Section 3, I present the no entry costs case as a benchmark. Section 4 generalizes the previous analysis in introducing asymmetric costs of participating in the auction. Then, Section 5 concludes. Finally, all proofs are in section 6.

[8] Fabra (2003) and Fabra, von der Fehr and Harbord (2006) also analyze discrete uniform price auction and compare them with discriminatoty price auctions.

[9] See Bulow and Klemperer (2002) to understand how rationing may raise the expected prices when common value objects are sold. This model is more complex than the Bulow and Klemperer's (2002) one, but the intuition is the same.

[10] A credible mechanism is a mechanism in which the seller use the announced pricing rule. Here, it means that the seller uses the rationing rule and don't always set the IPO price equal to the market clearing strategy. Such a deviation from the seller is not credible in the sense that bidders would predict it and would bid according to it.

2 The model

The uniform price auction model described below is in the lines of Bourjade (2009). A seller auctions a fixed quantity, Q, of a homogenous and perfectly divisible good with a uniform price auction. There are N rational and risk neutral bidders and small, non-strategic bidders. There is a continuum, $\overline{Q} \leq Q$, of non-strategic and non-competitive bidders characterized by their willingness to pay for the good, p_0, distributed on the interval $[0, v]$ with c.d.f. $F(.)$. All strategic bidders have the same information about the common value of one unit of the good, v.

Bidders simultaneously submit piecewise continuously differentiable demand schedules for the divisible good, $x_i(.)$, $i = 1, ...N$. $X(p) = \sum_{i=1}^{N} x_i(p)$ is thus the aggregate demand at price p. $X(p)$ is also piecewise continuously differentiable.

The seller is allowed to discount the offer price and to ration strategic bidders after having observed the bids when it is profitable for her to do so. Notice that, in equilibrium, the seller always sells the whole quantity. However, when the offer price is discounted, the aggregate demand exceeds the supply and the seller rations the quantity awarded to each bidder. In this model, rationing is therefore associated to price discounting.

I assume that the seller is not only interested in the auction's proceeds, also, in the number of winning bidders. This may give her incentives to ration the strategic bidders in order to favor non-strategic bidders.[11], [12] When she uses a rationing strategy, she selects an offer price $p^*(\alpha)$, lower than the market clearing price, at which strategic bidders only receive a fraction α of their demand for the good. $\alpha = 1$ corresponds to the case of no rationing, i.e. the seller sells the good at the market clearing price. When she selects a degree of rationing $\alpha < 1$, the seller allocate part of the quantity to non-strategic even though she has to discount the offer price to do so.

There are m types of bidders characterized by their asymmetric costs of participating in the auction. Bidders with lower costs may be considered as large bidders (or firms from the power sector for instance) who purchase repeatedly emission permits and do not need any intermediary, or as bidders who can easily find funds to finance the acquisition of the good or who need this quantity in order to balance their portfolios or who have a low cost of gathering information about the current value of the good because of their knowledge about the market. Higher cost bidders are smaller bidders (or firms from the cement or the steel industries) who need to hire an intermediary in order to participate in the auction or are bidders who have difficulties to unfreeze funds or already have large positions in this issue and so that raising their shareholding provides them with a too risky portfolio or who have a high cost of gathering information about the current value of the good because they are new entrants or small actors in this market.

[11] Rationning strategic bidders and allocating part of the quantity to strategic bidders may improve the liquidity in the secondary market, which is one of the main seller's objectives in IPOs, Treasury auctions or tradeable carbon permit auctions. I provide a more detailed discussion about rationing motivations in the Introduction.

[12] This preference towards dispersion in allocation is observed in practice. For instance, all interested parties as non-polluting entities or financial institutions may be allowed to participate to carbon auctions. In the U.S. Treasury auctions, non-competitive bidders have priority over the competitive bidders, see Back and Zender (1993) or Wang and Zender (2002).

Let type-i bidders be bidders who have entry costs, c_i,and let N_i be the number of type-i bidders, $i = 1, ..., m$. Without loss of generality, $c_m > c_{m-1} > ... > c_1 = 0$.

Moreover, bidders do not know the objective of the seller with certainty.[13]

According to these assumptions, the objective of the seller, when using degree of rationing α, is

$$W(\alpha) = p^*(\alpha)Q + \beta\alpha\overline{Q}\left[1 - F\left[p^*(\alpha)\right]\right]$$

where Q is the number of shares put up for sale, $p^*(\alpha)$ is the market clearing price when the degree of rationing is α and β is an unknown parameter representing the seller's preferences for ownership dispersion. When the market-clearing price is $p^*(\alpha)$, the objective of the seller is thus a weighted sum of a first term representing the proceeds from the auction, $p^*(\alpha)Q$, and a second one representing the seller's preferences for ownership dispersion as the quantity that will be allocated to non-strategic bidders with a degree of rationing α is $\alpha\overline{Q}\left[1 - F\left[p^*(\alpha)\right]\right]$.[14]

As noted above, I assume that bidders do not know the seller's preferences and only anticipate that β is distributed on $\left[0, \overline{\beta}\right]$ with c.d.f. $G(.)$.

Moreover, the objective of the seller is assumed to satisfy a single-crossing condition:

$$\frac{\partial W(\alpha, \beta)}{\partial\alpha\partial\beta} \leq 0 \Leftrightarrow \frac{\partial\left[\alpha\overline{Q}\left[1 - F\left[p^*(\alpha)\right]\right]\right]}{\partial\alpha} \leq 0$$

Intuitively, the quantity allocated to non strategic bidders has to be non increasing in the degree of rationing α.

The equilibrium price is now defined as[15]:

$$p^*(\alpha) = \max\{p \mid \alpha X(p) + \alpha\overline{Q}\left[1 - F(p)\right] = Q,\ p \geq 0\} \tag{1}$$

where $\alpha \in]0, 1]$ is the degree of rationing chosen by the seller after having observed bids.

The bidders being risk neutral, they submit a demand schedule which maximizes their expected profit under the market clearing condition, the expectation being with respect to α.

I only consider symmetric ex-post optimal Nash equilibria in pure strategies, i.e. if a bidder expects other bidders to submit their equilibrium strategy, his best response is also the equilibrium strategy. Moreover, given other bidders' demand schedules, for each α, bidders would not change their bids even though they were allowed to do it ex-post.

The timing of the game is the following. The seller announces that, depending on the market

[13] This may be explained by the market conditions which are better known by the seller than by the bidders. Another explanation may be that the seller's objective is perfectly known by the seller, but not by the bidders. These asymmetries of information between the seller and the bidders could explain the presence of uncertainty in the seller's objective.

[14] As there is a continuum $\overline{Q}$ of non-strategic bidders who are characterized by their willingness to pay for the good, p_0, which is distributed on the interval $[0, v]$ with c.d.f. $F(.)$, when the market-clearing price is p, the quantity that non-competitive bidders are willing to buy is $\overline{Q}\left[1 - F(p)\right]$.

[15] This specification assumes that the seller's reserve price is zero.

conditions, she can ration the bidders.Each bidder i then simultaneously submit a piecewise continuously differentiable demand schedule (bids). After observing bids, the seller chooses a degree of rationing, $\alpha \in [0,1]$ according to her objective. The seller then sets the equilibrium price p^* which can be lower than the market clearing price without rationing.Quantities are allocated: each bidder receives his rationed demand at the equilibrium price.

3 Rationing with no entry costs

As a benchmark, I first present the pure strategy ex-post optimal symmetric equilibrium of this game with no-entry costs.[16]

Proposition 1 (Bourjade (2009)) *Assume that the seller can ration the quantity awarded to strategic bidders after having observed bids. Then, strategic price discounting is equal to*

$$v - p^*(\alpha) = \frac{[v - p^*(1)] \left(Q - \overline{Q}\left[1 - F\left(p^*(1)\right)\right]\right) e^{\Gamma_2(p^*(1),1)}}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p(\alpha)\right)\right]\right) e^{\Gamma_2(p^*(\alpha),\alpha)} \alpha^{N_1+N_2-1}}$$

where $\Gamma_2(.,.)$ is defined in the Appendix and $p^*(1) \in [0,v]$ is the equilibrium price when there is no rationing ex-post and can take values

$$v\left[1 - \frac{\left(Q - \underline{\alpha}\overline{Q}\left[1 - F\left(p(\underline{\alpha})\right)\right]\right) e^{\Gamma_2(p^*(\underline{\alpha}),\underline{\alpha})} \underline{\alpha}^{N_1+N_2-1}}{\left(Q - \overline{Q}\left[1 - F\left(p^*(1)\right)\right]\right) e^{\Gamma_2(p^*(1),1)}}\right] \leq p^*(1) \leq v$$

Moreover, when the uncertainty about the seller's preferences for control is high enough (i.e. $\overline{\beta}$ is such that $\underline{\alpha}$ goes to 0), there is a unique equilibrium, $p^* = v$.

Proof. See Bourjade (2009) or the proof of Proposition 2 in the Appendix (assuming no entry costs) where all variables are defined. ■

This result establishes that when the seller's objective is uncertain, the set of equilibrium strategies and the ability of bidders to manipulate the equilibrium price are restricted. The inefficient equilibria, introduced by Wilson (1979) and Back and Zender (1993), can thus be eliminated by this rationing strategy.

4 Rationing with asymmetric entry costs

In this section, I study the effects of introducing asymmetric entry costs supported by the bidders on the equilibrium.

In this context, bidders only have a positive demand when their resulting profits are non negative. Therefore, there exists a pivotal price at which bidders with costs c_i quit the auction. In the neighborhood of this price, with a small price increase, bidders with lower costs can get

[16] See Bourjade (2009) for a complete analysis.

a substantial quantity increase because they will share the total quantity among themselves in eliminating higher costs bidders from the auction. This significantly changes the problem. Indeed, the number of participating (active) bidders varies when the equilibrium price varies.

4.1 Characterization of ex-post optimal equilibria with two types of bidders

In order to stress on the economic forces which drives the results with asymmetric entry costs, I first consider the case of two types of bidders. Type-2 bidders are therefore bidders who have high entry costs, $c_2 = c$, and type-1 bidders those who have low entry costs, $c_1 = 0$.

In this particular case, there only exists one pivotal price at which high cost bidders quit the auction.

Notice that in the Wilson (1979) model where the seller can not ration, bidders could still sustain any equilibrium price even if entry costs are introduced. Indeed, low cost bidders have no incentive to raise the price to the pivotal level as there always exists an equilibrium price for which the low cost bidders' profits are higher at this price than at the pivotal one. The intuition for this result is straightforward. As arbitrarily low prices can be sustained by the bidders, any increase in profits due to a larger received quantity is more than compensated in sustaining a lower equilibrium price. However, when rationing is allowed, the demand schedules' slope cannot be as flat as the bidders would want. This prevents them from sustaining too low equilibrium prices and to raise their profits in this way. Low cost bidders have now incentives to bid more aggressively in order to get a larger quantity in forcing high cost ones to quit the auction. The "quantity effect" compensates the "price effect".

I can now characterize the ex-post optimal symmetric Nash equilibria of the game when the bidders have asymmetric entry costs.

To solve for the equilibria of this game, I first derive the equilibrium demand schedules when all bidders participate to the auction, then, when only low cost bidders participate, and finally, I state the global equilibrium.

Due to the seller's objective, bidders anticipate that the degree of rationing that is used by the seller, α, is distributed over some non empty support $[\underline{\alpha}, 1]$, with c.d.f. $H(.)$. Indeed, as $p^*(\alpha)$ is strictly increasing[17], the optimal degree of rationing can be written as a decreasing function of β.

Given the support of β, I can therefore determine the support of α. $\beta = 0$ gives the higher bound of this support. In this case, the optimal α is the highest credible one, 1. Moreover, $\underline{\alpha} \in [0, 1]$. $\overline{\beta}$ allows me to state the lower bound $\underline{\alpha} = \max\{\alpha(\overline{\beta}), 0\}$. The support of credible degree of rationing is thus $[\underline{\alpha}, 1]$.

To solve the model, it is not necessary to determine $H(.)$. Indeed, as will be shown later, equilibrium are not sensitive to the distributional properties. However, it is straightforward to derive it from the seller's objective and $G(.)$. I can therefore make the analysis as if bidders had

[17] This is proved in the Appendix.

prior beliefs about the rationing scheme used by the seller and not about the relative importance accorded to the liquidity of the secondary market trading with respect to the proceeds from the auction.

Within this framework, low equilibrium prices do not necessarily result from bidders' market power because the seller may have incentives to sustain them in rationing. So, to analyze the effects of rationing against market power, I consider the slope of the bidders' demand schedules instead of the price level. Indeed, a flat demand schedule results from a bidder exerting his market power and a steep demand schedule reflects a more aggressive price competition.[18]

An ex-post optimal symmetric Nash equilibrium of this game is characterized by demand schedules $\{x_i(.)\}_{i=1,...,N}$ and an equilibrium price $p^*(\alpha)$ that equalize the aggregate demand for shares and the supplied quantity when the degree of rationing is α.

The equilibrium price is defined by (1) where $\alpha \in [\underline{\alpha}, 1]$ is the degree of rationing chosen by the seller after having observed the aggregate demand.

Given α, this price is uniquely defined when the aggregate demand is non increasing. The quantity $\alpha x_i(p^*(\alpha))$ is awarded to bidder i.

X_{-i} is the aggregate demand of all bidders but i.

We can now consider the bidder $i's$ program. The quantity $x_i(p^*(\alpha)) = \frac{Q}{\alpha} - \overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right] - X_{-i}(p^*(\alpha))$ and the price $p^*(\alpha)$ maximize the revenue of bidder i given $X_{-i}(.)$ and α because of ex-post optimality. $p^*(\alpha)$ is thus solution of the program[19]

$$p^*(\alpha) \in \arg\max_p \left\{(v-p)\left[Q - \alpha\overline{Q}\left[1 - F\left(p\right)\right] - \alpha X_{-i}(p)\right] - c_i\right\}$$

The first order condition of this program is

$$-Q + \alpha\overline{Q}\left[1 - F\left[p^*(\alpha)\right]\right] + \alpha X_{-i}(p) - (v-p)\left[\alpha X_{-i}'(p) - \alpha\overline{Q}f(p)\right] = 0.$$

The only difference with the previous section is that the aggregate demand of all bidders but i varies when the number of active bidders varies. More formally, when bidder i is a high cost bidder, $X_{-i}(.)$ is only defined when all bidders participate. Hence, $X_{-i}(.)$ takes the following form

$$X_{-i}(p) = (N_2 - 1)x_2(p) + N_1 x_1(p)$$

When bidder i is a low cost bidder, $X_{-i}(.)$ is defined by

$$X_{-i}(p) = \begin{cases} N_2 x_2(p) + (N_1 - 1)x_1(p) & \text{when all bidders participate.} \\ (N_1 - 1)x_1(p) & \text{when only low cost bidders participate.} \end{cases}$$

[18] Submitting flat demand schedules acts as an implicit veto to inhibit competition. If a bidder wants to get more shares, he is forced to sustain a large increase in price. Incentives to bid more aggressively are therefore reduced.

[19] Notice that bidders do not take α as given. However, ex-post optimality implies that, when choosing their bids, for each realization of α, they would not change their bids even though they were allowed to do so ex-post.

Again, if $p^*(.)$ is strictly increasing, then it is an ex-post optimal equilibrium. Moreover, there exists α_c such that when $\alpha < \alpha_c$, all bidders participate and when $\alpha > \alpha_c$, only low cost bidders participate. Thus, bidder $i's$ best response to $X_{-i}(p)$ are well defined for $p < \lim_{\alpha \to \alpha_c^-} p^*(\alpha)$ and for $p > \lim_{\alpha \to \alpha_c^+} p^*(\alpha)$.

The next step is to set the limits of $p^*(\alpha)$, on the left and on the right to ensure that no bidders have an incentive to deviate. This completely characterizes the global ex-post optimal equilibrium of the game.

This is stated in the following proposition:

Proposition 2 *When bidders have asymmetric entry costs, the equilibrium price is*

$$p^*(\alpha) = -\frac{k_i}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]\right) e^{\Gamma_i(p^*(\alpha),\alpha)} \alpha^{I-1}} + v,$$

when exactly $I = \sum_{l=1}^{i} N_l$ *bidders actively participate in the auction and where* k_1 *and* k_2 *are defined in the Appendix.*

Prices $p \in \left[v - \frac{c(N_1+N_2)}{\left(Q-\alpha\overline{Q}[1-F(p^*(\alpha))]\right)}, v - \frac{cN1}{\left(Q-\alpha\overline{Q}[1-F(p^*(\alpha))]\right)}\right]$ *cannot be sustain as equilibrium prices.*

Proof. See Appendix ■

These equilibrium demand schedules are represented in Figures (1), (2) and (3).

Notice that those equilibrium demand schedules are unique in the class of the pure strategy ex-post optimal symmetric equilibria.

While uniform price auctions admit a continuum of equilibrium prices, some of which being arbitrarily low, allowing the seller to ration bidders eliminates many of these "bad equilibria" when bidders have fixed entry costs. In these equilibria, the bidders' strategies are the same than without entry costs on each interval of prices in which the number of active bidders is constant. However, due to the asymmetric entry costs, each type quits the auction when his financial constraint is binding, i.e when the price exceeds the corresponding pivotal price.

Moreover, rationing prevents bidders from submitting non aggressive bids. Low cost bidders can now get a significant quantity increase by forcing high cost ones to quit the auction. But, their demand schedules' slope cannot be too flat otherwise the seller could increase the degree of rationing and then, high cost bidders would still participate to the auction. This would clear the previous effect. This creates an upward discontinuity in low cost bidders' demand schedules at the critical degree of rationing and makes the equilibrium demand schedules non monotonic. However, the aggregate demand is still non increasing when one only considers equilibrium prices which can be sustained.

Prices $p \in \left[v - \frac{c(N_1+N_2)}{\left(Q-\alpha\overline{Q}[1-F(p^*(\alpha))]\right)}, v - \frac{cN1}{\left(Q-\alpha\overline{Q}[1-F(p^*(\alpha))]\right)}\right]$ cannot be sustain as equilibrium prices in order to avoid deviations from other bidders. Indeed, if bidders would want to sustain such a price, it would be profitable for a low cost bidder to deviate from the equilibrium strategy and sustain a higher price. This bidder would get a larger quantity with a minor price increase. Thus, these prices can not be sustained as equilibrium prices.

These effects together enhance competition. Let me resume the intuition of the previous result. Strategic rationing prevents bidders from submitting too flat demand schedules. Moreover, close to the price at which high cost bidders quit the auction, with a small price increase, low cost bidders can get a substantial quantity increase. This restricts the ability of bidders to sustain too low price equilibria. The gain in profit due to more aggressive bidding compensates the loss due to a higher price.

With rationing and asymmetric entry costs, the "quantity effect" outweighs the "price effect", and, therefore, the inefficient equilibria from the basic uniform price auction cannot be sustained. When the difference in their asymmetric entry costs is low enough, it is therefore valuable for the seller to attract high cost bidders to the auction. However, the seller does not want to attract high bidders in order to increase the number of bidders as is usually the case in the standard auction literature. This strategy is valuable for the seller because this induces low cost bidders to bid more aggressively around the pivotal price in order to eliminate high cost ones. Consequently, this prevents low cost bidders to exert their market power and makes "collusive-seeming" equilibria less likely to emerge. This is stated in the following Corollary.

Corollary 3 *When the difference in the asymmetric entry costs is low enough, bidders' market power is reduced with respect to the case with no costs at all. It is therefore valuable for the seller to attract high cost bidders to the auction in order to induce low cost bidders to bid more aggressively.*

Proof. See Appendix. ■

I can now state a strong result which is in contrast with the theoretical uniform price auction's literature. When bidders are asymmetric, and rationing is allowed, there does not exist ex-post optimal equilibrium with non increasing demand schedules.

Remind that an equilibrium is ex-post optimal if, given other bidders' bid schedules, for each realization of the degree of rationing, α, bidders would not change their bids even though they were allowed to do it.

Corollary 4 *There does not exist ex-post optimal equilibrium of the uniform price auction with rationing and entry costs with non increasing demand schedules when the support of the uncertainty about the seller's objective is finite.*

Proof. See Appendix. ■

Notice that assuming that the support of the uncertainty about the seller's objective is finite is equivalent to assuming that $\underline{\alpha} > 0$.

This result is due to the fact that when high cost bidders quit the auction, there is an increase of the quantity of good available for low cost bidders. These low cost bidders have thus incentives to bid more aggressively when high cost bidders quit the auction than when they participate. This creates an upward jump in the low cost bidders' demand schedules.

To end this section, I characterize the equilibrium when the support of the uncertainty about the seller's objective is infinite or equivalently when the seller could use all degree of rationing in $]0,1]$. This corollary allows to compare my results with the theoretical literature about uniform price auctions which only considers monotonic equilibrium.

Corollary 5 *When the support of the uncertainty about the seller's objective is infinite, there is a unique equilibrium, $p^* = v$. Moreover, the demand schedules that sustain this equilibrium are monotonic.*

Proof. See Appendix. ■

Notice that even though bidders have asymmetric entry costs, there is a unique equilibrium which is the efficient one, $p^* = v$. This means that with asymmetric bidders, the seller may be induced to attract high cost bidders to the auction. Indeed, anticipating price discounting and rationing, high cost bidders participate and this pushes up the price. The seller may therefore get a higher price when high cost bidders compete than with only low cost bidders.

4.2 Characterization of ex-post optimal equilibria with m types of bidders

I apply the same approach as the one used for the 2 bidders' types case in order to characterize the equilibria with m types of bidders. Each type quits the auction when his financial constraint is binding. As there are more types of bidders, this creates more discontinuity points and thus the corresponding set of equilibrium prices is further reduced. Again, strategic rationing prevents bidders from submitting inelastic demand schedules because the slopes of the demand schedules are determined. Moreover, close to the price at which a bidders quit the auction, with a minor price increase, lower cost bidders can get a large quantity increase. This restricts the ability of bidders to sustain too low price equilibria.

Again, the bidders' strategy is the same than without asymmetric costs on each interval of prices in which the number of active bidders is constant. The equilibrium demand schedules are non monotonic, but the aggregate demand is non increasing when one only considers equilibrium prices that can be sustained.

This is stated in the following proposition.

Proposition 6 *When bidders have asymmetric entry costs, the equilibrium price is*

$$p^*(\alpha) = -\frac{k_i}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]\right) e^{\Gamma_i(p^*(\alpha),\alpha)}\alpha^{I-1}} + v,$$

when exactly $I = \sum_{l=1}^{i} N_l$ bidders actively participate in the auction and where $k_1, ..., k_n$ are defined in the Appendix.

For all i, prices $p \in \left[v - \frac{c_i(N_1+...+N_i)}{(Q-\alpha\overline{Q}[1-F(p^*(\alpha))])}, v - \frac{c_i(N_1+...+N_{i-1})}{(Q-\alpha\overline{Q}[1-F(p^*(\alpha))])}\right]$ *cannot be sustain as equilibrium prices.*

Proof. See Appendix. ∎

The equilibrium of this model is close to the one of the discrete multi-unit model[20] when the differences in the costs are low enough.

In the discrete multi-unit model, there is no collusive-like equilibria as in the continuous share auction of Wilson (1979). This result remains true in the limit, when the bid-step size goes to zero. Thus, the continuous share auction model is not a good approximation to the discrete model even though bid-steps are small enough.

The modified uniform price auction that I present in this paper may therefore reconcile these two branches of the literature. It can be viewed as a good approximation of the discrete model when bidders have asymmetric entry costs. Moreover, it is a more tractable model than the discrete multi-unit one. It should therefore be used to describe pollution permits, treasury or electricity markets.

5 Conclusion

An important part of the literature supports the use of discriminatory auctions when selling a divisible good because of the inefficient equilibria of the uniform price auctions. Recent contributions have shown that adjusting supply after observing bids eliminate some of these equilibria. This paper proves that uncertainty about the seller's strategy and asymmetric bidders reduces the bidders' market power and thus, eliminates many of these "bad equilibria." Results supporting the fact that uncertainty makes the outcome of uniform price auction more efficient are therefore extended to the case of asymmetric bidders. Moreover, the efficiency of uniform price auctions is shown to be enhanced by the introduction of a low enough entry cost supported by the bidders. Finally, I prove another result which is in contrast with the theoretical uniform price auction's literature. When bidders are asymmetric and when the seller's objective is uncertain, there does not exist ex-post optimal equilibria with non increasing demand schedules if the support of the uncertainty about the seller's objective is finite. However, when the support of the uncertainty about the seller's objective is infinite, there is a unique equilibrium: the efficient one and the demand schedules that sustain this equilibrium are monotonic.

In this paper, I have shown that rationing hurts bidders' market power. In order for rationing to be a credible selling mechanism, the seller must be able to precommit to effectively ration bidders. In what precedes, I assumed that the seller is not only interested in the proceeds from the auction, but, also, in the number of winning bidders.

However, a seller can make this strategy credible in developing a reputation for rationing. Indeed, an important feature of IPO, treasury, electricity or pollution permits auctions is that

[20] See Kremer and Nyborg (2004a) and (2004b), Fabra (2003) and Fabra, von der Fehr and Harbord (2006).

they are repeated frequently. The seller can use the previous auctions to make rationing credible. One may think that if the seller randomly ration shares in an auction, bidders may believe that she will play in the same way in the next auction. This makes them build prior beliefs about the rationing scheme used by the seller, anticipating that the degree of rationing used by the seller is distributed over some non empty support. This should make rationing a credible strategy without any possibility for bidders to exactly anticipate the exact degree of rationing. They are induced to use the same strategy that I characterize in this paper. Let's remark that if the seller does not ration at all, bidders anticipate it and use the same inefficient strategy as in Back and Zender (1993). So, even though the seller is only profit maximizing but interested in his long-term profits, rationing should be optimal in order to build a "rationing" reputation and, then, make profits in the subsequent auctions. However, sustaining a reputation to play a mixed strategy is an open question even though the previous behavior seems intuitive. It would be interesting to investigate this issue in future research.

6 Appendix

Proof of Proposition 2. Let Q be the quantity of good put for sale, N_i the total number of risk neutral strategic bidders with entry costs c_i, $x_i(p)$ the equilibrium strategy for bidder i, (piecewise continuously differentiable and downward sloping), and α the rationing scheme.

The market clearing price is determined such that $\alpha N x(p) + \alpha \overline{Q} \left[1 - F\left(p\right)\right] = Q$, when N bidders participate in the auction.

By assumption, $c_2 = c > c_1 = 0$.

Hence, I must consider two cases, the first one in which all bidders have a positive demand; and the second one in which only low cost bidders have a positive demand[21].

If $p^*(.)$, the equilibrium price, is strictly increasing, then it is an ex-post optimal equilibrium. Moreover, as $p^*(.)$ is strictly increasing, there exists α_c such that when $\alpha < \alpha_c$, all bidders participate and when $\alpha > \alpha_c$, only low cost bidders participate[22].

- I first consider the case in which all bidders actively participate.

I consider the program of the bidders. In this case, the number of active bidders is $N_1 + N_2$.

I thus only consider prices p such that type-i bidders' profits, $i = 1, 2$, are non negative.

For these prices, type-1 and type-2 bidders' demand schedules are the same, $x_1(p) = x_2(p)$ because their asymmetric costs are fixed costs.

All bidders demand shares for $\alpha \leq \alpha_c$. The prices I consider in this case are, thus, such that $\alpha \leq \alpha_c$.

[21] The equilibria that I characterize are the only symmetric "Demand" Function Equilibrium tracing through ex post optimal points. The proof is the same as in Step 2 of Proposition 5.

[22] α_c is defined such that the profits of high costs bidders is zero when $\alpha = \alpha_c$.

I now characterize the symmetric equilibrium of the game for $\alpha \leq \alpha_c$.

Let suppose that all bidders except bidder i, which is a high cost bidder, play their equilibrium strategy $x(p)$, and that bidder i plays the strategy $y(p)$.

The market clearing condition is:

$$\alpha\left[y(p) + (N_1 + N_2 - 1)x(p)\right] + \alpha\overline{Q}\left[1 - F\left(p\right)\right] = Q$$

Given α, the bidder i's residual supply is, then,

$$y(p) = \frac{Q}{\alpha} - (N_1 + N_2 - 1)x(p) - \overline{Q}\left[1 - F\left(p\right)\right]$$

Bidders being risk neutral, bidder i's profits are:

$$\begin{aligned} \pi_i &= (v-p)\alpha y(p) - c_i \\ &= (v-p)\left[Q - \alpha(N_1 + N_2 - 1)x(p) - \alpha\overline{Q}\left[1 - F\left(p\right)\right]\right] - c_i \end{aligned}$$

In a symmetric equilibrium, all bidders use the same strategy $x_i(p) = x(p)$, and $X_{-i}(p) = (N-1)x(p)$, for $i = 1, ..., N$.

Bidder i wants to sustain a price $p^*(\alpha)$ when the degree of rationing is α, in submitting a demand schedule $x_i(p)$. The first order condition is[23]

$$\begin{gathered} -Q + \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right] + \alpha(N_1 + N_2 - 1)x(p^*(\alpha)) \\ -(v - p^*(\alpha))\left[\alpha(N_1 + N_2 - 1)x'(p^*(\alpha)) - \alpha\overline{Q}f(p^*(\alpha))\right] = 0 \end{gathered} \tag{2}$$

The market-clearing condition is:

$$\alpha\left(N_1 + N_2\right)x(p^*(\alpha)) + \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right] = Q \tag{3}$$

Differentiating (3) with respect to α gives

$$\left(N_1 + N_2\right)x(p^*(\alpha)) + \overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right] + \alpha N\frac{dp^*(\alpha)}{d\alpha}x'(p^*(\alpha)) - \alpha\overline{Q}f(p^*(\alpha))\frac{dp^*(\alpha)}{d\alpha} = 0 \tag{4}$$

[23] Due to Step 2 of the Proof of Proposition 6, there are no other equilibria.

Putting this into (2):

$$-\alpha x(p^*(\alpha)) - (v - p^*(\alpha)) \left[\alpha(N_1 + N_2 - 1)x'(p^*(\alpha)) - \alpha\overline{Q}f(p^*(\alpha))\right] = 0$$

$$\Leftrightarrow -\frac{dp^*(\alpha)}{d\alpha} + (v - p^*(\alpha)) \left[\begin{array}{c} \frac{\alpha\overline{Q}f(p^*(\alpha))\frac{dp^*(\alpha)}{d\alpha} - \overline{Q}[1-F(p^*(\alpha))]}{Q-\alpha\overline{Q}[1-F(p^*(\alpha))]} \\ + \frac{N_1+N_2-1}{\alpha} + \frac{(N_1+N_2)\overline{Q}[1-F(p^*(\alpha))]}{Q-\alpha\overline{Q}[1-F(p^*(\alpha))]} \end{array} \right] = 0$$

$$\Leftrightarrow -\frac{\frac{dp^*(\alpha)}{d\alpha}}{(v - p^*(\alpha))} + \left[\frac{U'(p^*(\alpha))}{U(p^*(\alpha))} + \frac{N_1 + N_2 - 1}{\alpha} + \frac{d\Gamma_2\left(p^*(\alpha), \alpha\right)}{d\alpha}\right] = 0$$

where $U(p^*(\alpha)) = Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]$ is increasing from the single-crossing condition, and $\Gamma_2\left(p^*(\alpha), \alpha\right)$ is such that $\frac{d\Gamma_2(p^*(\alpha),\alpha)}{d\alpha} = \frac{(N_1+N_2)\overline{Q}[1-F(p^*(\alpha))]}{Q-\alpha\overline{Q}[1-F(p^*(\alpha))]}$

The solution of this differential equation is:

$$p^*(\alpha) = -\frac{k_2}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]\right) e^{\Gamma_2(p^*(\alpha),\alpha)} \alpha^{N_1+N_2-1}} + v \tag{5}$$

where k_2 is the corresponding integration's constant.

Second order conditions are also satisfied because the aggregate demand schedule is downward sloping on the interval of prices that can be sustained as equilibrium prices (see Step 2 of Proposition 5).

Notice that this formula works for all degree of rationing lower than α_c.

Therefore, the equilibrium price is completely determined for each degree of rationing lower than $\alpha_c = \left(\frac{k_2}{c(N_1+N_2)e^{\Gamma_2(p^*(\alpha_c),\alpha_c)}}\right)^{\frac{1}{N_1+N_2-1}}$, when the value k is given.

- I will, now go to the case in which only low cost bidders have a positive demand.

In this case, I only consider degree of rationing greater than α_c, the number of active bidders is, then, N_1.

The proof is nearly the same than in the previous case. The equilibrium price is now,

$$p^*(\alpha) = -\frac{k_1}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]\right) e^{\Gamma_1(p^*(\alpha),\alpha)} \alpha^{N_1-1}} + v \tag{6}$$

with $\alpha \in [\alpha_c, 1]$, $k_1 = [v - p^*(1)]\left(Q - \overline{Q}\left[1 - F\left(p^*(1)\right)\right]\right) e^{\Gamma_1(1)}$. $\Gamma_1\left(p^*(\alpha), \alpha\right)$ is such that $\frac{d\Gamma_1(p^*(\alpha),\alpha)}{d\alpha} = \frac{N_1\overline{Q}[1-F(p^*(\alpha))]}{Q-\alpha\overline{Q}[1-F(p^*(\alpha))]}$ and $p^*(1)$ is the equilibrium price with no rationing.

- I can then characterize the global equilibrium of the game.

$p_c = \lim_{\alpha \to \alpha_c^-} p^*(\alpha)$, the equilibrium price at which the profits of a high cost bidder when he participates to the auction is equal to the profits of the same bidder when he does not participate, satisfies the following equation:

$$p_c = -\frac{k_2}{\left[\left(Q-\alpha\overline{Q}\left[1-F\left(p^*(\alpha)\right)\right]\right)e^{\Gamma_2(p^*(\alpha),\alpha)}\right]^{\frac{k_2}{c(N_1+N_2)e^{\Gamma_2(p^*(\alpha),\alpha)}}}}+v$$

$$p_c = -\frac{c\left(N_1+N_2\right)}{\left(Q-\alpha\overline{Q}\left[1-F\left(p^*(\alpha)\right)\right]\right)}+v$$

I must, now, define the price, $\overline{p}=p^*(\alpha_c)$, such that low cost bidders have no incentives to deviate when $\alpha=\alpha_c$. $\overline{p}$ must satisfy the following equation

$$\Pi_1(p_c,\alpha_c^-)=\Pi_1(\overline{p},\alpha_c^+)$$

$$cN_1 = \frac{\left[c\left(N_1+N_2\right)\right]^{\frac{N_1-1}{N_1+N_2-1}}\left[v-p^*(1)\right]\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_1(1)}}{\left[e^{\Gamma_1(p^*(\alpha),\alpha)}\right]^{\frac{N_2}{N_1+N_2-1}}k_2^{\frac{N_1-1}{N_1+N_2-1}}}$$

$$k_2 = \frac{\left[c\left(N_1+N_2\right)\right]\left(\left[v-p^*(1)\right]\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_1(1)}\right)^{\frac{N_1+N_2-1}{N_1-1}}}{\left(cN_1\right)^{\frac{N_1+N_2-1}{N_1-1}}\left[e^{\Gamma_1(p^*(\alpha),\alpha)}\right]^{\frac{N_2}{N_1-1}}}$$

Notice that low cost bidders' demand schedules are not imposed by equilibrium conditions in $[p_c,\overline{p}]$ and that all prices in $[p_c,\overline{p}]$ are not equilibrium prices.

Moreover, as $p^*(.)$ is increasing in α and $\alpha\in[\underline{\alpha},1]$, a necessary and sufficient condition to get the positivity of the equilibrium price is given by $p^*\left(\underline{\alpha}\right)\geq 0$. That is

$$-\frac{k_2}{\left(\underline{\alpha}\right)^{N_1+N_2-1}}+v\geq 0$$

$$\Longleftrightarrow k_2\leq v\left(\underline{\alpha}\right)^{N_1+N_2-1}$$

This gives the following lower bound for the set of equilibrium prices:

$$p^*(1)\geq v\left[1-\left(\frac{N_1}{N_1+N_2}\right)\left(\frac{\left[\left(\frac{c(N_1+N_2)}{v}\right)e^{\Gamma_1(p^*(\alpha),\alpha)}\right]^{\frac{N_2}{N_1+N_2-1}}}{\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_1(1)}}\right)\left(\underline{\alpha}\right)^{N_1-1}\right]. \qquad (7)$$

■

Proof of Corollary 3. Using the result of Proposition 1 from Bourjade (2009), the lower bound for the set of equilibrium prices with no entry costs is:

$$v\left[1-\frac{\left(Q-\underline{\alpha}\overline{Q}\left[1-F\left(p(\underline{\alpha})\right)\right]\right)e^{\Gamma_2(p^*(\underline{\alpha}),\underline{\alpha})}\underline{\alpha}^{N_1+N_2-1}}{\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_2(p^*(1),1)}}\right]\leq p^*(1)\leq v,$$

and using (7), the lower bound for the set of equilibrium prices with asymmetric entry costs is:

$$p^*(1) \geq v\left[1-\left(\frac{N_1}{N_1+N_2}\right)\left(\frac{\left[\left(\frac{c(N_1+N_2)}{v}\right)e^{\Gamma_1(p^*(\alpha),\alpha)}\right]^{\frac{N_2}{N_1+N_2-1}}}{\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_1(1)}}\right)\left(\underline{\alpha}\right)^{N_1-1}\right].$$

Consequently, when the entry costs of high cost bidders are small enough,

$$c \leq \frac{v}{(N_1+N_2)}\left(\left[\left(\frac{N_1+N_2}{N_1}\right)\left(Q-\underline{\alpha}\overline{Q}\left[1-F\left(p^*(\underline{\alpha})\right)\right]\right)\right]^{\frac{N_1+N_2-1}{N_2}}\left(e^{\Gamma_1(\underline{\alpha})}\right)^{\frac{N_1-1}{N_2}}\right)\left(\underline{\alpha}\right)^{N_1+N_2-1}$$

then, the set of equilibrium prices is reduced in a case with N_1 bidders with no costs and N_2 bidders with costs c compared to a case with N_1+N_2 bidders with no entry costs.

I can therefore conclude that the set of equilibrium prices is reduced when the difference in bidders' entry costs is small enough. ■

Proof of Corollary 4. For an equilibrium to be monotonic, the demand schedule of a low cost bidder must be decreasing in $\overline{p}$. Indeed, high cost bidders' demand schedules are decreasing over $[0, p^*(1)]$ and low cost bidders' ones are decreasing over $[0, \overline{p}[$ and $]\overline{p}, p^*(1)]$.

$$\lim_{p\to\overline{p}^-} x_1(p) = \frac{Q-\alpha_c\overline{Q}\left[1-F\left(\overline{p}\right)\right]}{\alpha_c\left(N_1+N_2\right)} \leq \lim_{p\to\overline{p}^+} x_1(p) = \frac{Q-\alpha_c\overline{Q}\left[1-F\left(\overline{p}\right)\right]}{\alpha_c N_1}$$

Notice that this inequality may not be satisfied when $k=0$.However, this would imply that $\alpha_c = 0$ which is impossible as $\alpha \geq \underline{\alpha} > 0$.

Hence, there does not exist monotonic equilibria when $\underline{\alpha} > 0$. ■

Proof of Corollary 5. If the seller could use all degree of rationing in $]0,1]$, the only monotonic equilibrium inverse demand schedules of low cost bidders is $p^*(x) = v$, for all x.

Indeed their equilibrium inverse demand schedules is

$$p^*(x) = -\frac{\left[v-p^*(1)\right]\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_L(1)}}{\left(Q-x\overline{Q}\left[1-F\left(p^*(x)\right)\right]\right)e^{\Gamma_L(p^*(x),x)}x^{N_1-1}}+v$$

for all $x>0$.

And as in all monotonic equilibria, one must have

$$k = \left[v-p^*(1)\right]\left(Q-\overline{Q}\left[1-F\left(p^*(1)\right)\right]\right)e^{\Gamma_L(1)} = 0,$$

this proves the result.

One can remark that in equilibrium, high cost bidders demand no shares whatever the price is. ■

Proof of Proposition 6. I decompose the proof in 5 steps.

In Step 1, I characterize the equilibrium when only bidders with costs lower than c_{i+1} have a positive demand.

In Step 2, I state necessary and sufficient conditions for demand schedules to form a "Demand" Function Equilibrium (DFE) tracing through ex-post optimal points when the financial constraints of i bidders are non binding and that the financial constraints of the other $N-i$ are binding for prices in $\left[\underline{p_i}, \overline{p_i}\right]$.

Finally, in Step 3 and 4, I characterize the lower bound $\underline{p_i}$ and the higher bound $\overline{p_i}$, and I give the ex-post optimal equilibria of the game.

Step 1 I consider the case in which only bidders with costs lower than c_{i+1} have a positive demand.

In this case, I only consider degree of rationing in the interval $]\alpha_i, \alpha_{i+1}]$ for which the number of active bidders is exactly $N_1 + ... + N_{i+1}$. Where α_i is defined as in the 2 bidders' types case by $\alpha_i = \left[\frac{k_i}{c_i(N_1+...+N_i)e^{\Gamma_i(p^*(\alpha_i),\alpha_i)}}\right]^{\frac{1}{N_1+...+N_i-1}}$.

The proof is the same as in the 2 bidders' types case. The equilibrium price is,

$$\overline{p_{i+1}}(\alpha) = -\frac{k_{i+1}}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]\right) e^{\Gamma_{i+1}(p^*(\alpha),\alpha)}\alpha^{N_1+...+N_{i+1}-1}} + v$$

with $\alpha \in \left]\alpha_i, \alpha_{i+1}\right]$, where $\Gamma_{i+1}\left(p^*(\alpha), \alpha\right)$ is equal to $\Gamma\left(p^*(\alpha), \alpha\right)$ with $N =_1 + ... + N_{i+1}$ and k_{i+1} is a constant from the differential equation.

Step 2 The demand schedules $x(.)$ form a symmetric "Demand" Function Equilibrium tracing through ex-post optimal points if and only if (1) $x(.)$ satisfies the first order condition of the problem together with the Market Clearing condition on each interval $\left[\underline{p_i}, \overline{p_i}\right]$ such that the financial constraints of i bidders are non binding and the other $N-i$ are not, (2) the aggregate demand is non increasing on the interval of prices that can be sustained as equilibrium prices, (3) $\alpha i x(.) + \alpha\overline{Q}\left[1 - F(.)\right]$ is non increasing for all i and (4) no bidder has incentives to deviate at each $\underline{p_i}$ or $\overline{p_i}$ for all i.

- Sufficiency: Assume that the financial constraints of i bidders are non binding and that the financial constraints of the other $N-i$ are binding for prices in $\left[\underline{p_i}, \overline{p_i}\right]$. There are only i bidders who participate actively in the auction.

Since the aggregate demand is non increasing for all realized equilibrium price it intersects the fixed supply at a unique point for each α.

Moreover, the demand schedules satisfy first order condition for ex-post profit maximization when the other firms choose their equilibrium strategy.

Both conditions together implies that the second derivative of a bidder's profit, Π'', is negative for all $p^*(\alpha)$.

Indeed, the first order condition of a bidder problem is

$$\alpha x(p^*(\alpha)) = -(v - p^*(\alpha))\left[\alpha(i-1)x'(p^*(\alpha)) - \alpha\overline{Q}f(p^*(\alpha))\right]$$

Differentiating this equation with respect to p and combining it with the second order condition of bidder i, I get

$$\Pi_i^{''}(p^*(\alpha)) = \alpha i x^{'}(p^*(\alpha)) - \alpha\overline{Q} f(p^*(\alpha)) < 0$$

This is clearly non positive when and $\alpha ix(.) + \alpha\overline{Q}[1 - F(.)]$ is non increasing..

Global second order conditions for ex-post profit maximization are satisfied everywhere on $[\underline{p_i}, \overline{p_i}]$. Thus, the demand schedules form a “Demand” Function Equilibrium tracing through ex-post optimal points on $[\underline{p_i}, \overline{p_i}]$.

Moreover, as no bidder has incentives to deviate at each $\underline{p_i}$ or $\overline{p_i}$ for all i, the demand schedules form a DFE tracing through ex-post optimal points on $[0, v]$.

- Necessity:

Satisfaction of the first order condition of the problem together with the Market Clearing condition is a necessary condition for a supply function to trace through ex-post optimal points.

Moreover, if $\alpha ix(p^*(\alpha)) + \alpha\overline{Q}[1 - F(p^*(\alpha))] \geq 0$, then $\Pi^{''}(p^*(\alpha)) \geq 0$.

Therefore, the demand schedules $x(.)$ cannot be a symmetric DFE.

Step 3 I now determine the price $\overline{p_i}$ which is the highest price for which the financial constraints of i bidders are non binding and the financial constraints of the other $N - i$ are binding.

$\overline{p_i}$ is the highest price such that bidders with costs $c_1, ..., c_{i+1}$ participate to the auction and bidders with costs $c_{i+2}, ..., c_N$ do not participate.

$\overline{p_i}$ is thus the price such that bidders with costs c_{i+1} participate when $p \leq \overline{p_i}$ and do not participate when $p > \overline{p_i}$.

Consequently, $\overline{p_i}$ is the price at which the profits of a bidder with costs c_{i+1} when he participates to the auction is equal to the profits of the same bidder when he does not participate. This gives:

$$\overline{p_i} = -\frac{c_{i+1}[N_1 + ... + N_{i+1}]}{\left(Q - \alpha\overline{Q}[1 - F(p^*(\alpha))]\right)} + v$$

Step 4 Let α_i, the degree of rationing such that when $\alpha \leq \alpha_i$, bidders with costs c_{i+2} participate and when $\alpha > \alpha_i$, only bidders with costs lower than c_{i+1} participate.

As $p^*(.)$, the equilibrium price, is strictly increasing, then α_i exists for all i.

Moreover, I have $\alpha_i = \left[\frac{k_i}{c_i(N_1+...+N_i)e^{\Gamma_i(p^*(\alpha_i),\alpha_i)}}\right]^{\frac{1}{N_1+...+N_i-1}}$

The price $\underline{p_i}$ is defined by $\underline{p_i} = \overline{p_{i+1}}(\alpha_i)$, such that bidders with costs strictly higher than c_{i+1} have no incentives to deviate when $\alpha = \alpha_i$.

Remember that $\overline{p_{i+1}}(\alpha) = -\frac{k_{i+1}}{\left(Q - \alpha\overline{Q}[1-F(p^*(\alpha))]\right)e^{\Gamma_{i+1}(p^*(\alpha),\alpha)}\alpha^{N_1+...+N_{i+1}-1}} + v$ is the equilibrium price when the degree of rationing is α and only bidders with costs strictly lower than c_{i+1} participate.

$\underline{p_i}$ must satisfy the following equation:

$$\Pi_i(\overline{p_{i+1}}, \alpha_i^-) = \Pi_i(\underline{p_i}, \alpha_i^+)$$

$$\Longleftrightarrow k_{i+2} = \frac{\left[c_{i+2}\left(N_1 + ... + N_{i+2}\right)\right]}{\left[c_{i+2}\left(N_1 + ... + N_{i+1}\right)\right]^{\frac{N_1+...+N_{i+1}-1}{N_1+...+N_{i+1}}} \left[e^{\Gamma_{i+1}(p^*(\alpha),\alpha)}\right]^{\frac{N_{i+2}}{N_1+...+N_{i+1}-1}}} \left[k_{i+1}\right]^{\frac{N_1+...+N_{i+2}-1}{N_1+...+N_{i+1}-1}}$$

I finally get

$$\underline{p_i} = -\frac{c_{i+2}\left[N_1 + ... + N_{i+1}\right]}{\left(Q - \alpha\overline{Q}\left[1 - F\left(p^*(\alpha)\right)\right]\right)} + v$$

■

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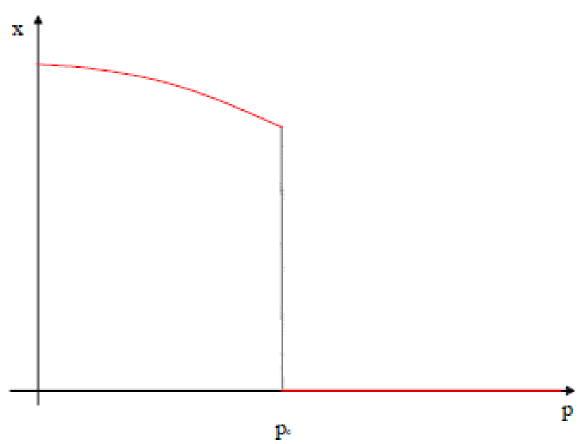

Figure 1: Equilibrium demand schedule of a high cost bidder.

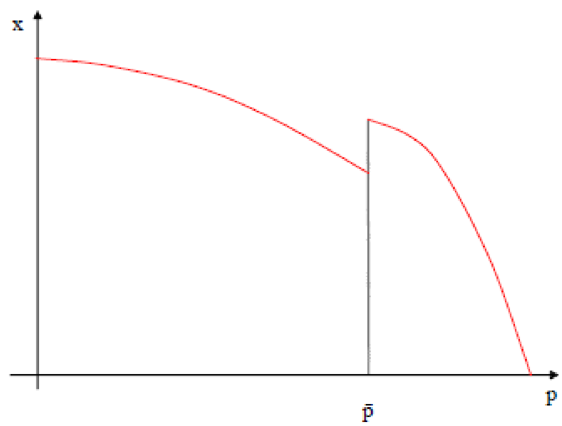

Figure 2: Equilibrium demand schedule of a high cost bidder.

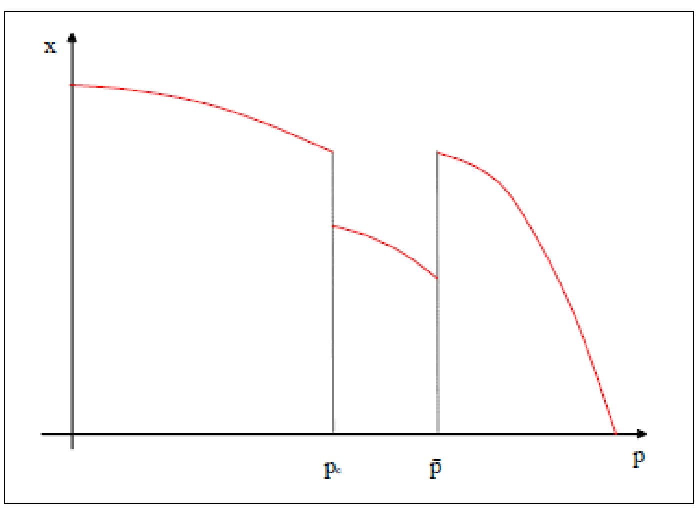

Figure 3: Aggregate equilibrium demand schedule.