# Optimal endogenous growth with natural resources

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**Abstract:** This paper considers an optimal endogenous growth model where production function is assumed to exhibit increasing returns to scale in technological change and two energy resources are not perfect substitutes. Natural resources, labors and physical capital are used in the final goods sector and in the accumulation of knowledge. Based on results in calculus of variations, a direct proof of existence of optimal solution is provided. Analytical solutions for the planner case,balanced growth paths and steady states are found for a specific CRRA utility and Cobb-Douglas production function. Equilibrium stability and transitional dynamics are also analyzed. It is possible to have a long-run growth where both energy resources are consumed simultaneously along the equilibrium path. As the law of motion of the technological change is not concave reflecting the increasing return of scale so that the Arrow-Mangasarian sufficiency conditions do not apply, we provide directly a sufficient condition.

**Keywords:** Endogenous optimal growth, transitional dynamics, energy transitions, renewable resource, non-renewable resource.

**JEL Classification:** C61, D51, E13.

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# 1 Introduction

There is a large theoretical literature investigates the factors influencing the increasing scarcity of natural resources and assess their impact on the allocation of production factors and on the changes they bring to the world economy. Recently, because of concerns about global climate change and its impacts on human well-being, a growing interest is deserving to the possibility of using alternative energy transition and technological change to reduce energy consumption from the combustion of fossil fuels. In particular, an important question is whether growth is sustainable in the presence of natural resource scarcity and appropriate environment policies of substitution mechanisms of alternative energy and technological progress. Under the general umbrella of growth model, we specially pay more attention to the issues arising in the allocation of resources over an infinite time horizon where the production function is assumed to exhibit increasing returns to scale in endogenous technological change and two types energy resources are not perfect substitutes. Infinite time horizon optimization is relevant when one deals with sustainable development since one is concerned with the very long term consequences of socio-economics decisions. More precisely, the exploration of the infinite time horizon optimization, with the characterization of stability properties of the dynamical systems used to represent economic equilibrium process, contributes to address such important research questions in endogenous growth and natural resource economics : It is possible to have a long-run growth where both energy resources are consumed simultaneously along the equilibrium path? Which role can biomass / renewable resources play in energy generation? What is the impact of environmental concerns on growth, and in particular, how are the level, the paths or the growth rates of crucial variable such as capital, output, consumption, labors and technological change levels. Is steady state equilibrium is indeed optimal along with balanced growth paths? The paper also explores the mathematical contribution of the dynamic optimization theory to study the optimal solutions of natural resources in non-convex optimal economic growth models.

Most of existing studies focus separately renewable and non-renewable resources. Recently, the roles of renewable and non-renewable resources have been simultaneously analyzed, e.g. Tahvonen and Salo (2001), Gerlagh and van der Zwaan (2003), Tsur and Zemel (2003), André and Cerdá (2006), Grimaud and Rougé (2004, 2005, 2008), Growiec and Schumacher (2008). Bretschger (2006) provided an empirical validation of the balanced growth path derived from an endogenous growth model with energy. Almost these papers mainly study the substitutability of natural resources (substitution between man-made and natural capitals, substitution between renewable and non-renewable resources) in which the usual concave production functions exhibit constant returns to scale in endogenous technological change or conditions of exogenous technological changes that ensure a sustainable economic growth.

In line with this strand of literature, this paper investigates a more technical issue in a framework where technological change is endogenous and the production employs labor, physical capital, and both type of renewable and non-renewable energy resources. The production function is assumed to exhibit increasing returns to scale in endogenous technological change, we present a rigorous proof of the existence of the optimal solution of a non-convex model, which is often assumed in the many papers in the literature. As always, the arguments for existence of solutions rely on compactness of feasible set and some form of continuity of objective function. We first prove the uniformly boundedness of feasible set (assumptions in d'Albis et al, 2002) that deduces the Lebesgue uniformly integrability. The theorem of Dunford-Pettis ( Dunford-Schwartz (1967)) which characterizes the Lebesgue uniformly integrability and the relatively weak compactness of feasible set is needed in the proof. Then we prove the set of feasible consumption paths is compact. Combined with compactness, upper semi-continuous of objective function is all that is necessary for existence of a maximum.

Most of theoretical literature on energy transition is based on the assumption that two resources are perfect substitutes, renewable substitutes can be produced under constant marginal cost and a backstop technology can produce unlimited amount of a renewable substitute at a constant cost per unit (Gerlagh (2011)). In such model resources use has two distinct phases: in the first phase, only the exhaustible resource is used, and in the second phase, the renewable substitute completely replace the use of fossil fuels. By this construction there is no phase where both resources are consumed simultaneously. However in reality, renewable resources such as solar energy cannot entirely replace petroleum in a number of uses at the present level of technological change. We then relax the assumption of perfect substitutability. Our results show that there exists a steady state growth equilibrium where both resources are consumed simultaneously along the equilibrium path and technological change is growing without bound. We also analyze the transitional dynamics and the stability of equilibrium.

As Arrow sufficiency conditions can not apply when Hamiltonian is not jointly concave in state variables, we prove a new type of transversality conditions to obtain a direct sufficient condition for any path satisfies the necessary conditions is optimal. This is crucial as when we check the maximality of the Hamiltonian we can write it can decompose it into two parts: the first just relies on the control variables and we have concavity in the objective function in control variables, and thus, using standard results the difference between the candidate solution and any other solution is non-negative; and a term that depends on the co-state and the state variables as given above. Recall, the non-concavity in the problem arises from the law of evolution of state variables only. As this term converges to zero, we are able to obtain sufficiency of the first order conditions. Thus, three things turn out to be important in our problem: the boundedness of the state and control and hence, co-state variables; concavity of objective in control variables; and the ability to separate the control and state/co-state variables in the Hamiltonian. As a result the steady states are indeed locally optimal.

The paper is organized as follows. After the Introduction, in Section 2 we introduce the general optimal endogenous growth model. The existence of an optimal solution is shown in Section 3. Section 4 proves the existence of steady state growth equilibrium, finds analytical optimal growth rates of all variables and analyze the transitional dynamics. In section 5 we show that the steady state is indeed locally optimal by proving a direct sufficient condition.

# 2 The model

The model can be heuristically described as follows. The aggregate output produced from the labor, physical capital and two types of natural resources: non-renewable resources (e.g., fossil fuels) and renewable resources (solar, thermal, biomass, etc.). The final product is shared between consumption and investment in physical capital. The representative consumer derives her utility from consumption. The production function takes the form

$$Y = A^{\theta} f(K, L_Y, Q, R)$$

where $A$, $L_Y, K, Q$ and $R$ represent the technological level, labor input, physical capital input, non-renewable resource (or fossil energy), and renewable resource (or non-fossil energy), respectively. As the technical progress can be increasing return to scale in the production function, the positive constant $\theta$ and may get the value greater than 1 (see Groth and Schou (2007) or Groth and Schou (2002) for the discussion about strictly endogenous growth). As it is not necessary, the time index is not included for simplifying our notation.

We assume the law of motion of technological change is

$$\dot{A} = \Psi(A, L_A)$$

where $L_A$ is labor employed for research and $\Psi$ is a knowledge production function. Normalizing the total flow of labor we have

$$L_Y + L_A = 1.$$

The final output can be allocated between consumption and investment (or capital accumulation)

$$\dot{K} = Y - C - \delta K = I - \delta K,$$

where $\delta \in (0, 1)$ is the depreciate rate of the stock of capital.

It is standard that the dynamics of stock non-renewable resource is following

$$\dot{S}_Q = -Q,$$

where $S_{Q_t}$ is the stock of exhaustible resource at time $t$. It follows from this equation and non-negative restriction on $Q$ that

$$\int_0^{\infty} Q dt \leq S_{Q_0}.$$

The dynamics of stocks of renewable is

$$\dot{S}_R = h(S_R) - R$$

where $h$ is a regeneration function.

The representative consumer's utility function is given by

$$U = \int_0^{\infty} u(C) e^{-\rho t} dt$$

## 3 Existence of optimal solution

In this section, we prove the existence of solution to the social problem (P):

$$\max \int_0^\infty u(C)e^{-\rho t}dt$$

subject to

$$\begin{aligned} \dot{S}_R &= h(S_R) - R, & (1)\\ \dot{S}_Q &= -Q, & (2)\\ \dot{A} &= \Psi(A, L_A), & (3)\\ \dot{K} &= A^\theta f(K, L_Y, Q, R) - C - \delta K, & (4)\\ L_A + L_Y &= 1 & (5) \end{aligned}$$

and $C \geq 0, K \geq 0, A \geq 0, 0 \leq L_Y \leq 1, 0 \leq L_A \leq 1$, given $A_0, K_0, S_{Q_0}, S_{R_0}$.

As always, the arguments for existence of solutions rely on compactness of feasible set and some form of continuity of objective function. We first prove the uniformly boundedness of feasible set (assumptions in d'Albis et al, 2002) that deduces the Lebesgue uniformly integrability. The theorem of Dunford-Pettis ( Dunford-Schwartz (1967)) which characterizes the Lebesgue uniformly integrability and the relatively weak compactness of feasible set is needed in the proof. Then we prove the set of feasible consumption paths is compact. Combined with compactness, upper semi-continuous of objective function is all that is necessary for existence of a maximum.

Let us denote by $L^1(e^{-\rho t})$ is the set of function $f$ verifying $\int_0^\infty |f(t)|\, e^{-\rho t}dt < \infty$. Recall that $f_i(t) \in L^1(e^{-\rho t})$ weakly converges to $f(t) \in L^1(e^{-\rho t})$ for the topology $\sigma(L^1(e^{-\rho t}), L^\infty)$ (written as $f_i \rightharpoonup f$ ) if and only if for every $\Psi \in L^\infty$, $\int_0^\infty f_i q e^{-\rho t}dt$ converges to $\int_0^\infty f\Psi e^{-\rho t}dt$ as $i \to \infty$. ( written as $\int_0^\infty f_i\Psi e^{-\rho t}dt \longrightarrow \int_0^\infty f\Psi e^{-\rho t}dt$ ).

When writing $f_i \longrightarrow f^*$ we mean that for every $t \in [0, \infty), \lim_{i\to\infty} f_i(t) = f^*(t)$.

We make the following assumptions:

H1. *The function* $u(C) : R_+ \to R$ *is strictly concave, increasing and continuous.*

H2. *Functions* $f(K, L_Y, Q, R) : R_+^4 \to R_+$ *is continuously differentiable, increasing on all arguments, concave in* $L_Y, Q, R$ *and* $\lim_{K\to+\infty} f_K \leq 0$. Moreover, it satisfies the Inada conditions $\lim_{K\to 0} f_K = \lim_{L_Y\to 0} f_{L_Y} = \lim_{Q\to 0} f_Q = \lim_{R\to 0} f_R = 0$.

H3. *Functions* $\Psi(A, L_A) : R_+^2 \to R_+$ *is continuously differentiable, increasing in both arguments.Moreover, there exists a constant* $b$ *such that*

$$\Psi(A, L_A) \leq bA.$$

H4. *Functions* $h(S_R) : R_+ \to R_+$ *is continuously differentiable increasing and there exists a constant* $m$ *such that* $h(S_R) \leq mS_R$.

H5. *There exists* $\kappa, \varsigma, \pi \neq \infty$ such that $-\kappa \leq \dot{K}/K$ and $-\varsigma \leq \dot{S}_R/S_R, -\pi \leq \dot{S}_Q/S_Q$.

H6. $\rho > \max\{b, m, b\theta\}$.

H1-H4 are standards. We do not need $f$ is concave in capital. Assuming $\Psi(A, L_A) \leq bA$ has been used in Chichilnisky (1981) and it is weaker than the standard assumption $\lim_{A\to\infty} \frac{\partial\Psi(A,L_A)}{\partial A} = 0$. It means that after certain levels of technical change the technology is constrained in its knowledge capital represented by the depreciation parameter $b$. Similarly for assumption on function $h$.

Assumption H5 is reasonable. It implies that it is not possible that the growth rate of physical capital or stock of renewable resource converges to $-\infty$ rapidly and is weaker than those used in the literature where $\kappa$ is a physical depreciation rate (Chichilnisky (1981), d'Albis et al (2008)). Let us define the net investment : $I = \dot{K} - \delta K = A^\theta f(K, L_Y, Q, R) - C$. Then H5 implies there exist $\kappa \geq 0, \kappa \neq \infty$ such that $I + (\kappa - \delta)K \geq 0$. Thus if the standard assumption of non-negative investment holds (that means capital goods cannot be converted back into consumption goods) then H5 holds with $\kappa = \delta$. Therefore assumption non-negative investment is stronger than H.5 ($\kappa$ can take any value except for infinity. This assumption will be used to prove the boundedness of variables by some exponential functions which belong to $L^1(e^{-\rho t})$. H6 is similar to A4 in d'Albis et al (2008) which ensures a finite value of objective function and the maximal growth rate of the output is less than discount rate.

Now, the main idea for the proof of existence is to show that the control variables and derivatives of state variables weakly converge in a weak topology, while the state variables converge pointwise. The problem is that even if we have a weakly convergent sequence, the limit point may not be feasible. For pointwise convergent sequences, the continuity is all that is necessary to prove the feasibility. Therefore, concavity is not needed for state variables. We will show that the limit point is indeed optimal in the original problem. For weakly convergent sequence, Mazur's Lemma is used to change into pointwise convergence. Jensen's inequality is used to eliminate the convex-combination-coefficients to prove the feasibility. Thus, concavity with respect to control variables is crucial.

**Lemma 1** *Let us denote by $\mathcal{K} = (L_A, L_Y, Q, R, S_Q, S_R, A, K, C)$ the feasible path from $A_0, K_0, S_{Q_0}, S_{R_0}$ which satisfies (1)-(5) and $C \geq 0, K \geq 0, A \geq 0, Q \geq 0, R \geq 0, 0 \leq L_Y \leq 1, 0 \leq L_A \leq 1$. Then*

*i) $\mathcal{K}$ is relatively weak compact in $L^1(e^{-\rho t})$.*

*ii) State variables $S_Q, S_R, A, K$ are absolutely continuous.*

**Proof.** i) By (1) and assumption H4 we have $\dot{S}_R \leq h(S_R) \leq mS_R$ and we get $\dot{S}_R/S_R \leq m$. Thus, there exists $\overline{S}$ such that

$$\begin{aligned} 0 &\leq S_R \leq \overline{S}e^{mt}, \\ \dot{S}_R &\leq m\overline{S}e^{mt}. \end{aligned} \tag{6}$$

Thus, $S_R$ belongs to the space $L^1(e^{-\rho t})$ since

$$0 \leq \int_0^\infty S_R e^{-\rho t} dt \leq \overline{S} \int_0^\infty e^{(m-\rho)t} dt < +\infty.$$

According to H4, $-\dot{S}_R \leq \varsigma S_R \leq \overline{\varsigma S} e^{mt}$. It follows from (6) that $\left|\dot{S}_R\right| \leq \max\{m\overline{S}, \varsigma\overline{S}\}e^{mt}$ and

$$\int_0^\infty \left|\dot{S}_R\right| e^{-\rho t} dt \leq \max\{m\overline{S}, \varsigma\overline{S}\} \int_0^\infty e^{(m-\rho)t} dt < +\infty.$$

Since $0 \leq R = h(S_R) - \dot{S}_R \leq (m + \varsigma)\overline{S}e^{mt}$. Therefore we have

$$0 \leq \int_0^\infty R e^{-\rho t} dt \leq (m+\varsigma)\overline{S} \int_0^\infty e^{(m-\rho)t} dt < +\infty.$$

It follows from (2) that $0 \leq \int_0^t Q_s ds = -\int_0^t \dot{S}_{Q_s} ds = S_{Q_0} - S_Q \leq S_{Q_0}$. Thus $\left|\dot{S}_Q\right| = Q$, $S_Q \leq S_{Q_0}$ and $Q = -\dot{S}_Q \leq \pi S_Q$. We then have

$$\begin{aligned} \int_0^\infty S_Q e^{-\rho t} dt &\leq S_{Q_0} \int_0^\infty e^{-\rho t} dt < +\infty \\ \int_0^\infty Q e^{-\rho t} dt &= \int_0^\infty \left|\dot{S}_Q\right| e^{-\rho t} dt \leq \pi S_{Q_0} \int_0^\infty e^{-\rho t} dt < +\infty. \end{aligned}$$

Since $\lim_{K \to +\infty} f_K(K, 1, S_{Q_0}, S_{R_0}) \leq 0$, for any $\zeta \in (0, \rho - b\theta)$ there exist a constant $B_0$ such that

$$f(K, L_Y, Q, R) \leq B_0 + \zeta K.$$

It follows that

$$\dot{K} \leq B_0 + \zeta K.$$

Multiply by $e^{-\zeta s}$ we get $e^{-\zeta s}\dot{K} - \zeta K e^{-\zeta s} \leq B_0 e^{-\zeta s}$. Then we get

$$e^{-\zeta t} K = \int_0^t \frac{\partial(e^{-\zeta s} K)}{\partial s} ds + K_0 \leq \int_0^t B_0 e^{-\zeta s} ds + K_0 = \frac{-B_0 e^{-\zeta t}}{\zeta} + \frac{B_0 + \zeta K_0}{\zeta}.$$

This implies that there exists constant $B_1$ such that $K \leq B_1 e^{\zeta t}$. Hence $\int_0^\infty K e^{-\rho t} dt \leq \int_0^\infty B_1 e^{(\zeta - \rho)t} dt < +\infty$.

Furthermore, since $-\dot{K} \leq \kappa K$ and $\dot{K} \leq B_0 + \zeta K \leq B_0 + \zeta B_1 e^{\zeta t}$, there exist a constant $B_2$ such that $\left|\dot{K}\right| \leq B_2 e^{\zeta t}$. Thus,

$$\int_0^\infty \left|\dot{K}\right| e^{-\rho t} dt \leq \int_0^\infty B_2 e^{(\zeta - \rho)t} dt < +\infty.$$

Since $\Psi(A, L_A) \leq bA$, we have $\dot{A}/A \leq b$. There exists a constant $D_1$ such that $A \leq D_1 e^{bt}$. Moreover, we have $0 \leq \dot{A} \leq bA \leq D_1 e^{bt}$. Therefore, $A$, $\left|\dot{A}\right|$ belong to $L^1(e^{-\rho t})$ because

$$\begin{aligned} 0 &\leq \int_0^\infty A e^{-\rho t} dt \leq D_1 \int_0^\infty e^{(b-\rho)t} dt < +\infty \\ 0 &\leq \int_0^\infty \left|\dot{A}\right| e^{-\rho t} dt \leq D_1 \int_0^\infty e^{(b-\rho)t} dt < +\infty. \end{aligned}$$

As $-\dot{K} \leq \kappa K$, we have

$$\begin{aligned} C &\leq A^\theta f(K, L_Y, Q, R) + (\kappa - \delta) K \\ &\leq {D_1}^\theta e^{b\theta t}(B + \zeta K) + (\kappa - \delta) B_1 e^{\zeta t} \\ &\leq {D_1}^\theta e^{b\theta t}(B + \zeta B_1 e^{\zeta t}) + (\kappa - \delta) B_1 e^{\zeta t} \\ &= {D_1}^\theta \zeta B_1 e^{(b\theta + \zeta)t} + {D_1}^\theta B e^{b\theta t} + (\kappa - \delta) B_1 e^{\zeta t}. \end{aligned}$$

Thus, we can choose a positive constant $D_2 \geq {D_1}^\theta B + {D_1}^\theta \zeta B_1 + (\kappa - \delta) B_1$. Then

$$C \leq D_2 e^{(b\theta + \zeta)t} < D_2 e^{\rho t},$$

which implies

$$0 \leq \int_0^\infty C e^{-\rho t} dt < +\infty.$$

We have proven that $\mathcal{K}$ is uniformly bounded on $L^1(e^{-\rho t})$.

Moreover, $\lim_{a \to \infty} \int_a^\infty K e^{-\rho t} dt \leq \lim_{a \to \infty} \int_a^\infty B_1 e^{(\zeta - \rho)t} dt = 0$. This property is true for other variables in $\mathcal{K}$. Therefore $\mathcal{K}$ satisfies Dunford-Pettis theorem and it is relatively compact in the weak topology $\sigma(L^1(e^{-\rho t}), L^\infty)$.

ii) We have shown that every states variables and their derivatives lie in $L^1(e^{-\rho t})$. Thus they belong to Sobolev space which are integrable and absolutely continuous. ( Maz'ja, 1985). ■

Since $\mathcal{K}$ is relatively compact in the weak topology $\sigma(L^1(e^{-\rho t}), L^\infty)$, a sequence $X_i$ in $\mathcal{K}$ has convergent subsequences (denote by $X_i$ for simplicity of notation) which weakly converge to limit points in $L^1(e^{-\rho t})$.

The following Lemma shows that any weakly convergent sequence of control variables in $\mathcal{K}$ has a sequence of convex combinations of its members that converges pointwise to the same limit while the limit of weak convergence coincide with limit of pointwise convergence for state variables.

**Lemma 2** *i)Let state variables $(K, A, S_R, S_Q)_i$ in $\mathcal{K}$ and suppose that $(K, A, S_R, S_Q)_i \rightharpoonup (K^*, A^*, S_R^*, S_Q^*)$.*

*Then $(K, A, S_R, S_Q)_i \to (K^*, A^*, S_R^*, S_Q^*)$ as $i \to \infty$ and $(\dot{K}, \dot{A}, \dot{S}_R, \dot{S}_Q)_i \rightharpoonup (\dot{K}^*, \dot{A}^*, \dot{S}_R^*, \dot{S}_Q^*)$ for the the topology $\sigma(L^1(e^{-\rho t}), L^\infty)$.*

*ii) In addition, suppose that $\mathbf{Z}_i = (R, Q, \dot{K}, \dot{A}, \dot{S}_R, \dot{S}_Q)_i$ in $\mathcal{K}$ and $\mathbf{Z}_i \rightharpoonup \mathbf{Z}^* = (R^*, Q^*, \dot{K}^*, \dot{A}^*, \dot{S}_R^*, \dot{S}_Q^*)$ in $L^1(e^{-\rho t})$ then there exists a sequence of sets of real numbers $\{\omega_{i(n)} \mid i = n, ..., \mathcal{N}(n)\}$ such that $\omega_{i(n)} \geq 0$ and $\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1$ such that the sequence $(v_n)_{n \in N}$ defined by the convex combination $v_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \mathbf{Z}_i$ converges pointwise to $\mathbf{Z}^*$ as $n \to \infty$, i.e., for every $t \in [0, \infty)$, $\lim_{n \to \infty} v_n(t) = \mathbf{Z}^*(t)$. It is clearly that, since $(L_A, L_Y,)_{i(n)} \in [0, 1]$, $(L_A, L_Y,)_{i(n)} \to (L_A^*, L_Y^*)$ as $n \to \infty$.*

**Proof.** For any state variable $X_i \in \{(K, A, S_R, S_Q)_i\}$ in $\mathcal{K}$, by hypothesis $X_i \rightharpoonup X^*$. We first claim that, for $t \in [0, \infty)$, $\int_0^t X_i dt \to \int_0^t X^* dt$. Note that $X_i \rightharpoonup X^*$ for the topology $\sigma(L^1(e^{-\rho t}), L^\infty)$ if and only if for every $Y \in L^\infty$, $\int_0^\infty X_i Y e^{-\rho t} dt \to \int_0^\infty X^* Y e^{-\rho t} dt$.

Pick any $t$ in $[0, \infty)$ and let

$$Y(s) = \begin{cases} \frac{1}{e^{-\rho t}} \text{ if } s \in [0, t] \\ 0 \text{ if } s > t. \end{cases}$$

Therefore $Y(s) \in L^{\infty}$ and we get $\int_0^t X_i(s)ds = \int_0^{\infty} X_i(s)Y(s)e^{-\rho s}ds \to \int_0^{\infty} X^*(s)Y(s)e^{-\rho s}ds = \int_0^t X^*(s)ds$ .

Given that $K_i \rightharpoonup K^*$ and $\dot{K}_i \rightharpoonup y$ weakly for some $y$ in $L^1(e^{-\rho t})$, by the claim above, for all $t \in [0, \infty)$ we have $\int_0^t K_i ds \to \int_0^t yds$ . This implies, for a fix $t$, $K_i \to \int_0^t yds + K_0$. Thus $\int_0^t yds + k_0 = K^*$. Therefore $\dot{K}^* = y$ or $\dot{K}_i \rightharpoonup \dot{K}^*$. The same reasoning applies for $(A, S_R, S_Q)_i$ in $\mathcal{K}$.

ii) A direct application of Mazur's Lemma. ∎

We are now able to prove the existence of solution to the to the social planner's problem. The difficulty is that while we have convergence of the feasible sequences, as only the closure of the feasible set is closed since $\mathcal{K}$ is only weakly compact, we do not know that the limit point is in fact feasible. This is shown below.

**Theorem 1** *Under Assumptions H.1-H.7, there exists a solution to the social planner's problem.*

**Proof.** Since $u$ is concave, for any $\bar{c} > 0$, $u(C) - u(\bar{c}) \leq u'(\bar{c})(C - \bar{c})$. Thus, if $C \in L^1(e^{-\rho t})$ then $\int_0^{\infty} u(C)e^{-\rho t}dt$ is well defined because

$$\int_0^{\infty} u(C)e^{-\rho t}dt \leq \int_0^{\infty} [u(\bar{c}) - u'(\bar{c})\bar{c}]e^{-\rho t}dt + u'(\bar{c}) \int_0^{\infty} Ce^{-\rho t}dt < +\infty.$$

Let us define $W = \sup_{C \in \mathcal{K}} \int_0^{\infty} u(C)e^{-\rho t}dt$. Assume that $W > -\infty$ (otherwise the proof is trivial). Let $C_i \in \mathcal{K}$ be the maximizing sequence of $\int_0^{\infty} u(C)e^{-\rho t}dt$ so $\lim_{i \to \infty} \int_0^{\infty} u(C_i)e^{-\rho t}dt = W$.

Since $\mathcal{K}$ is relatively weak compact, suppose that $C_i \rightharpoonup C^*$ for some $C^*$ in $L^1(e^{-\rho t})$. By Mazur's Lemma, there is a sequence of convex combination

$$x_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} C_{i(n)} \to C^*, \omega_{i(n)} \geq 0, \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1.$$

Because $u$ is concave, we have

$$\begin{aligned} \limsup_{n \to \infty} u(x_n) &= \limsup_{n \to \infty} u(\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} C_{i(n)}) \\ &\leq \limsup_{n \to \infty} [u(C^*) + u'(C^*)(\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} C_{i(n)} - C^*)] = u(C^*). \end{aligned}$$

Since this holds for almost $t$, integrate w.r.t $e^{-\rho t}dt$ to get

$$\int_0^{\infty} \limsup_{n \to \infty} u(x_n)e^{-\rho t}dt \leq \int_0^{\infty} u(C^*)e^{-\rho t}dt < +\infty.$$

Using a reverse Fatou's lemma we yield

$$\limsup_{n \to \infty} \int_0^{\infty} u(x_n)e^{-\rho t}dt \leq \int_0^{\infty} \limsup_{n \to \infty} u(x_n)e^{-\rho t}dt \leq \int_0^{\infty} u(C^*)e^{-\rho t}dt. \tag{7}$$

Moreover, by Jensen's inequality we get

$$\limsup_{n \to \infty} \int_0^{\infty} u(x_n)e^{-\rho t}dt \geq \limsup_{n \to \infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \int_0^{\infty} u(C_{i(n)})e^{-\rho t}dt. \tag{8}$$

But since $\int_0^{\infty} u(C_{i(n)})e^{-\rho t}dt \to W$, (7) and (8) imply $\int_0^{\infty} u(C^*)e^{-\rho t}dt \geq W$.

So it remains to show that $C^*$ is feasible.

The task is now to show that there exists some $(K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ in $\mathcal{K}$ such that $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ satisfies (1)-(4).

Consider a feasible sequence $(K, L_A, L_Y, A, R, Q, S_R, S_Q)_{i(n)}$ in $\mathcal{K}$ associated with $C_{i(n)}$. According to Lemma2 and Jensen's inequality we have

$$\begin{aligned}
C^* &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} C_{i(n)} \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} [A_i^\theta f(K_i, L_{Y_i}, Q_i, R_i) - \delta K_i - \dot{K}_i] \\
&= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} [\lim_{n\to\infty} A_i^\theta f(\lim_{n\to\infty} K_{i(n)}, \lim_{n\to\infty} L_{Y_i}, Q_i, R_i) - \delta \lim_{n\to\infty} K_{i(n)}] - \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{K}_i \\
&\leq A^{*\theta} f(K^*, L_Y^*, \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} Q_i, \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} R_i) - \delta K^* - \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{K}_i \\
&= A^{*\theta} f(K^*, L_Y^*, Q^*, R^*) - \delta K^* - \dot{K}^*.
\end{aligned}$$

Therefore,

$$C^* \leq A^{*\theta} f(K^*, L_Y^*, Q^*, R^*) - \delta K^* - \dot{K}^*.$$

Applying a similar argument and using Jensen's inequality we get

$$\begin{aligned}
\dot{A}^* &= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{A}_{i(n)} = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \Psi(\lim_{n\to\infty} A_i, \lim_{n\to\infty} L_{A_i}) = \Psi(A^*, L_A^*), \\
\dot{S}_R^* &= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{S}_{R_i} = \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} (h(S_{R_i}) - R) = h(S_R^*) - R^*, \\
\dot{S}_Q^* &= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{S}_{Q_i} = -\lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} Q_i = -Q^*
\end{aligned}$$

Therefore, $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ satisfies (1)-(4).

The proof is done. ■

d'Albis et al (2008) also establish existence of an optimal solution in an abstract model. Our proof is direct and more constructive with less stringent assumptions. Indeed, the Lemma 1 proves the boundedness assumptions in their model. We have proven that the control variables and derivative of state variables weakly converge in the weak topology $\sigma(L^1(e^{-\rho t}), L^\infty)$, while the state variables pointwise converge. And for pointwise converge sequence, the continuity is all that is necessary to prove the feasibility. Therefore, concavity is not needed for the state variables. Related results on the existence of solution can be found in Chichilnisky (1981) who used the theory of Sobolev weighted space and imposed a Caratheodory condition on utility function.

# 4 Existence of Steady State Growth Equilibrium

Let denote $\sigma_C = -\frac{CU_{Cc}}{U_C}$ the elasticity of marginal utility and $F = A^\theta f(K, L_Y, Q, R)$. We have proven that (P) has a solution denoted by $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$. Then the solution are characterized by the Kuhn-Tucker conditions, Euler equations and transversality conditions.

The current-value Hamiltonian is

$$\begin{aligned}
H(C, K, Q, R, L_Y, A) &= u(C) + \lambda(mS_R - R) - \mu Q \\
&\quad + \nu(F - C - \delta K) + \omega b A^\phi (1 - L_Y)
\end{aligned}$$

where $\lambda, \mu, \nu, \omega$ are four costate variables.

The first order conditions $\frac{\partial H}{\partial C}=0$, $\frac{\partial H}{\partial Q}=0$, $\frac{\partial H}{\partial R}=0$, $\frac{\partial H}{\partial L_Y}=0$ yield

$$\nu = U_C, \tag{9}$$
$$\mu = vF_Q, \tag{10}$$
$$\lambda = vF_R, \tag{11}$$
$$\omega = \frac{vF_{L_Y}}{bA^{\phi}}. \tag{12}$$

From Euler equations $\frac{\partial H}{\partial K}=\rho\nu-\dot{\nu}$, $\frac{\partial H}{\partial S_R}=\rho\lambda-\dot{\lambda}$, $\frac{\partial H}{\partial S_Q}=\rho\mu-\dot{\mu}$, and $\frac{\partial H}{\partial A}=\rho\omega-\dot{\omega}$ we get

$$\frac{\dot{\nu}}{v} = \rho - F_K - \delta \tag{13}$$
$$\frac{\dot{\mu}}{\mu} = \rho \tag{14}$$
$$\frac{\dot{\lambda}}{\lambda} = \rho - m \tag{15}$$
$$\dot{\omega} = (\rho - b\phi A^{\phi-1}(1-L_Y))\omega - vF_A.$$

The transversality conditions are

$$\lim_{t\to+\infty} \lambda S_R^* e^{-\rho t} = \lim_{t\to+\infty} \mu S_Q^* e^{-\rho t} = \lim_{t\to+\infty} \nu K^* e^{-\rho t} = \lim_{t\to+\infty} \omega A^* e^{-\rho t} = 0. \tag{16}$$

Moreover, it is easy to see that, at the optimum, the Hotelling rules are satisfied:

$$\rho + \sigma_C \frac{\dot{C}^*}{C^*} = \frac{\dot{F}_Q}{F_Q}$$
$$\rho + \sigma_C \frac{\dot{C}^*}{C^*} = \frac{\dot{F}_R}{F_R} + h_{S_R}.$$

Before analyzing the full dynamic system, we look at the characterization of a balanced optimal growth path. Our model corresponds to a system with four state variables, we specify a set of restrictions imposed on preferences and production technology in order to have analytical results. We make the following assumptions:

H7.

$$u(C) = \begin{cases} \frac{C^{1-\varepsilon}-1}{1-\varepsilon}, \text{ if } \varepsilon \neq 1, \\ \ln C \text{ if } \varepsilon = 1. \end{cases},$$

H8. $f(L_Y, K, Q, R) = L_Y^{\gamma} K^{\xi} Q^{\alpha} R^{\beta}$ where $\gamma, \xi, \alpha, \beta \geq 0, \gamma+\xi+\alpha+\beta = 1$.

H9. $\Psi(A, L_A) = bA^{\phi}L_A$ where $b > 0, 0 < \phi \leq 1$

H10. $h(S_R) = mS_R, m > 0$.

This specification satisfies H1-H6. H9 is widely used in the literature (see, e.g., Aghion and Howitt, 1998, Jones, 2006). Assumption H10 allows for constant infinite growth for renewable which is not realistic for ecological restriction. However, it may be reasonable for a type of of non-fossil energy such as solar energy, wind energy or nuclear energy in which the stock of alternative energy can be considered as infinity.

Let $g_\chi = \dot{\chi}/\chi$ denote the growth rate of any variable $\chi$. We shall summarize the macroeconomic equilibrium in terms of five variables: $x = F/K, y = C/K, z = Q/S_Q, u = R/S_R, q = L_Y A^{\phi-1}, r = A^{\phi-1}$ from which other equilibrium rates $g_F, g_K, g_C, g_{L_Y}, g_{L_A}, g_A, g_Q, g_{S_Q}, g_R$, and $g_{S_R}$ can be derived as in the following proposition.

**Proposition 1** *The optimal growth rates take the following values*

$$
\begin{aligned}
g_A &= b(r-q),\\
g_K &= x-y-\delta,\\
g_C &= \frac{\xi x-\delta-\rho}{\varepsilon},\\
g_{S_Q} &= -z,\\
g_{S_R} &= m-u,\\
g_Q &= -y+\frac{b\theta r}{\xi}+\frac{m\beta+(1-\xi)\delta}{\xi},\\
g_R &= -y+\frac{b\theta r}{\xi}+\frac{m(\beta+\xi)+(1-\xi)\delta}{\xi},\\
g_{L_Y} &= -y+\frac{b\theta r}{\xi}+\frac{b\theta}{\gamma}q+\frac{m\beta+(1-\xi)\delta}{\xi},\\
g_F &= \xi x-y+\frac{b\theta r}{\xi}+\frac{m\beta+\delta(\alpha+\beta+\gamma)}{\xi}-\delta,\\
g_{L_A} &= \frac{q}{q-r}g_{L_Y}.
\end{aligned}
$$

**Proof.** By (9) and $\dot{A}/A = bA^{\phi-1}(1-L_Y)$ we get

$$\dot{\omega} = (\rho-\phi g_A)\omega - U_C F_A. \tag{17}$$

The transversality conditions are

$$\lim_{t\to+\infty}\lambda S_R e^{-\rho t} = \lim_{t\to+\infty}\mu S_Q e^{-\rho t} = \lim_{t\to+\infty}\nu K e^{-\rho t} = \lim_{t\to+\infty}\omega A e^{-\rho t} = 0.$$

From the identities $\dot{S}_R = mS_R - R$, $\dot{S}_Q = -Q$ and $\dot{K} = F - C - \delta K$, we obtain

$$
\begin{aligned}
g_{S_Q} &= -z,\\
g_{S_R} &= m-u,\\
g_K &= x-y-\delta,\\
g_A &= b(r-q).
\end{aligned}
$$

Since $F = A^\theta L_Y^\gamma K^\xi Q^\alpha R^\beta$, we have

$$F_K = \xi F/K = \xi x,$$

$$
\begin{aligned}
\frac{\dot{F}_Q}{F_Q} &= \theta g_A+\gamma g_{L_Y}+\xi g_K+(\alpha-1)g_Q+\beta g_R, && (18)\\
\frac{\dot{F}_R}{F_R} &= \theta g_A+\gamma g_{L_Y}+\xi g_K+\alpha g_Q+(\beta-1)g_R, && (19)\\
\frac{\dot{F}_{L_Y}}{F_{L_Y}} &= \theta g_A+(\gamma-1)g_{L_Y}+\xi g_K+\alpha g_Q+\beta g_R && (20)
\end{aligned}
$$

Equation (9) together with (13) yield

$$\rho - \frac{\dot{U}_C}{U_C} = F_K - \delta = \xi x - \delta. \tag{21}$$

It is easy to check that

$$\frac{\dot{U}_C}{U_C} = -\varepsilon\frac{\dot{C}}{C} = -\varepsilon g_C. \tag{22}$$

Thus,

$$g_C = \frac{\xi x-\delta-\rho}{\varepsilon}.$$

By logarithmic differentiation (10) and together with (14) we have

$$\frac{\dot{F}_Q}{F_Q} = \rho - \frac{\dot{U}_C}{U_C} = \xi x - \delta.$$

It follows from (11) and (15) that

$$\frac{\dot{F}_R}{F_R} = \rho - \frac{\dot{U}_C}{U_C} - m = \xi x - \delta - m.$$

From (12) and (17) we get

$$\begin{aligned}
\frac{\dot{U}_C}{U_C} + \frac{\dot{F}_{L_Y}}{F_{L_Y}} - \phi g_A &= \rho - \phi g_A - \frac{U_C F_A}{\omega} = \\
\rho - \phi g_A - \frac{U_C F_A}{U_C F_{L_Y}} A^{\phi} &= \rho - \phi g_A - \frac{\theta F/A}{\gamma F/L_Y} A^{\phi} = \\
\rho - \phi g_A - \frac{\theta}{\gamma} L_Y A^{\phi - 1} &= \rho - \phi g_A - \frac{b\theta}{\gamma} q.
\end{aligned}$$

Thus,

$$\frac{\dot{F}_{L_Y}}{F_{L_Y}} = \rho - \frac{\dot{U}_C}{U_C} - \frac{b\theta}{\gamma} q = \xi x - \delta - \frac{b\theta}{\gamma} q.$$

Now, it follows from (18)-(20) that we have a system of equations with three variables $g_Q$, $g_R$, and $g_{L_Y}$:

$$\begin{aligned}
\gamma g_{L_Y} + (\alpha - 1) g_Q + \beta g_R &= \xi y - b\theta(r - q) + (\xi - 1)\delta = T_1, && (23) \\
\gamma g_{L_Y} + \alpha g_Q + (\beta - 1) g_R &= \xi y - m - b\theta(r - q) + (\xi - 1)\delta, && (24) \\
(\gamma - 1) g_{L_Y} + \alpha g_Q + \beta g_R &= \xi y - \frac{b\theta}{\gamma} q - b\theta(r - q) + (\xi - 1)\delta. && (25)
\end{aligned}$$

From (23)-(24) we have $-g_Q + g_R = m$. From (23)-(25) we get $\gamma g_{L_Y} = b\theta q + \gamma g_Q$. Replacing this equation into (23), we have

$$(\gamma + \alpha - 1) g_Q + \beta g_R = T_1 - b\theta q$$

We then have two equations to find $g_Q$ and $g_R$:

$$\begin{aligned}
g_Q &= \frac{m\beta + b\theta q - T_1}{\xi} = \frac{b\theta r - \xi y + m\beta + (1 - \xi)\delta}{\xi}, \\
g_R &= g_Q + m = \frac{b\theta r - \xi y + m(\beta + \xi) + (1 - \xi)\delta}{\xi},
\end{aligned}$$

and

$$g_{L_Y} = \frac{b\theta}{\gamma} q + g_Q = \frac{b\theta r - \xi y + m\beta + (1 - \xi)\delta}{\xi} + \frac{b\theta}{\gamma} q.$$

By log-differentiating $F = A^{\theta} L_Y^{\gamma} K^{\xi} Q^{\alpha} R^{\beta}$ we get

$$\begin{aligned}
g_F &= \theta g_A + \gamma g_{L_Y} + \xi g_K + \alpha g_Q + \beta g_R = \\
&= \gamma g_{L_Y} + (\alpha - 1) g_Q + \beta g_R + \theta g_A + \xi g_K + g_Q \\
&= T_1 + \theta g_A + \xi g_K + g_Q \\
&= \xi y + (\xi - 1)\delta + \xi(x - y - \delta) + \frac{b\theta r - \xi y + m\beta + (1 - \xi)\delta}{\xi} \\
&= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(1 - 2\xi)}{\xi} \\
&= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(\alpha + \beta + \gamma)}{\xi} - \delta.
\end{aligned}$$

Finally, since $L_Y = \frac{q}{r}$,

$$g_{L_A} = \frac{\dot{L}_A}{L_A} = -\frac{\dot{L}_Y}{1 - L_Y} = \frac{L_Y}{L_Y - 1} g_{L_Y} = \frac{q}{q - r} g_{L_Y}.$$

■

A steady state satisfies that all rates of growth are constant. Let $\chi^*$ and $g_{\chi}^*$ denote respectively the value and the growth rate of any variable $\chi$ at the steady state.

**Proposition 2** *There exists a positive steady state equilibrium growth rates which take the following values*

$$
\begin{aligned}
g_Q^* &= g_{S_Q}^* = -y^* + \frac{b\theta r^*}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi},\\
g_R^* &= g_{S_R}^* = -y^* + \frac{b\theta r^*}{\xi} + \frac{m(\beta+\xi) + (1-\xi)\delta}{\xi},\\
g_F^* &= g_K^* = g_C^* = \frac{\xi x^* - \delta - \rho}{\varepsilon},\\
g_{L_Y}^* &= g_{L_A}^* = 0,\\
g_A^* &= b(r^* - q^*),
\end{aligned}
$$

*where if $\phi = 1$ then*

$$
\begin{aligned}
x^* &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi(1-\xi)},\\
y^* &= \frac{(\varepsilon-\xi)(b\theta + m\beta + \delta(1-\xi)) + \xi(1-\varepsilon)\delta + \rho]}{\varepsilon\xi(1-\xi)},\\
q^* &= [y^* - \frac{m\beta + (1-\xi)\delta + b\theta}{\xi}]\frac{\gamma}{\theta b},\\
r^* &= 1,
\end{aligned}
$$

*and if $\phi < 1$ then $x^*, y^*, q^*, r^*$ are given by*

$$
\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta,\\
(\xi - 1)x^* - y^* + \delta + \frac{b\theta q^* + m\beta + \delta(1-2\xi)}{\xi} &= 0, \qquad (26)\\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta q^*}{\xi} + \frac{b\theta}{\gamma}q^* &= 0, \qquad (27)\\
r^* &= q^*.
\end{aligned}
$$

**Proof.** At the steady state, $g_A^* = b(r^* - q^*)$ is constant. Therefore, since $g_C^*$ and $g_K^*$ are constant, it follows that $x^*$ and $y^*$ are constant. It also follows from constant $g_Q^*$ that $r^*$ is constant. Thus, $q^*$, $z^*$, and $u^*$ are also constant. Since $x = F/K$ and $y = C/K$, we have $g_C^* = g_F^* = g_K^*$. Moreover, $L_Y^* = q^*/r^*$ is constant, which implies that $L_A^*$ is constant. So, we get

$$g_{L_Y}^* = g_{L_A}^* = 0.$$

Since $r^* = A^{*\phi-1}$ is constant, we have $(\phi - 1)g_A^* = 0$ or

$$(\phi - 1)b(r^* - q^*) = 0.$$

This equation together with $g_C^* = g_K^*$, $g_F^* = g_K^*$, and $g_{L_Y}^* = 0$, yield

$$
\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta,\\
\xi x^* - y^* + \frac{b\theta r^* + m\beta + \delta(1-2\xi)}{\xi} &= x^* - y^* - \delta,\\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta r^*}{\xi} + \frac{b\theta}{\gamma}q^* &= 0,\\
(\phi - 1)(r^* - q^*) &= 0.
\end{aligned}
$$

We consider two cases:

a) If $\phi = 1$, $r^* = A^{*\phi-1} = 1$, we have

$$
\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta\\
\xi x^* - y^* + \frac{b\theta + m\beta + \delta(1-2\xi)}{\xi} &= x^* - y^* - \delta\\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta}{\xi} + \frac{b\theta}{\gamma}q^* &= 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
x^* &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi(1-\xi)}, \\
y^* &= \frac{(\varepsilon-\xi)(b\theta + m\beta + \delta(1-\xi)) + \xi(1-\varepsilon)\delta + \rho]}{\varepsilon\xi(1-\xi)}, \\
q^* &= [y^* - \frac{m\beta + (1-\xi)\delta + b\theta}{\xi}]\frac{\gamma}{\theta b}.
\end{aligned}
$$

b) If $\phi \neq 1$, $A^* = (r^*)^{1/\phi-1}$, and then $r^* = q^*$. Hence, we have three equations which determine the optimal growth rates at the steady state

$$
\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta, \\
\xi x^* + \frac{b\theta q^* + m\beta + \delta(1-2\xi)}{\xi} &= x^* - \delta, \\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta q^*}{\xi} + \frac{b\theta}{\gamma}q^* &= 0.
\end{aligned}
$$

Finally, since $\dot{S}_Q/S_Q = -Q/S_Q$ and $g^*_{S_Q}$ is constant, we have $g^*_Q = g^*_{S_Q}$. Similarly, we have $g^*_R = g^*_{S_R}$. ■

We now study the dynamic behavior of the nonlinear system characterized by the behavior of the linearized system around the steady state.We shall summarize the macroeconomic equilibrium in terms of four stationary variables, $x = F/K, y = C/K, z = Q/S_Q, u = R/S_R, q = A^{\phi-1}L_Y, r = A^{\phi-1}$, from which other equilibrium rates can be derived as in Proposition 1. Let us denote $\mathbf{h} = (x, y, z, u, q, r)$. In the neighborhood of the steady state we can write $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$ where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$ and $\mathbf{J}$ is the Jacobian matrix evaluated at the steady state :

$$
\mathbf{J} = \begin{pmatrix}
\partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial u & \partial\dot{x}/\partial q & \partial\dot{x}/\partial r \\
\partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial z & \partial\dot{y}/\partial u & \partial\dot{y}/\partial q & \partial\dot{y}/\partial r \\
\partial\dot{z}/\partial x & \partial\dot{z}/\partial y & \partial\dot{z}/\partial z & \partial\dot{z}/\partial u & \partial\dot{z}/\partial q & \partial\dot{z}/\partial r \\
\partial\dot{u}/\partial x & \partial\dot{u}/\partial y & \partial\dot{u}/\partial z & \partial\dot{u}/\partial u & \partial\dot{u}/\partial q & \partial\dot{u}/\partial r \\
\partial\dot{q}/\partial x & \partial\dot{q}/\partial y & \partial\dot{q}/\partial z & \partial\dot{q}/\partial u & \partial\dot{q}/\partial q & \partial\dot{q}/\partial r \\
\partial\dot{r}/\partial x & \partial\dot{r}/\partial y & \partial\dot{r}/\partial z & \partial\dot{r}/\partial u & \partial\dot{r}/\partial q & \partial\dot{r}/\partial r
\end{pmatrix}.
$$

By investigating the number of negative eigenvalues of the Jacobian, we obtain.

**Proposition 3** *There is a saddle point of stability.*

**Proof.** *Case 1.* If $\phi = 1$. In this case $r = 1$, we just need to analyze the dynamic system of $x, y, z, u, q$. By logarithmic differentiation and using Proposition 1 we get

$$
\begin{aligned}
\dot{x} &= (g_F - g_K)x = [(\xi - 1)x + \frac{b\theta}{\xi} + \frac{m\beta + \delta(1-\xi)}{\xi}]x, \\
\dot{y} &= (g_C - g_K)y = [\frac{(\xi - \epsilon)x + (\varepsilon - 1)\delta - \rho}{\varepsilon} + y]y, \\
\dot{z} &= (g_Q + z)z = (-y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + z)z, \\
\dot{u} &= (g_R + u - m)u = (-y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + u)u, \\
\dot{q} &= (g_{L_Y})q = (-y + \frac{b\theta}{\xi} + \frac{b\theta}{\gamma}q + \frac{m\beta + (1-\xi)\delta}{\xi})q.
\end{aligned}
$$

The dynamics of $\mathbf{h} = (x, y, z, u, q, r)$ is described by the system above. From the theory of linear approximation we know that in the neighborhood of the steady state, the dynamic behavior of the nonlinear system is characterized by the behavior of the linearized system around the steady state $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$ where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$ and $\mathbf{J}$ is the Jacobian matrix evaluated at the steady state, i.e.

$$
\mathbf{J} = \begin{pmatrix}
\partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial u & \partial\dot{x}/\partial q \\
\partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial z & \partial\dot{y}/\partial u & \partial\dot{y}/\partial q \\
\partial\dot{z}/\partial x & \partial\dot{z}/\partial y & \partial\dot{z}/\partial z & \partial\dot{z}/\partial u & \partial\dot{z}/\partial q \\
\partial\dot{u}/\partial x & \partial\dot{u}/\partial y & \partial\dot{u}/\partial z & \partial\dot{u}/\partial u & \partial\dot{u}/\partial q \\
\partial\dot{q}/\partial x & \partial\dot{q}/\partial y & \partial\dot{q}/\partial z & \partial\dot{q}/\partial u & \partial\dot{q}/\partial q
\end{pmatrix}.
$$

Note that $x^*, y^*, z^*, u^*, q^*$ are stationary variables, i.e. if $\dot{x} = f(\mathbf{h})x$ then $f(\mathbf{h}^*) = 0$ and

$$\frac{\partial \dot{x}}{\partial x}(\mathbf{h}^*) = \frac{\partial f(\mathbf{h}^*)}{\partial x} x^*.$$

Thus, we get the Jacobian matrix

$$\mathbf{J} = \begin{pmatrix} (\xi-1)x^* & 0 & 0 & 0 & 0 \\ (\frac{\xi-\varepsilon}{\varepsilon})y^* & y^* & 0 & 0 & 0 \\ 0 & -z^* & z^* & 0 & 0 \\ 0 & -u^* & 0 & u^* & 0 \\ 0 & -q^* & 0 & 0 & \frac{b\theta}{\gamma}q^* \end{pmatrix}.$$

The characteristic roots $\lambda_k$, $k = 1, .., 5$, are the solutions of the characteristic equation $|\mathbf{J} - \lambda \mathbf{U}| = 0$ where $\mathbf{U}$ is the $5 \times 5$ unit matrix. We can write at $\mathbf{h}^*$ that

$$[\frac{b\theta}{\gamma}q^* - \lambda_5][u^* - \lambda_4][z^* - \lambda_3][y^* - \lambda_2][((\xi-1)x^* - \lambda_1] = 0$$

It is easy to see that there is only $\lambda_1 = (\xi - 1)x^* < 0$ while the others are positive.

*Case 2.* If $\phi \neq 1$, we must analyze the dynamic system of $x, y, z, u, q$ and $r = A^{\phi-1}$. Since $\dot{r}/r = (\phi-1)g_A$, we know that $g_A^* = 0$ which implies that $r$ is a stationary variable. Moreover, $r^* = q^*$. We have

$$\begin{aligned}
\dot{x} &= (g_F - g_K)x = [(\xi-1)x + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(1-\xi)}{\xi}]x, \\
\dot{y} &= (g_C - g_K)y = [\frac{(\xi-\epsilon)x + (\varepsilon-1)\delta - \rho}{\varepsilon} + y]y, \\
\dot{z} &= (g_Q + z)z = (-y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + z)z, \\
\dot{u} &= (g_R + u - m)u = (-y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + u)u, \\
\dot{q} &= ((\phi-1)g_A + g_{L_Y})q = ((\phi-1)b(r-q) - y + \frac{b\theta r}{\xi} + \frac{b\theta}{\gamma}q + \frac{m\beta + (1-\xi)\delta}{\xi})q, \\
\dot{r} &= [(\phi-1)g_A]r = [b(\phi-1)(r-q)]r.
\end{aligned}$$

It is easy to get

$$J = \begin{pmatrix} (\xi-1)x^* & 0 & 0 & 0 & 0 & \frac{b\theta}{\xi}x^* \\ \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* & 0 & 0 & 0 & 0 \\ 0 & -z^* & z^* & 0 & 0 & \frac{b\theta}{\xi}z^* \\ 0 & -u^* & 0 & u^* & 0 & \frac{b\theta}{\xi}u^* \\ 0 & -q^* & 0 & 0 & [(1-\phi)b + \frac{b\theta}{\gamma}]q^* & [(\phi-1)b + \frac{b\theta}{\xi}]q^* \\ 0 & 0 & 0 & 0 & b(1-\phi)r^* & b(\phi-1)r^* \end{pmatrix}.$$

The characteristic roots $\lambda_k$, $k = 1, .., 6$ are the solutions of the characteristic equation

$$|\mathbf{J} - \lambda \mathbf{U}| = 0 \qquad (28)$$

where $\mathbf{U}$ is the $6 \times 6$ unit matrix. Equation (28) is equivalent to

$$(z^* - \lambda)(u^* - \lambda)\det M = 0$$

where

$$M = \begin{pmatrix} (\xi-1)x^* - \lambda & 0 & 0 & \frac{b\theta}{\xi}x^* \\ \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* - \lambda & 0 & 0 \\ 0 & -q^* & [(1-\phi)b + \frac{b\theta}{\gamma}]q^* - \lambda & [(\phi-1)b + \frac{b\theta}{\xi}]q^* \\ 0 & 0 & b(1-\phi)r^* & b(\phi-1)r^* - \lambda \end{pmatrix}.$$

We then get two positive solutions, $\lambda = z^*$ and $\lambda = u^*$ immediately. We have

$$\begin{aligned}
\det M(\lambda) &= ((\xi-1)x^* - \lambda)(y^* - \lambda) \begin{vmatrix} ((1-\phi)b + \frac{b\theta}{\gamma})q^* - \lambda & ((\phi-1)b + \frac{b\theta}{\xi})q^* \\ b(1-\phi)r^* & b(\phi-1)r^* - \lambda \end{vmatrix} \\
&- \frac{b\theta}{\xi}x^* \begin{vmatrix} \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* - \lambda & 0 \\ 0 & -q^* & [(1-\phi)b + \frac{b\theta}{\gamma}]q^* - \lambda \\ 0 & 0 & b(1-\phi)r^* \end{vmatrix} \\
&= ((\xi-1)x^* - \lambda)(y^* - \lambda)\det N(\lambda) + \frac{b\theta}{\xi}\frac{(\xi-\varepsilon)}{\varepsilon}b(1-\phi)x^*y^*q^*r^*,
\end{aligned}$$

where

$$\det N(\lambda) = \begin{vmatrix} ((1-\phi)b + \frac{b\theta}{\gamma})q^* - \lambda & ((\phi-1)b + \frac{b\theta}{\xi})q^* \\ b(1-\phi)r^* & b(\phi-1)r^* - \lambda \end{vmatrix}.$$

Hence, $\det M(\lambda)$ is a polynomial of degree 4,

$$\det M(\lambda) = H(\lambda) = \lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda^2 + c_4\lambda + c_5$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be solutions of $H(\lambda) = 0$. Then by Viete's theorem, we get

$$\begin{aligned}
H(0) &= \lambda_1\lambda_2\lambda_3\lambda_4 = \\
&= (\xi-1)x^*y^*[(\frac{b\theta}{\gamma} + \frac{b\theta}{\xi})b(\phi-1)r^*q^*] - \frac{b\theta}{\xi}x^*\frac{(\xi-\varepsilon)}{\varepsilon}y^*b(\phi-1)q^*r^* \\
&= [b(\phi-1)r^*q^*b\theta x^*y^*][\frac{(\xi-1)(\xi+\gamma)}{\gamma} - \frac{(\xi-\varepsilon)}{\varepsilon}] \\
&= [b(\phi-1)r^*q^*b\theta x^*y^*][\frac{(\xi-1)\xi\varepsilon + (\xi-1)\gamma\varepsilon - \gamma(\xi-\varepsilon)}{\gamma\varepsilon}] \\
&= [b(\phi-1)r^*q^*b\theta x^*y^*][\frac{(\xi-1)\xi\varepsilon + \xi\gamma(\varepsilon-1)}{\gamma\varepsilon}] \\
&= [b(1-\phi)\theta r^*q^*bx^*y^*][\frac{\xi(\alpha+\beta)\varepsilon + \gamma)}{\gamma\varepsilon}] > 0.
\end{aligned}$$

It follows from (26) and (27) that

$$(\xi-1)x^* + y^* = \frac{b\theta q^*}{\gamma}.$$

Therefore, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = (\xi-1)x^* + y^* + [(1-\phi)b + \frac{b\theta}{\gamma}]q^* + b(\phi-1)r^* = \frac{2b\theta q^*}{\gamma} > 0$ and equation $H(\lambda) = 0$ has either two negative solutions or zero negative solution.

If $\xi < \varepsilon$, it follows that $H((\xi-1)x^*) = \frac{b\theta}{\xi}\frac{(\xi-\varepsilon)}{\varepsilon}b(1-\phi)x^*y^*q^*r^* < 0$.

It results that $H((\xi-1)x^*)H(0) < 0$ and there is a negative $\lambda \in ((\xi-1)x^*, 0)$. Since $\lim_{\lambda\to-\infty} H(\lambda) > 0$, there is another negative $\lambda \in (-\infty, (\xi-1)x^*)$. ■

**Proposition 4** *There can be a long-run growth where both resources are consumed simultaneously along the equilibrium path.*

**Proof.** As Inada conditions are satisfied, two resources used are strictly positive optimal solutions. Therefore two resources are used simultaneously in finite time. It follows from transversality conditions that $\lim_{t\to\infty} \mu S_Q^* e^{-\rho t} = 0$ where $\mu = \mu(0)e^{\rho t}$ and $S_Q^*(t) = S_Q^*(0)e^{g_Q^* t}$. Then $\lim_{t\to\infty} \mu(0)S_Q^*(0)e^{g_Q^* t} = 0$ which implies $g_Q^* < 0$.

Similarly, since $\lim_{t\to\infty} \lambda S_R^* e^{-\rho t} = 0$ where $\lambda = \lambda(0)e^{(\rho-m)t}$, we get $\lim_{t\to\infty} \lambda(0)S_R^*(0)e^{(g_R^*-m)t} = 0$ or $g_R^* < m$. From Proposition 2, we have shown that there exists a steady state $(R^s(t), Q^s(t))$ of renewable energy resource and non-renewable energy resource where the rates of growth $g_R^* = \frac{\dot{R}^s(t)}{R^s(t)}$ and $g_Q^* = \frac{\dot{Q}^s(t)}{Q^s(t)}$ are constant. Indeed, $R^s(t) = R_0 e^{g_R^* t} > 0$ and $Q^s(t) = Q_0 e^{g_Q^* t} > 0$. Along such a path, since $g_Q^* < 0$ and $g_R^* < m$, long-run growth of non-renewable resource must be nonnegative but approach zero as $t \to \infty$ and renewable resource grows with an exponential bound, $R^s(t) < R_0 e^{mt}$. Moreover, when $\phi < 1$, steady state of technological change is constant since $A^s(t) = A_0 e^{b(r^*-q^*)t} = 0$. As production function exhibits increasing returns to scale in technological change and two resources are imperfect substitutes, there can be a long-run growth where both resources are consumed simultaneously. ■

## 5 Sufficient conditions

We have shown that a solution to the social planner's problem and the steady state growth equilibrium exists. And the steady state can be a saddle point of stability. Note that $\theta$ may be greater than 1, the maximal Hamiltonian is not concave in every state variable so the Arrow or Mangasarian sufficiency theorem does not apply in our model. In such an endogenous natural resources dynamic model with non-concave maximal Hamitonian, Kuhn-Tucker first-order conditions together with transversality conditions are necessary and sufficient conditions for an optimal solution is still a conjecture. We show directly that it satisfies the Leitmann-Stalford (1971) sufficiency conditions, and hence the steady state is optimal. We recall the Leitmann-Stalford theorem (1971) for sufficiency conditions for non-cacave Hamiltonian.

**Theorem 2** *Consider problem*

$$\max \int_0^\infty e^{-\rho t} g_0(x(t), z(t))$$

*subject to*

$$\dot{x}(t) = g(x(t), z(t)), x(0) = x_0, z(t) \in Z$$

*Define the current-value Hamiltonian*

$$H(x, z, \Lambda) = g_0(x(t), z(t)) + < \Lambda, g(x(t), z(t)) >$$

*where* $< \Lambda, g(x(t), z(t)) >$ *is the inner product in* $R^n$*.Let* $z^* \in Z$*, let* $x^* = x(z^*)$ *be the corresponding trajectory, and let* $\Lambda : [0, \infty) \to R^n$ *be absolutely continuous. Let following conditions be fulfilled for every* $u \in Z$ *and* $x = x(z)$

$$i) \int_0^\infty e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) + < \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t) >]dt \geq 0,$$

*and*

$$ii) \lim_{t\to\infty} e^{-\rho t}\langle \Lambda, x^*(t) - x(t)\rangle \leq 0.$$

*Then* $(x^*, z^*)$ *is an optimal solution.*

If the co-state variable is non zero, then the $\Lambda(t)$ above can be set equal to the co-state variable that satisfies the first order necessary conditions (Remark 3.1, Leitmann and Stalford (1971)).

**Proposition 5** *Let denote state variable* $x^*(t) = (S_R^*, S_Q^*, K^*, A^*)$ *where* $x_0^* = (S_{R_0}, S_{Q_0}, K_0, A_0)$ *given and control variable* $z^*(t) = (C^*, Q^*, R^*, L_Y^*, L_A^*)$*. Assume that* $U_C, F_R, F_{L_Y}, F_K, F_A$ *belong to* $L^1(e^{-\rho t})$ *when* $x^*(t), z^*(t)$ *lie in* $L^1(e^{-\rho t})$*.If there exists* $\Lambda(t) = (\lambda(t), \mu(t), \nu(t), \omega(t))$ *satisfy conditions (9)-(16) then* $(x^*, z^*)$ *be optimal solution.*

**Proof.** Since $S_Q \to 0$ as $t \to \infty, \lim_{t\to\infty} e^{-\rho t}\mu(S_Q^* - S_Q) = \lim_{t\to\infty} \mu_0(S_Q^* - S_Q) = 0$ . It follows from Lemma 1 co-state variables $v, \lambda, \omega$ are absolutely continuous and belong to $L^1(e^{-\theta t})$. It follows from Assumption H2 that there exists $\hat{k}$ such that $K \leq \hat{k}$. As $\nu(t) \in L^1(e^{-\theta t})$, we have $\lim_{t\to\infty} e^{-\rho t}v(K^* - K) = 0$. Similarly, we can prove that have $\lim_{t\to\infty} e^{-\rho t}\Lambda(x^* - x) = 0$ where $x \in (S_R^*, A^*)$.

Recall the current Hamiltonian

$$\begin{aligned} H(\Lambda, x, z) &= u(C) + \lambda(mS_R - R) - \mu Q \\ &\quad + \nu(F - C - \delta K) + \omega b A^\phi(1 - L_Y) \end{aligned}$$

For any $x(t), z(t)$ feasible with $x(0) = x_0^*$. We have

$$\begin{aligned} &\int_0^\infty e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t))] + \langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle]dt = \\ = &\int_0^\infty e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x^*(t), z(t), \Lambda(t))]dt \\ &+ \int_0^\infty e^{-\rho t}[H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) + \langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle]dt \end{aligned}$$

Looking at the first term, using (9)-(15) and concavity of $U, F$ w.r.t control variables as $F(z^*)-F(z) \geq F'(z^*)(z^*-z)$, we easily obtain

$$H(x^*(t), z^*(t), \Lambda(t)) - H(x^*(t), z(t), \Lambda(t)) \geq 0$$

Thus,

$$\int_0^\infty e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x^*(t), z(t), \Lambda(t))]dt \geq 0. \tag{29}$$

Since

$$\begin{aligned} H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) &= < \Lambda, g(x^*(t), z(t)) - g(x(t), z(t)) > \\ &= < \Lambda, \dot{x}^*(t) - \dot{x}(t)) >, \end{aligned}$$

the next term can be read as

$$\begin{aligned} H(\Lambda, x, z) &= u(C) + \lambda(mS_R - R) - \mu Q \\ &\quad +\nu(F - C - \delta K) + \omega b A^\phi(1 - L_Y) \end{aligned}$$

$$\begin{aligned} &\int_0^\infty e^{-\rho t}[H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t))]dt + \int_0^\infty e^{-\rho t}\langle\dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle dt \\ = &\int_0^\infty e^{-\rho t}\langle\Lambda, \dot{x}^*(t) - \dot{x}(t)\rangle + \int_0^\infty e^{-\rho t}\langle\dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle dt \\ = &\int_0^\infty e^{-\rho t}[\lambda(\dot{S}_R^* - \dot{S}_R) + \mu(\dot{S}_Q^* - \dot{S}_Q) + v(\dot{K}^* - \dot{K}) + \omega(\dot{A}^* - \dot{A})]dt + \int_0^\infty e^{-\rho t}[(\dot{\lambda} - \rho\lambda)(S_R^* - S_R) \\ &+(\dot{\mu} - \rho\mu)(S_Q^* - S_Q) + (\dot{v} - \rho v)(K^* - K) + (\dot{\omega} - \rho\omega)(A^* - A]dt \end{aligned} \tag{30}$$

Since $e^{-\rho t}(\dot{\lambda} - \rho\lambda) = \frac{d(e^{-\rho t}\lambda)}{dt}$ we have

$$\begin{aligned} &\int_0^\infty e^{-\rho t}[\lambda(\dot{S}_R^* - \dot{S}_R) + (\dot{\lambda} - \rho\lambda)(S_R^* - S_R)]dt \\ = &\int_0^\infty [\frac{d(e^{-\rho t}\lambda)(S_R^* - S_R)}{dt}]dt = \lim_{t\to\infty} e^{-\rho t}\lambda(S_R^* - S_R) - \lambda(0)(S_{R_0}^* - S_{R_0}) \\ = &\lim_{t\to\infty} e^{-\rho t}\lambda(S_R^* - S_R) = 0. \end{aligned}$$

With the same reasoning for the second and third terms in (30) we have

$$\begin{aligned} &\int_0^\infty e^{-\rho t}[H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) + \langle\dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle]dt \\ = &\lim_{t\to\infty} e^{-\rho t}\langle\Lambda, x^*(t) - x(t)\rangle \leq 0 \end{aligned} \tag{31}$$

It follows from (29) and (31) the condition (i) in Theorem 2 is satisfied. ■

As the steady state satisfies the necessary conditions, we have shown that it is indeed optimal. Thus, we have the following result.

**Theorem 3** *All the steady states are locally optimal.*

The Leitmann-Stalford theorem is a powerful result but has not been used much in the economics literature as the conditions are difficult to verify. To use the Leitmann and Stalford conditions we show that from Lemma 1 that control and state variables are bounded and belong to $L^1(e^{-\rho t})$. This implies that the co-state variables, $\Lambda \in L^1(e^{-\rho t})$. This enables us to show the condition $lim_{t\to\infty} e^{-\rho t}\langle\Lambda, x^*(t) - x(t)\rangle = 0$. This is crucial as when we check the maximality of the Hamiltonian we can write it can decompose it into two parts: the first just relies on the control variables and we have concavity in the objective function in control variables, and thus, using standard results the difference between the candidate solution and any other solution is non-negative; and a term that depends on the co-state and the state variables as given above. Recall, the non-concavity in the problem arises from the law of evolution of state variables only. As this term converges to zero, we are able to obtain sufficiency of the first order conditions. Thus, three things turn out to be important in our problem: the boundedness of the state and control and hence, co-state variables; concavity of objective in control variables; and the ability to separate the control and state/co-state variables in the Hamiltonian.

In conclusion, this paper is an attempt to explore theoretically the interaction between growth, endogenous technical change, and consumptions of renewable and non-renewable resources. When production function exhibits increasing returns to scale in technological change and two resources are imperfect substitutes, there can be a long-run growth where both resources are consumed simultaneously along the equilibrium path. An proof of existence optimal solutions and sufficient conditions are provided when the Arrow-Mangasarian conditions do not apply. We show directly a sufficient conditions and deduce that the steady states satisfy the necessary conditions are indeed locally optimal. We also characterize the BGP together with the transitional dynamics. It would be of particular interest, in a further work, to consider externalities such as pollution, extraction and backstop use costs in order to improve the realism of the model.

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