Spatial Dynamics of Green Corridors

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Abstract

Forest management aims at building ecological networks that minimize the impacts on timber production. We formalize the construction of ecological networks in forest environments as the optimal control dynamic graph-theoretic problem. The ecological network is based on a set of bioreserves and patches linked by green corridors. The network is defined as a graph, in which bioreserves, which host the species, are represented by the target nodes provided with an attraction function. The role of patches, which are represented by the unmarked nodes, is to receive the species before redirecting them to bioreserves. We consider two cases: a case of complete graph, where the ecological network is fully connected, and a case of incomplete graph, where the ecological network is partially connected. We use an ecologically and economically weighted Mahalanobis distance when dealing with the species’ migration through the grid. In both cases, at the equilibrium, the ecological network maintains its connectedness while minimizing the opportunity costs of timber production. Our simulations show that taking into account the opportunity costs of timber production is essential to determine the economic soundness of the ecological project, but the optimality threshold depends on the type of network that is envisaged.

Keywords: Bioeconomics, Resource Management, Graph Theory, Optimal Control, Spatial Dynamics, Green Corridors, Forestry, Timber Production, Opportunity Costs

JEL Classification: C61, C65, C67, N5, Q57

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1 Introduction

Ancient land-use changes and running anthropisation of areas lead to the ecological fragmentation of territories and habitats, partly explaining the current biodiversity erosion. The supply of bioreserves and ecological corridors has been developed in order to ensure the species’ viability through minimizing the damages of habitat fragmentation and allowing for the species’ migration. Indeed, public policies have emphasized the need to ensure the interconnectivity between selected bioreserve sites. For example, following the conclusions of civil society players to the Senate and the National Assembly in France, the Environment Round Table Act has been voted (CGDD 2009). It stresses that the protection of biodiversity and ecosystems be undertaken through the green and blue belt networks, meant to facilitate the species circulation. In that event, the implementation of nature conservation policies is based on contracts between public authorities and landowners (Anthon et al. 2010).

In this paper, we are interested in the way the forest landowner addresses the question of foregone gain in wood production, provided her obligation to play the role of biodiversity producer: a role that she delegates to the forest manager. Anthon et al. (2010) have shown that when conservation measures are correlated with forest management, the contractual measures, such as those observed in the Natura 2000 policy, imply distorted transfers; ergo the public authorities tend to overcompensate the landowners. What is more, the authors find that conservation contracts can have a positive or negative impact on forest management costs, thereby over- or undercompensating for the cost of fulfilling their contractual obligations.

The need to connect the areas of ecological significance has given impetus to the design of optimal ecological corridors (Noss 1987). The models of reserve site selection, which guarantee that the targeted species in an area are protected (Margules et al. 1988, Underhill 1994), as well as the models of constrained number of parcels, which maximize the number of protected species (Camm et al. 1996, Church 1996), have been proposed. Provided that the spatial location of a protected parcel is critical, spatial attributes have been incorporated via the optimal selection of ecological or green corridors (Williams and Snyder 2005, Conrad et al. 2012). Some of the key findings from the scientific literature indicate that connectivity preservation measures should have clear biodiversity conservation objectives; plan measures on the basis of functional structure; be based on ecological science; only increase connectivity when necessary; consider the cost effectiveness; and treat landscape connectivity as a dynamic property.

The connectivity within ecological networks – by means of green corridors – should provide sufficient conditions in policy preservation of species and habitats; the networks should facilitate the adaptation of vulnerable species to local or broader environmental disturbances (Williams et al. 2005). For instance, suitable channels bring about the species’ mobility (Schmitt and Seitz 2002) and allow them to escape from all
kinds of threats (McEuen 1993, Andreassen et al. 1996). Timely, the corridors have to be implemented by taking the precaution of verifying that the sizes and the number of the reserve sites – essential to the survival of species – are large enough. In this case, the network approach has the advantage of being conveniently controllable by the forest managers. Besides, Smith et al. (2009) assert that optimal policies must be determined over the subsets of connected systems.

Despite the increasing number of papers and studies documenting the benefits of corridors, the theoretical framework for the optimal settling of corridors has been addressed quite recently. Conrad et al. (2012) have conducted an extensive review on this topic. Models have focused on minimizing the number of parcels for a quantified preservation objective (Cerdeira et al. 2005, Rothley and Rae 2005, Fuller et al. 2006, Önal and Briers 2006). In line with this, Berthoud (2004), Önal and Briers (2006) and Conrad et al. (2012) use graph theory to find a species-covering connected subgraph within a geographic area.

Establishing ecological corridors in forest environments induces a series of opportunity costs, especially in timber production. These costs can come from the restriction in cuts to parts of the forest, the establishment of a number of dead trees with the aim to create senescence islets or ageing patches, or from the performance of forestry work to ensure some optimal forest cover in the woods. Taking into account the economic variables, such as the costs, tends to increase the efficiency of the network design (Wunder 2005, Naidoo and Adamowicz 2006). Such as pointed out by Conrad et al. (2012), economic-based studies seek to procure conservation parcels, subject to a budget constraint, that maximize the number of protected species (Ando 1998, Costello and Polasky 2004, Polasky et al. 2001) or maximize the environmental benefits of the selected sites (Ferraro 2003, Messer 2006, Newburn et al. 2006). The set of parcels becomes different and greater environmental benefits are obtained when the reserve site selection incorporates the cost and the budget constraints (Naidoo et al. 2006, Conrad et al. 2012).

Forest management aims at building ecological networks that minimize the impacts on timber production, for the multifunctional role of forests is to both provide wood products and environmental services. We model an ecological network composed of bioreserves and patches connected by the green corridors. The network is defined as a graph, in which bioreserves, which host the species, are represented by the target nodes provided with an attraction function. The role of patches, which are represented by the unmarked nodes, is to receive the species before redirecting them to bioreserves. We consider two cases: a case of complete graph, where the ecological network is fully connected, and a case of incomplete graph, where the ecological network is partially connected.

According to Williams et al. (2012), conservation issues must be assessed as combinations, rather than individually, to the point that landscape-scale programs occasion
computational challenges. Their paper investigates an ecological metric design in order to achieve some desired spatial configurations. Instead of the standard Euclidean distance, and in line with these authors, we also decide to use an ecologically and economically weighted Mahalanobis distance when dealing with the species’ migration through the grid. By taking into account the ecosystem complementarity, the Mahalanobis distance enables to complexify the paths through the network and thus to add more realism to the graph.

The aim of the paper is to analyze the ecological network under the spectrum of graph theory and to obtain the graph-theoretic properties of control. The paper tries notably to answer the questions raised in Urban and Keitt (2001) on the influence of distance and dispersal capacities of species on the landscape connectivity. Contrary to Conrad et al. (2012), who study a budget constrained optimization in the design of optimal wildlife corridors, we propose a connectivity constrained optimization and value the financial resources necessary for that purpose. We first find that the connectivity between the areas, set up by the forest manager, depends on their ecological similarity. The dynamic optimal control of the ecological network goes through a Hamiltonian system, where the program of the forest landowner consists in minimizing the performance function defined over the distances between the ecological areas respectful of the species’ transportation capacities and weighted by the opportunity costs of timber production. The program is subjected to the consensus dynamics exerted on the nodes. Our results show that, at the equilibrium, the ecological network maintains its connectedness.

When the graph is complete, the optimality conditions impose that, along the time path, the forest owner’s willingness to pay for establishing and preserving the connectivity between bioreserves and patches be strictly greater than her willingness to pay for establishing and preserving the connectivity between the patches, the latter have been being strictly positive. When the graph is incomplete, the optimality conditions impose that the forest owner’s willingness to pay for establishing and preserving the connectivity between bioreserves and patches be equal to the negative of her willingness to pay for establishing and preserving the connectivity between the patches, that is, her willingness to receive compensation for neglecting the patch connectivity. Our simulations show that both the resources mobilizables by the foresters and the opportunity costs that affect the timber production reveal which kind of ecological network should be implemented.

After this starting section, we present the graph-theoretic framework in Section 2. The dynamic controllability of the ecological network is modeled in Section 3. Section 4 is devoted to illustrating simulation examples. Section 5 concludes.
2 Graph Theory Model

Following Dragicevic and Sinclair-Desgagné (2013), we consider network closed-loop systems in which the system outputs are used as the system inputs. Let $N$ be the number of nodes or the number of areas of ecological significance. Let $x_i \in \mathbb{R}^2$ denote the relative position of a node or an ecological area $i$ to other nodes or ecological areas in the network. The set of all possible positions of the dynamic system is the configuration space. It is spanned by the stack vector of all the control inputs $x = [x_1^T, ..., x_N^T]^T$ which denotes the aggregated state of the network as an involution. In our case, the involution implies the maintenance of the network connectedness. The stack vector reflects the environmental policy space toward the network, for the control inputs are amassed and configured so as to arrive to the objective. The trajectory of a node obeys the following model

$$\dot{x} = k_i, \ i \in \Lambda = \{1, \ldots, N\}$$

where $k_i$ denotes the control input for each node. The interaction between the nodes’ dynamics is realized through the control input for each node $k_i = -Lx$, with $L$ the Laplacian matrix representation of the network.

The ecosystem manager identifies the subset of target nodes or bioreserves $\Lambda^t$ and the subset of unmarked nodes or patches $\Lambda^u$, such that

$$\Lambda^t \cup \Lambda^u = \Lambda \text{ and } \Lambda^t \cap \Lambda^u = \emptyset$$

The number of nodes in each subset is given by $|\Lambda^t| = N_t$ and $|\Lambda^u| = N_u$ respectively. From the forest manager’s point of view, nodes obtain and provide utility $u_{ij}$ from and to other nodes$^2$. This utility depends on the following criteria.

**Definition 1** For $\forall i, j \in \Lambda$, $d_{ij} = |x_i - x_j|$, and for $\Lambda_i = \{j \in \Lambda: 0 < d_{ij} \leq c_{ij}\}$

$$u_{ij} := \begin{cases} 
  c_{ij} - d_{ij} & \text{if } j \in \Lambda_i, \ cov(x_i, x_j) > 0 \\
  0 & \text{if } j \notin \Lambda_i, \ cov(x_i, x_j) \leq 0
\end{cases}$$

where $c_{ij}$ and $d_{ij}$ respectively stand for the species transportation capacity and the Euclidean distance between the ecological sites. Indeed, we have to take into account the various species’ capacities of movement to measure the potential utility between two connected nodes. Moreover, $cov(x_i, x_j)$ is the covariance reflecting the ecological

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$^1$In closed-loop control, the control input to the system is the reference value for the system output. It represents the desired operating value of the output. In our case, it is the coordinate of a node for a given network topology.

$^2$Although the existence of bioreserves and patches could in itself be valued through utility, and their connections defined as sources of positive externalities, we assume that only connections provide utility, knowing that the ecological areas predate the construction of the network.
similarity between the areas of ecological significance. Not only the coordinates of areas express their spatial positions, they also unfold their environmental features. A sufficiently close but unsuitable environment for survival cannot provide any utility to the migratory species\(^3\). In light of the dynamic analysis of the ecological network, we need a statistic representation of the degree to which the coordinates of the network vary together. Such as shown in the study by Bell et al. (2000), the rate of increase of the general and specific variances with distance between forest sites reflects the environmental structure. According to their results, the general environmental variance between forest sites provides a measure of environmental heterogeneity.

All nodes within the utility domain of a node form its utility set \(\Lambda_i\). In our case, the utility nodes’ set reflects the ecological network.

**Definition 2** For \(j \in \Lambda\), the network utility domain \(U_i(\Lambda) = \sum_j u_{ij}\) is the sum of utility domains contained in the network utility set.

The network utility can thus be interpreted as the set of nodes, reasonably similar, separated by distances which satisfy the migratory species’ transportation capacity constraints.

### 2.1 Complete Graph Case

Interactions in the system are represented by a graph \(\Gamma = \{V, E\}\) which defines the topology. The graph consists of vertices \(V = \{1, ..., N\}\) indexed by the node members, where \(i\) and \(j\) represent two neighboring connected nodes, and the set of edges \(E = \{(i, j) \in V \times V \mid i \in \Lambda_j\}\) that represent the inter-node interactions. In the model, the grid of green corridors stands for these interactions. The set of edges \(E\) and the graph \(\Gamma\) vary in finite time for \(t \in [0, T]\).

Fig. 1 illustrates the forest environment with the passage from forest mapping to graph theory.

**Fig. 1 about here.**

#### 2.1.1 Distance

We define the Mahalanobis distance \(m(I)_{ij}\) between two random nodes \(i\) and \(j\) as

\[
m(I)_{ij} = \left[(x_i - x_j)^T S^{-1} (x_i - x_j)\right]^{\frac{1}{2}} I
\]

\(^3\)As portrayed in the work by Williams et al. (2012), habitat connectivity is difficult to deliver with standard metrics and requires knowledge of how biodiversity responds over time. This requires maps of habitat suitability, current habitat state, and a model to simulate dispersal and movement of the species.
where $|x_i - x_j| \leq c_{ij}$ such that utility flows between the nodes. We know from the literature that the standard Euclidean distance becomes irrelevant with various properties of the hosting environments\(^4\). Therewith, $S^{-1}$ is the inverse of the covariance matrix reflecting the ecological similarity between the areas.

As the forest manager implements the ecological grid, it immobilizes the standing timber. These economic opportunity costs $I$, computed from the market value $p$, are the sum of the sacrificed timber production, both in the ecological areas, where $\alpha = 1$, and the corridors, where $\alpha \in [0, 1]$. Put differently

$$I = p \left[ (1 + \alpha) \sum x_i + (1 - \alpha) \sum x_j \right]$$

(4)

2.1.2 Dynamics

Following Gustavi et al. (2010), the patch (unmarked node) dynamics is given by the Laplacian-based control strategy (consensus) differential equation, meaning that each patch evolves in the direction of the average position of its utility nodes. Put differently, the rate of change of each node’s position is governed by the sum of its relative positions with respect to other nodes. We have

$$\dot{x}_i = -Nx_i + \sum_{k \in \Lambda} x_k$$

(5)

Its annulment yields the equilibrium position under the patch dynamics.

Lemma 1 The consensus equilibrium under the patch dynamics is equal to $x_i = \frac{1}{N} \sum x_k$. The proof is in the appendix.

The bioreserve (target node) dynamics is also based on the consensus equation and on the attraction function which depends on the size of $x_j$ delimited by its frontier $g$. The rationale is that the greater the distance between $x_j$ and $g$, the greater the attraction\(^5\).

We have

$$\dot{x}_j = -Nx_j + \sum_{k \in \Lambda} x_k + f(|g - x_j|)$$

(6)

where $f(|g - x_j|)$ is the attraction function toward the rest of the network. Its annulment also yields the equilibrium position under the target node dynamics.

Lemma 2 The consensus equilibrium under the bioreserve dynamics is equal to $x_j - \frac{f(|g - x_j|)}{N} = \frac{1}{N} \sum x_k$.

\(^4\)The covariance can be compared to the scaling function of habitat quality in Urban and Keitt (2001).

\(^5\)In order to capture the source strength of the ecological area, we only consider its size and, contrary to Urban and Keitt (2001), do not consider the fluxes dispersal based on a probability of dispersal.
The proof is in the appendix. Equalizing the equilibria via $\frac{1}{N} \sum x_k$ brings us to the following theorem.

**Theorem 1** Given the bioreserve position $x_j$ and its attraction function $f(|g - x_j|)$, the fully connected network equilibrium is

$$x_i = x_j - \frac{f(|g - x_j|)}{N}$$

The proof is in the appendix.

According to Theorem 1, the position of the patch depends on the position of the bioreserve by cause of the fact that bioreserves lead the ecological network. However, an additional term appears, that is, the magnitude of attraction of the bioreserve on the patch. The following remarks can be made.

**Remark 1** The magnitude of attraction of the bioreserve on the patch is proportional to its size and inversely proportional to the number of ecological areas.

**Remark 2** When the equilibrium position under the patch dynamics is attained, the coordinates of the ecological areas do not alter the network topology.

2.1.3 Connectivity

The connectivity relation $\dot{m}(I)_{ij}$ defines the preservation of network connectedness, enabling the preservation of the ecological network as it has been defined by the forest manager. In other words, the network does not disconnect in time. The condition for nodes $i$ and $j$ to evolve connected is $\dot{m}(I)_{ij} \leq 0$. When the latter is true, it proves that the convex hull containing the nodes is invariant and, therefore, that the network is stable in the sense of Lyapunov. The time derivative $\dot{m}(I)_{ij}$ may not be defined when $m(I)_{ij} = 0$, so the time derivative of the squared distance shall be considered instead (Gustavi et al. 2010). It depends on the dynamics of nodes $i$ and $j$ and equals

$$\dot{m}(I)_{ij}^2 = 2 (x_i - x_j)^T S^{-1} \left( x_i - x_j \right) 2I$$

2.1.4 Connectivity between Random Patches

For two random nodes $i \in \Lambda^u$ and $j \in \Lambda^u$, the connectivity is defined by

$$\dot{m}(I)_{ij}^2 = 4I (x_i - x_j)^T S^{-1} [-N (x_i - x_j)]$$

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6 This result is comparable to Lemma 1 in Gazi and Passino (2004).
7 In reality, the occurrence of climatic events or silvicultural work can modify the relative coordinates of nodes, with reference to other nodes, which would put the ecological network out of equilibrium. For instance, some bioreserves and patches can be entirely or partially destroyed, reconfiguring the initial ecological network.
The condition for nodes to evolve connected being $\dot{m}(I)_{ij} \leq 0$, we fall on a configuration exclusively dependent on the covariance, i.e. the ecological similarity between the patches. If $S^{-1} \geq 0$, i.e. if the inverse of the covariance matrix is positive-semidefinite, which implies that the patches are either pariable or mutually neutral, their connectivity is guaranteed. Indeed, every invertible positive-semidefinite matrix yields an inverse that is also positive-semidefinite (Pease 1965).

**Corollary 1** Necessary and sufficient condition for patches to evolve connected is their ecological similarity.

### 2.1.5 Connectivity between Random Bioreserves and Patches

For two random nodes $i \in \Lambda^n$ and $j \in \Lambda^t$ and for $f (|g - x_j|) \neq 0$, the connectivity is defined by

$$\dot{m}(I)_{ij}^2 = 4I (x_i - x_j)^T S^{-1} [-N (x_i - x_j) - f (|g - x_j|)]$$

(9)

The condition for nodes to evolve connected being $\dot{m}(I)_{ij} \leq 0$, we fall on a configuration dependent on both the covariance and the sign of $-N (x_i - x_j) - f (|g - x_j|)$, i.e. the ecological similarity between bioreserves and patches and their vicinity. If $S^{-1} \geq 0$ and if $-N (x_i - x_j) - f (|g - x_j|) < 0$, the latter being true, we have $\dot{m}(I)_{ij} \leq 0$ and the connectivity by the corridor is ensured.

**Corollary 2** Necessary and sufficient condition for bioreserves and patches to evolve connected is their ecological similarity.

Corollaries 2 and 3 instruct us that the connectivity of an ecological network built on the consensus protocol be dependent on a simple criterion of ecological similarity between the areas. At this level of analysis, in the absence of optimal control, the opportunity cost of timber production does not count.

### 2.2 Incomplete Graph Case

In the previous subsection, we have considered a complete graph. We next analyze the case of an incomplete graph and no longer consider that all target nodes are connected to all unmarked nodes. Let $\Lambda_i^j \subset \Lambda^t$ be the subset of bioreserves that are connected to a patch $i \in \Lambda^t$ with $|\Lambda_i^j| = N_{ii}$. Let $\Lambda_j^i \subset \Lambda^n$ be the subset of patches connected to a bioreserve $j \in \Lambda^n$ with $|\Lambda_j^i| = N_{uj}$.

Fig. 2 illustrates the ecological network in a forest environment through an incomplete graph.

Fig. 2 about here.
2.2.1 Dynamics

The dynamics for an arbitrary patch can be written as

\[ \dot{x}_i = -N (x_i - x_j) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k \] (10)

Its annulment yields the equilibrium position under the patch dynamics.

**Lemma 3** The consensus equilibrium under the patch dynamics is equal to
\[ x_i - x_j = \frac{1}{N} \sum_{k \in \Lambda_i} x_k - \frac{1}{N} \sum_{k \in \Lambda_j} x_k. \]

The proof is in the appendix.

For an arbitrary bioreserve, we have

\[ \dot{x}_j = -N (x_i - x_j) + \sum_{k \in \Lambda_i^w} x_k - \sum_{k \in \Lambda_j^w} x_k + f(|g - x_j|) \] (11)

Its annulment yields the equilibrium position under the bioreserve dynamics.

**Lemma 4** The consensus equilibrium under the bioreserve dynamics is equal to
\[ x_i - x_j = \frac{1}{N} \sum_{k \in \Lambda_i^w} x_k - \frac{1}{N} \sum_{k \in \Lambda_j^w} x_k + \frac{f(|g - x_j|)}{N}. \]

The proof is in the appendix.

Equalizing the equilibria via \( x_i - x_j \) yields the following theorem.

**Theorem 2** At the partially connected network equilibrium, the net weight of connections between bioreserves and the rest of the network and the attraction function \( f(|g - x_j|) \) equals the net weight of connections between the patches and the rest of the network or

\[ \left( \sum_{k \in \Lambda_i^w} x_k - \sum_{k \in \Lambda_j^w} x_k \right) = \left( \sum_{k \in \Lambda_i^w} x_k - \sum_{k \in \Lambda_j^w} x_k \right) - f(|g - x_j|) \]

The proof is in the appendix.

According to Theorem 2, for the incomplete network to be at the equilibrium, the net weight of connectivity between the patches and the rest of the network has to be equal to the net weight of connectivity between the bioreserves and the rest of the network reduced by their forces of attraction.

**Remark 3** In the partially connected ecological network, the magnitude of attraction of the bioreserves on the network is proportional to the overweight of bioreserves over patches.

We will now derive general conditions for the subgraphs to remain connected.
2.2.2 Connectivity

Whether the graph is complete or incomplete, the time derivative $\dot{m}(I)_{ij}$ still defines the preservation of network connectedness. Once again, the derivative of the squared distance shall be considered. Like in the previous case, it equals

$$\dot{m}(I)_{ij}^2 = 2 (x_i - x_j)^T S^{-1} \left( x_i - x_j \right) 2 I$$

(12)

2.2.3 Connectivity between Random Patches

For random nodes $i, j \in \Lambda^u$ and $i, j \in \Lambda^t$, the connectivity is defined by

$$\dot{m}(I)_{ij}^2 = 8 I (x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) + \sum_{k \in \Lambda^t_i} x_k - \sum_{k \in \Lambda^t_j} x_k \right]$$

(13)

The condition for nodes to evolve connected is $\dot{m}(I)_{ij} \leq 0$. Given that the unmarked nodes always outnumber the target nodes, we fall on a configuration which sign exclusively depends on the covariance.

**Corollary 3** Necessary and sufficient condition for patches to evolve connected is their ecological similarity.

2.2.4 Connectivity between Random Bioreserves and Patches

For random nodes $i, j \in \Lambda^u$ and $i, j \in \Lambda^t$ and for $f \left( |g - (x_i, x_j)| \right) \neq 0$, the connectivity is defined by

$$\dot{m}(I)_{ij}^2 = 8 I (x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) - f \left( |g - (x_i, x_j)| \right) + \sum_{k \in \Lambda^t_i} x_k - \sum_{k \in \Lambda^t_j} x_k - \sum_{k \in \Lambda^u_i} x_k + \sum_{k \in \Lambda^u_j} x_k \right]$$

(14)

The condition for nodes to evolve connected being $\dot{m}(I)_{ij} \leq 0$, we fall on a configuration dependent on both the covariance and the sign of the connection evolution. Bearing in mind that the unmarked nodes outnumber the target nodes, the expression in parenthesis is negative.

**Corollary 4** Necessary and sufficient condition for bioreserves and patches to evolve connected is their ecological similarity.
3 Optimal Control Problem

Following the methodology by Mesbahi and Egerstedt (2010), the ecological network performance function in finite time is such that for all \(0 \leq t \leq T\)

\[
J = \int_0^T \sum m(I)_{ij} \, dt
\] (15)

In absence of path constraints on the control variables, a continuous time optimal control problem can be defined as the equilibrium process problem, that is, the issue of maintaining the connectivity between \(x_i\) and \(x_j\) in a finite amount of time while satisfying the equilibria. Given that the forest landowner seeks to minimize the impact of the ecological network on timber production, while satisfying the species’ transport capacities, the optimal control problem is formulated as the minimization of the Mahalanobis distances separating the network nodes weighted by the opportunity costs, while preserving the network connectedness under the consensus dynamics applied to the set of reserve sites selected by the forest manager.

Put differently

\[
\min_{x_i, x_j} J
\] (16)

subject to first-order dynamic constraints

\[
\dot{x}_i, \dot{x}_j
\] (17)

3.1 Complete Graph Case

In order to solve the optimal control problem, we must form the Hamiltonian. The system is subjected to the laws of evolution of \(x_i\) and \(x_j\). Therewith, \(\lambda^T\) and \(\mu^T\) are the costate variables or the shadow values of patches and bioreserves obtained by strengthening the connectivity constraints. The Hamiltonian represents the impact of the behavior of \(x_i\) and \(x_j\) on the network topology. We define it as follows

\[
H = \begin{cases} 
4I \left( x_i - x_j \right)^T S^{-1} \left[ -N \left( x_i - x_j \right) \right] \\
+4I \left( x_i - x_j \right)^T S^{-1} \left[ -N \left( x_i - x_j \right) - f \left( |g - x_j| \right) \right] \\
+\lambda^T \left[ -Nx_i + \sum x_k \right] \\
+\mu^T \left[ -Nx_j + \sum x_k + f \left( |g - x_j| \right) \right]
\end{cases}
\] (18)

The solutions are determined by the following boundary conditions
\[
\begin{align*}
  x_{io} &= x_{j0} - f\left(\frac{g - x_{j0}}{N}\right) \\
  x_{ir} &= x_{jr} - f\left(\frac{g - x_{jr}}{N}\right)
\end{align*}
\]

\[(19)\]

### 3.1.1 Optimality Conditions

The first-order optimality conditions are

\[
\frac{\partial H}{\partial x_i} = 4I (x_i - x_j)^T S^{-1} [-N] + 4I (x_i - x_j)^T S^{-1} [-N] - \lambda^T N = 0
\]

from which we obtain

\[
x_i = x_j - \lambda (8I)^{-1} S
\]

where \(\lambda\) represents the shadow value for preserving the connectivity between the patches, and \((8I)^{-1}\) is the timber immobilization rate, that is, the timber loss per unit of corridor built toward the patch.

In parallel, as we normalize \(f(1) = 1\), we obtain

\[
\frac{\partial H}{\partial x_j} = 4I (x_i - x_j)^T S^{-1} [N] + 4I (x_i - x_j)^T S^{-1} [N - 1] + \mu^T [-N + 1] = 0
\]

which yields

\[
x_i = x_j + \frac{N - 1}{2N - 1} \mu (4I)^{-1} S
\]

where \(\mu\) embodies the shadow value for preserving the connectivity between bioreserves and patches, \((N - 1)/(2N - 1)\) is the pressure coming from the attraction function on the ecological network, and \((4I)^{-1}\) is the timber loss per unit of corridor built toward the bioreserve. It can be noticed that the timber immobilization rate is higher with bioreserves than with patches. This is due to their larger sizes and greater weights in the network.

From the foregoing, we obtain

\[
x_j - \lambda (8I)^{-1} S = x_j + \frac{N - 1}{2N - 1} \mu (4I)^{-1} S
\]

\[(24)\]

And

\[
x_i + \lambda (8I)^{-1} S = x_i - \frac{N - 1}{2N - 1} \mu (4I)^{-1} S
\]

\[(25)\]

Piecing the precedent expressions together yields

\[
N = \frac{2\mu - \lambda}{2(\mu - \lambda)}
\]

\[(26)\]
Given that $N > 0$ by construction, this resumes to

$$\mu > \lambda > 0$$  \hfill (27)

The costate variables $\lambda$ and $\mu$ have the role of rationing the use of the state variables (Lyon 1999). They reveal the forest owner’s preferences for the green corridors. These inequalities are part of the initial conditions on the choices of costate variables for the system control.

**Lemma 5** The bioreserve shadow value has to be strictly greater than the patch shadow value, the latter being strictly positive.

The proof is in the appendix.

In other words, if the forest owner is not willing to pay a positive amount of money for establishing and preserving the connectivity between the patches, and if her willingness to pay for establishing and preserving the connectivity between bioreserves and patches is not strictly greater than her willingness to pay related to the safeguarding of the patches, establishing the patches and building the fully connected ecological network will be economically suboptimal.

Finally, the boundary conditions, which enable to track the scale of change of the network performance function per unit change in the nodes’ coordinates, are

$$\dot{\lambda} = 8INS^{-1} (x_i - x_j) + 4IS^{-1} f (|g - x_j|) + \lambda N$$  \hfill (28)

And

$$\dot{\mu} = -8INS^{-1} (x_i - x_j) - 4IS^{-1} f (|g - x_j|) + \mu (N - 1)$$  \hfill (29)

By dint of the first-order necessary optimality conditions, we can now set up the control of the system.

### 3.1.2 Solution

By letting $z = [x_i^T, x_j^T, \lambda^T, \mu^T]^T$ reflect the network state, we obtain the following Hamiltonian system

$$\dot{z} = Mz$$  \hfill (30)

where
The first-order optimality conditions are

\[ M = \begin{bmatrix} 0 & 1 & -\left(8I\right)^{-1}S & \left(4I\right)^{-1} \frac{N-1}{2N-1}S \\ 1 & 0 & \left(8I\right)^{-1}S & -\left(4I\right)^{-1} \frac{N-1}{2N-1}S \\ 8INS^{-1} & -4I(2N+1)S^{-1} & N & 0 \\ -8INS^{-1} & 4I(2N+1)S^{-1} & 0 & N - 1 \end{bmatrix} \]

**Theorem 3** The complete ecological network state \( z \) optimally evolves according to the coordinates \( M \).

The proof is in the appendix.

Let the initial network state be given by \( z_0 = [x_{i_0}^T, x_{j_0}^T, \lambda_0^T, \mu_0^T]^T \) where \( x_{i_0} = x_{j_0} - f(|g - x_{j_0}|)/N \). The problem is to fix \( \lambda_0 \) and \( \mu_0 \) in such a way that, through their choices, we get

\[ x_i(T) = x_{jT} - \frac{f(|g - x_{jT}|)}{N} = x_{iT} \]  

(31)

Let us recall that the choices are constrained by \( \mu_0 > \lambda_0 > 0 \).

### 3.2 Incomplete Graph Case

With the incomplete graph, the Hamiltonian becomes

\[ H = \begin{cases} 8I(x_i - x_j)^T S^{-1} \left[ -N(x_i - x_j) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k \right] \\ +8I(x_i - x_j)^T S^{-1} \left[ -N(x_i - x_j) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k + f(|g - x_j|) \right] \\ +\lambda^T \left[ -N(x_i - x_j) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k \right] \\ +\mu^T \left[ -N(x_i - x_j) - f(|g - x_j|) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k + \sum_{k \in \Lambda_j^c} x_k \right] \end{cases} \]  

(32)

#### 3.2.1 Optimality Conditions

The first-order optimality conditions are

\[ \frac{\partial H}{\partial x_i} = 8I(x_i - x_j)^T S^{-1} [-N] + 8I(x_i - x_j)^T S^{-1} [-N] - \lambda^T N + \mu^T N = 0 \]

(33)

from which we obtain

\[ x_i = x_j + (\mu - \lambda)(16I)^{-1}S \]

(34)

where \( \mu - \lambda \) displays the net shadow value for preserving the connectivity between patches that are connected to bioreserves, as compared to the maintenance of corridors solely linking the patches.

In parallel, as we normalize \( f(1) = 1 \), we obtain

\[ \frac{\partial H}{\partial x_j} = 8I(x_i - x_j)^T S^{-1} [N] + 8I(x_i - x_j)^T S^{-1} [N - 1] + \lambda^T N + \mu^T [-N + 1] = 0 \]

(35)
which yields

\[ x_i = x_j + \frac{\mu(N - 1) - \lambda N}{2N - 1} (8I)^{-1} S \]  

(36)

where \( [\mu(N - 1) - \lambda N] / (2N - 1) \) unveils the net shadow value for preserving the connectivity between bioreserves that are connected to patches, given the overall weight of patches in the ecological network.

Rewriting the previous expressions holds out the following output

\[ \mu = -\lambda \]  

(37)

**Lemma 6** The bioreserve shadow value has to be equal to the negative of the patch shadow value.

The proof is in the appendix.

The landowner’s willingness to receive compensation for neglecting the patch connectivity has to be equal to her willingness to pay for establishing and preserving the connectivity between bioreserves and patches. Thereupon, if the forest owner is not willing to transfer the financial resources – employed for preserving the connectivity between the patches – to ensure the connectivity between bioreserves and patches, controlling the partially connected ecological network is nonoptimal.

The boundary conditions are

\[ \dot{\lambda} = 16INS^{-1} (x_i - x_j) + 8IS^{-1} f (|g - x_j|) + \lambda N - \mu N \]  

(38)

And

\[ \dot{\mu} = -16INS^{-1} (x_i - x_j) - 8IS^{-1} f (|g - x_j|) - \lambda N + \mu (N - 1) \]  

(39)

All the optimality conditions being identified, the control of the system is as follows.

### 3.2.2 Solution

By letting \( w = [x_i^T, x_j^T, \lambda^T, \mu^T]^T \) reflect the network state, we obtain the following Hamiltonian system

\[ \dot{w} = Pw \]  

(40)

where
\[
P = \begin{bmatrix}
0 & 1 & (\mu - 1)(16I)^{-1}S & (8I)^{-1}\frac{N(1-\lambda)}{2N-1}S \\
1 & 0 & -(\mu - 1)(16I)^{-1}S & -\frac{(8I)^{-1}N(1-\lambda)}{2N-1}S \\
16INS^{-1} & -8I(2N+1)S^{-1} & N & -N \\
-16INS^{-1} & 8I(2N+1)S^{-1} & -N & N - 1
\end{bmatrix}
\]

**Theorem 4** The incomplete ecological network state \( w \) optimally evolves according to the co-
ordinates \( P \).

The proof is in the appendix.

Let the initial network state be given by \( w_0 = [x_{i0}^T, x_{j0}^T, \lambda_0^T, \mu_0^T]^T \). In order to control the network, the task consists in fixing \( \lambda_0 \) and \( \mu_0 \) such that

\[
x_i(T) - x_j(T) = \left( \sum_{k \in \Lambda_j} x_{kT} - \sum_{k \in \Lambda_i} x_{kT} \right) = \left( \sum_{k \in \Lambda_j'} x_{kT} - \sum_{k \in \Lambda_i'} x_{kT} \right) - f(|g - x_{jt}|)
\]

This time, the choices of \( \lambda_0 \) and \( \mu_0 \) are constrained by \( \mu_0 = -\lambda_0 \).

## 4 Simulations

Based on the properties and conditions previously obtained, the aim of this section is to illustrate the behaviors of simplified ecological networks through simulations. The conditions derived in Sections 2 and 3 guarantee the convergence to the objectives and the maintenance of connectivity.

### 4.1 Complete Graph Case

To illustrate the behavior of a basic complete network of nine nodes that compose the ecological network, two of the nodes, meaning bioreserves, represent the target nodes with dynamics given by (6) and the remaining seven unmarked nodes, such as various patches, follow the dynamics given by (5). In all simulations, the attraction function is defined as \( f_i(g_i - x_i) = a(g_i - x_i) \) with \( a = -2 \) given that the bioreserve has to attract against the weights of the patches. The hypothetical distribution of coordinates is \( x(0) = [10 10 8 8 8 8 8 8 8] \), where \( N_B = \{1, 2\} \) and \( N_P = \{3, 4, 5, 6, 7, 8, 9\} \). We suppose that the ecological network is fully connected at \( t = 0 \). Let us expose the network to the Laplacian laws of motion.

The first stage consists in building the distance matrix from the vector of distances defined above (Table 1). In parallel, we suppose a variance-covariance matrix as follows (Table 2). The second stage deals with computing the product between the transpose of the distance matrix and the inverse of the variance-covariance matrix. The
third stage consists in multiplying the results from the second stage with the transpose of the distance matrix. The final stage results in the computation of the Mahalanobis coordinates by taking the root of the modulus of the diagonal elements of the matrix previously obtained (Table 3).

We repeat the same operation for $t \in [0,50]$. The evolution of the Mahalanobis distances is shown in Fig. 3. We can observe that Theorem 1 is verified along the time path, for the barycenters of the network at $t = 0$ and $t = 50$ are similar, that is, $8.27 \simeq 8.10$. For instance, $x_{P_i} \simeq x_{B_j}$. At $t = 50$, $x_{B_1} = 8$, $x_{B_2} = 5$ and $x_{P_i} = 9$. Thereby, $9 \simeq 8 - \frac{2(8-5)}{9} = 8.7$, which signifies that the position of the patch depends on the position of the bioreserve and its magnitude of attraction.

Despite the oscillation of nodes subjected to the Laplacian consensus dynamics, the graph coordinates and connectedness are stationnary starting from $t = 25$. As a matter of fact, the network faces a temporary disconnection during the oscillatory phase. For example, at $t = 17$, the average distance between the Mahalanobis coordinates of $B_2$ and the rest of the network nodes is of 14, inducing a serious risk of disconnection. A comparable example can be taken at $t = 21$, where $x_{B_1}$ is distanced by other nodes by 9. A part of the network could thus disappear, which, in turn, would dismantle the network full connectivity. Given that the positive-semidefinite matrix yields a positive-semidefinite matrix inverse, the connectivity is ensured. In our simulations, the inverse of the covariance matrix yields the following eigenvalues $\text{eig}(S^{-1}) = \{95, 8, 1, 0.9, 0.8, 0.2, 0.1, 0.03, 0^{-}\}$.

Next to the analysis of the Mahalanobis coordinates in time, let us take a closer look at the optimal control conditions for different levels of opportunity costs. We thus decide to test the network robustness with $z_0 = [x_{B_1}, x_{P_1}, \lambda, \mu]$. We test four scenarios which differ by their opportunity costs.

Provided the Hamiltonian system, for $z_0 = [8,8,1,2]$ and an opportunity cost of $I = 1$ Eur, the outputs are as follows.

$$ M = \begin{bmatrix} 0 & 1 & -0.06 & 0.06 \\ 1 & 0 & 0.06 & -0.06 \\ 1.70 & -1.80 & 9.00 & 0 \\ -1.70 & 1.80 & 0 & 8.00 \end{bmatrix} $$
In Fig. 4, it can be acknowledged that the Mahalanobis coordinates are similar with and without the optimal control. The optimal control secured the coordinates of the initial ecological network, in that the absence of opportunity costs did not affect the network coordinates. For example, the differences of coordinates between $B_1$ and $P_1$ at $t = 21$ fall from 9 to 5, which ensures the network continuity. Even though the optimality constraint of $\mu > \lambda$ is verified, the needs to implement the optimal control drive up the shadow values. Thereby, when deciding to control and optimize the ecological network, the forest owner’s willingness to pay for the patches has to be 9 times greater than without control. For the bioreserves, her willingness to pay is found to be on average 8 times greater when she decides to control them. Therefore, the optimal control of the fully connected ecological network, i.e. the maintenance of the green corridors, implies the need to provide sufficient financial resources.

Fig. 4 about here.

Fig. 5 about here.

For $z_0 = [8, 8, 1, 2]$ and an opportunity cost of $I = 10$ Eur, Fig. 5 shows that the Mahalanobis coordinates with optimal control remain reasonably close to those without the optimal control, at the levels of shadow values observed in the previous case. Up to $I = 10$ Eur, building the ecological network by connecting bioreserves and patches via green corridors is still optimally-sound in view of the sacrifice of timber valued at the market price.

Fig. 6 about here.

In the third case, where $I = 100$ Eur, the Mahalanobis coordinates are no longer similar, such as shown in Fig. 6. The optimal control coordinates prove to be asymmetric. Given that the program seeks to minimize the distances between the areas, the trajectories now diverge. This is due to high opportunity costs that have an impact on the connectivity preservation, both between bioreserves and patches and between the patches. In point of fact, in order to control the evolution of the ecological network, the forest owner’s willingness to pay for optimizing the distances between patches ought to be 11 times greater than without control. More importantly, the shadow values for patches start exceeding the shadow values for bioreserves. So as to preserve the network connectedness, the landowner now has to allocate greater amounts of resources to maintain the connectivity between patches than to conserve the ex-situ conservation reserves. At this level of wood-production sacrifice and with $\mu < \lambda$, it becomes irrelevant to control the ecological network.

Fig. 7 about here.
In the final case, where $I = 1,000$ Eur, Fig. 7 illustrates even more pronounced differences between the Mahalanobis coordinates with and without the optimal control. The mean shadow value meant to preserve connectivity between patches ($\bar{\lambda} = 86.136$) is 29 times greater in case of network control, which makes their role in the network predominant, while it becomes negative ($\bar{\mu} = -38.36$) for the connectivity preservation between bioreserves and patches. It means that the financial resources to be spent in maintaining the connectivity between the patches that bridge the ecological reserves now end up at the expense of the preservation of bioreserves. The suboptimality of control at this level of opportunity costs is all the more straightforward.

For more synthesis, Fig. 8 illustrates the behavior of mean shadow values given the rising levels of opportunity costs. We see that the threshold level of economic suboptimality, where $\bar{\mu} < \bar{\lambda}$, is rapidly reached. Withal, it takes high opportunity costs for the shadow values to tumble.

For more synthesis, Fig. 8 illustrates the behavior of mean shadow values given the rising levels of opportunity costs. We see that the threshold level of economic suboptimality, where $\bar{\mu} < \bar{\lambda}$, is rapidly reached. Withal, it takes high opportunity costs for the shadow values to tumble.

The simulations we have performed indicate that the inclusion of opportunity costs while valuing the relevance of fully connected ecological networks by means of green corridors largely decides on the remainder of the project, even when the latter benefits from the foresters’ willingmesses to control.

**Result 1** If the economic optimality is measured upon the criterion of sacrifice in timber production, the nonoptimality of control of fully connected ecological networks looms at low opportunity costs.

From the reported result, it can be recommended to set up and control fully connected ecological networks only when the foregone timber production is worthless.

### 4.2 Incomplete Graph Case

Let us now consider the case of an initial incomplete graph, that is, some bioreserves and patches are not connected. The dynamics of the unmarked nodes and the target nodes are respectively defined by (10) and (11). In detail, we still consider nine nodes, but, unlike the previous case, $N_{Bi} = \{1\}$, $N_{Bj} = \{2\}$, $N_{Pi} = \{3, 4, 5\}$ and $N_{Pj} = \{6, 7, 8, 9\}$. $B_i$ thus connects with $P_i$, while $B_j$ connects with $P_j$. The Laplacian dynamics applied to target and unmarked nodes yield the following evolution of the Mahalanobis coordinates.

We have $\frac{1}{N_i}\Sigma_i = 8.26$ at $t = 0$ and $\frac{1}{N_i}\Sigma_i = 19.14$ at $t = 50$. Likewise, $\frac{1}{N_j}\Sigma_j = 8.27$ at $t = 0$ while $\frac{1}{N_j}\Sigma_j = 17.36$ at $t = 50$. Whereas the coordinates of the ecological network
evolve, the gap between the barycenters of the subgraphs along the time path stays negligible, such as shown in Fig. 9. Despite a temporary disconnection at $t = 8$, the incomplete graph preserves its connectedness.

As regards the network equilibrium, we initially observe $x_{B_1} = 8$, $x_{B_2} = 8$ and $\Sigma_{P_i} = 33$ and $\Sigma_{P_j} = 25$. Correspondingly, $33 - 25 = 8 > (8 - 8) - (-2 \times (8 - 8)) = 0$. Provided the attraction functions, the gap between the subgraphs’ weights is of 8. At $t = 50$, $x_{B_1} = 19$, $x_{B_2} = 10$, $\Sigma_{P_i} = 77$ and $\Sigma_{P_j} = 58$, giving the following inequality $77 - 58 = 27.3 > (19 - 10) - (-2 \times (19 - 10)) = 19.3$. Once more, the gap amounts to 8. Theorem 2 is not distinctly verified due to the fact that the number of patches connected to $B_1$ is lower than the number of patches connected to $B_2$. Nevertheless, the weight of this extra patch being 8, and provided the gap constancy in time, we can reasonably assert that the equilibrium of the incomplete graph, be it slightly imbalanced, is verifiable.

For $w_0 = [8, 8, -4, 4]$ and an opportunity cost of $I = 1$ Eur, the system outputs are

$$P = \begin{bmatrix} 0 & 1 & 0.03 & -0.04 \\ 1 & 0 & -0.03 & 0.04 \\ 3.41 & -3.60 & 9.00 & -9.00 \\ -3.41 & 3.60 & -9.00 & 8.00 \end{bmatrix}$$

The simulations performed for the incomplete graph case attest that the control of an incomplete graph is much more resource-demanding. Indeed, some connections are very costly to safeguard because of their exclusivity in maintaining the subgraphs connected: their conservancy becomes fundamental. This is all the more true in a case of decentralized network, where the coordinates’ update is no longer achieved across the whole network, but only across the available nearby nodes.

As stated in Lemma 6, the control of the incomplete graph goes through the transfer of financial resources allocated to maintaining the connectivity between the patches to those safeguarding the connectivity between bioreserves and patches. For the bioreserves, the forest owner’s willingness to pay is found to be 18 times greater when she decides to control the incomplete ecological network. The amounts of resources that the forester must be willing to devote to optimization and control of an incomplete network are by far higher (Figs. 10–13) than those observed in a fully connected network. And yet, unlike the previous case, the existence of nonoptimality with low opportunity costs is no longer observed. Indeed, we observe the stability of shadow values over a large array of opportunity costs.

Fig. 10 about here.

Fig. 11 about here.

Fig. 12 about here.

Fig. 13 about here.
Fig. 14 shows that the threshold level of economic suboptimality, where $-\hat{\mu} = \bar{\lambda}$, comes about at high opportunity costs. Nonetheless, and like in the previous case, it takes high opportunity costs for the shadow values to tumble.

What comes out is that controlling partially connected ecological networks is more costly than controlling fully connected networks, which can be deduced from the simple observation of shadow values in respective network configurations, but should be undertaken when the foregone timber production is significant, for the costate variables soar at high opportunity costs.

Result 2 If the economic optimality is measured upon the criterion of sacrifice in timber production, the nonoptimal control of partially connected ecological networks looms at high opportunity costs.

5 Conclusion

The ecological fragmentation of territory, due to land-use changes, particularly the expansion of anthropic areas, led to the species’ decline. In order to minimize the damages of habitat fragmentation and to ensure the species’ viability, the supply of bioreserves and ecological corridors has been recommended. We modeled an ecological network of bioreserves and patches, connected by the green corridors, which act as inputs and outputs while running the consensus protocol. This approach enabled us to show that the network connectivity depends on the ecological similarity between the areas of ecological significance. We then applied the optimal control properties to a network topology in order to obtain the graph-theoretic characterization of controllability. By doing so, we allowed for the ecological network control. To nuance the last statement, this work has to be considered as initiatory, provided that graph theory is a schematized representation of network patterns.

The solution to the optimal control program of the landowner is a matrix equation, verifying the network equilibrium, which depends on the positions and the weights of both the bioreserves and the patches. At the equilibrium, the ecological network set up by the forest manager maintains its connectedness while minimizing the impact on timber production on which depend the forest owner’s revenues. In the margin, the optimality conditions impose that the owner’s willingnesses to pay be of certain types, where the latter depend on the network at stake.

The simple simulations we have conducted report that taking into account the opportunity costs of wood production is essential to determine the economic soundness of the ecological project. In this manner, we find that optimizing and controlling partially connected ecological networks is more onerous than optimizing and controlling
fully connected ecological networks. Even so, when the sacrifice in timber production is significant, only partial connections between the areas of ecological significance should be established. Following these findings, case studies related to forest environments should be done. For example, one could value the levels of shadow values, through market-based mechanisms such as auctions, and assess the opportunity costs, through the inventory of forest stands, with a view to reveal the landowners’ optimality thresholds.

The future avenues of research can be classified in two categories. The first category concerns the migratory species, that is, the introduction of population dynamics and migratory flows, in order to estimate the needed sizes of bioreserves and the number of patches that would prevent from the occurrence of congestion. The second category involves the graph topology, through the implementation of probabilistic corridors on random graphs, by reason of the stochastic processes in nature. As the risk of occurrence of exogeneous events, such as storms, fires, pathologies, and floods steps up, the probability of existence of the ecological network in its initial configuration is threatened. The second category also pertains to the introduction of an evolutive variance-covariance matrix, so as to seize the impacts of climate change on the biophysical properties of ecological sites.
References


Appendix

Eq. (4)

\[ I = \sum px_i + \sum px_j + \alpha \sum p(x_i - x_j) \]
\[ = \sum px_i + \sum px_j + \alpha \sum px_i - \alpha \sum px_j \]
\[ = p \sum x_i + p \sum x_j + \alpha p \sum x_i - \alpha p \sum x_j \]
\[ = p \left[ (1 + \alpha) \sum x_i + (1 - \alpha) \sum x_j \right] \]

Eq. (5)

\[ \dot{x}_i = -\sum_{k \in \Lambda^u} (x_i - x_k) - \sum_{k \in \Lambda^t} (x_i - x_k) \]
\[ = -N_u x_i + \sum_{k \in \Lambda^u} x_k - N_t x_i + \sum_{k \in \Lambda^t} x_k \]
\[ = -(N_u + N_t) x_i + \sum_{k \in \Lambda} x_k \]
\[ = -N x_i + \sum_{k \in \Lambda} x_k \]

Proof of Lemma 1.

\[ -N x_i + \sum_{k \in \Lambda} x_k = 0 \]
\[ -N x_i = -\sum_{k \in \Lambda} x_k \]
\[ x_i = \frac{1}{N} \sum_{k \in \Lambda} x_k \]

We know that Eq. (5) is well defined. Hence the equilibrium point is unique.
\begin{align*}
\dot{x}_j &= - \sum_{k \in \Lambda^u} (x_j - x_k) - \sum_{k \in \Lambda^t} (x_j - x_k) + f(|g - x_j|) \\
&= -N_u x_j + \sum_{k \in \Lambda^u} x_k - N_t x_j + \sum_{k \in \Lambda^t} x_k + f(|g - x_j|) \\
&= -(N_u + N_t) x_j + \sum_{k \in \Lambda} x_k + f(|g - x_j|) \\
&= -N x_j + \sum_{k \in \Lambda} x_k + f(|g - x_j|)
\end{align*}

Proof of Lemma 2.

\begin{align*}
-N x_j + \sum_{k \in \Lambda} x_k + f(|g - x_j|) &= 0 \\
-N x_j + f(|g - x_j|) &= -\sum_{k \in \Lambda} x_k \\
N x_j - f(|g - x_j|) &= \sum_{k \in \Lambda} x_k \\
x_j - \frac{1}{N} f(|g - x_j|) &= \frac{1}{N} \sum_{k \in \Lambda} x_k
\end{align*}

We know that Eq. (6) is well defined. Hence the equilibrium point is unique. ■

Proof of Theorem 1. Notwithstanding the fact that the dynamics of target and unmarked nodes are distinctive, the unmarked nodes update their coordinates from those of the target nodes. We thus have \( \lim_{t \to T} x_i(t) = g \ \forall i \in \Lambda \), such as shown by Dragicevic and Sinclair-Desgagné (2013). Accordingly,

\begin{align*}
\frac{1}{N} \sum_{k \in \Lambda} x_k &= \frac{1}{N} \sum_{k \in \Lambda} x_k \\
x_i &= x_j - \frac{1}{N} f(|g - x_j|)
\end{align*}

The proof is straightforward by Lemma 1 and Lemma 2. ■

Eq. (8)

\begin{align*}
\dot{m}(I)_{ij}^2 &= 2 (x_i - x_j)^T S^{-1} \left[ -N x_i + \sum_{k \in \Lambda} x_k - \left( -N x_j + \sum_{k \in \Lambda} x_k \right) \right] 2I \\
&= 4I (x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) \right]
\end{align*}
Eq. (9)

\[
\dot{m}(I)_{ij}^2 = 2 (x_i - x_j)^T S^{-1} \left[ -N x_i + \sum x_k - \left( -N x_j + \sum x_k + f(|g - x_j|) \right) \right] 2 I
\]

Eq. (10)

\[
\dot{x}_i = -\sum_{k \in \Lambda^u} (x_i - x_k) - \sum_{k \in \Lambda^t_i} (x_i - x_k) + \sum_{k \in \Lambda^u} (x_j - x_k) + \sum_{k \in \Lambda^t_j} (x_j - x_k)
\]

Proof of Lemma 3.

\[
-N (x_i - x_j) + \sum_{k \in \Lambda^t_i} x_k - \sum_{k \in \Lambda^t_j} x_k = 0
\]

\[
-N (x_i - x_j) = -\sum_{k \in \Lambda^t_i} x_k + \sum_{k \in \Lambda^t_j} x_k
\]

\[
N (x_i - x_j) = \sum_{k \in \Lambda^t_i} x_k - \sum_{k \in \Lambda^t_j} x_k
\]

\[
x_i - x_j = \frac{1}{N} \left( \sum_{k \in \Lambda^t_i} x_k - \sum_{k \in \Lambda^t_j} x_k \right)
\]

We know that Eq. (10) is well defined. Hence the equilibrium point is unique. 

Eq. (11)
\[
\dot{x}_j = - \sum_{k \in \Lambda^t} (x_j - x_k) - \sum_{k \in \Lambda^u_j} (x_j - x_k) + \sum_{k \in \Lambda^u_i} (x_i - x_k) + f(|g - x_j|)
\]
\[
= -N_t x_j - N_u x_j + N_t x_i + \sum_{k \in \Lambda^t} x_k + \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k + f(|g - x_j|)
\]
\[
= -N_t (x_i - x_j) - N_u (x_i - x_j) + \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k + f(|g - x_j|)
\]
\[
= -N (x_i - x_j) + \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k + f(|g - x_j|)
\]

Proof of Lemma 4.

\[-N (x_i - x_j) + \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k + f(|g - x_j|) = 0\]

\[-N (x_i - x_j) = - \sum_{k \in \Lambda^u_j} x_k + \sum_{k \in \Lambda^u_i} x_k - f(|g - x_j|)\]

\[N (x_i - x_j) = \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k + f(|g - x_j|)\]

\[x_i - x_j = \frac{1}{N} \left( \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k + f(|g - x_j|) \right)\]

We know that Eq. (11) is well defined. Hence the equilibrium point is unique. ■

Proof of Theorem 2. Despite the fact that the dynamics of target and unmarked nodes are distinctive, the unmarked nodes update their coordinates from those of the target nodes. We thus have \(\lim_{t \to T} x_i(t) = g\ \forall i \in \Lambda\), as in Dragicevic and Sinclair-Desgagné (2013). Correspondingly,

\[\frac{1}{N} \sum_{k \in \Lambda^t} x_k - \frac{1}{N} \sum_{k \in \Lambda^u_j} x_k = \frac{1}{N} \sum_{k \in \Lambda^u_j} x_k - \frac{1}{N} \sum_{k \in \Lambda^u_i} x_k + \frac{1}{N} f(|g - x_j|)\]

\[\left( \sum_{k \in \Lambda^u_j} x_k - \sum_{k \in \Lambda^u_i} x_k \right) - f(|g - x_j|)\]

The proof is straightforward by Lemma 3 and Lemma 4. ■

Eq. (13)
\[ \dot{m}(I)_{ij}^2 = 2(x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k - \left( -N (x_j - x_i) + \sum_{k \in \Lambda_j} x_k - \sum_{k \in \Lambda_i} x_k \right) \right] \]

\[ = 8I (x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) + \sum_{k \in \Lambda_i^a} x_k - \sum_{k \in \Lambda_j^a} x_k \right] \]

Eq. (14)

\[ \dot{m}(I)_{ij}^2 = 2(x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) + \sum_{k \in \Lambda_i} x_k - \sum_{k \in \Lambda_j} x_k - \left( -N (x_j - x_i) + \sum_{k \in \Lambda_j} x_k - \sum_{k \in \Lambda_i} x_k + f \right) \right] \]

\[ = 8I (x_i - x_j)^T S^{-1} \left[ -N (x_i - x_j) - f (|g - x_j|) + \sum_{k \in \Lambda_i^a} x_k - \sum_{k \in \Lambda_j^a} x_k - \sum_{k \in \Lambda_i} x_k + \sum_{k \in \Lambda_j} x_k \right] \]

Proof of Lemma 5. Given (21) and (23), we fall on

\[ -\lambda (8I)^{-1} S = \frac{N - 1}{2N - 1} \mu (4I)^{-1} S \iff \frac{1}{2} \lambda = \left| \frac{N - 1}{2N - 1} \mu \right| \iff \frac{1}{2} \lambda = \frac{N - 1}{2N - 1} \mu \]

\[ \iff 2 \mu (N - 1) = \lambda (2N - 1) \iff 2N (\mu - \lambda) = 2 \mu - \lambda \iff N = \frac{2 \mu - \lambda}{2 (\mu - \lambda)} \]

Since \( N > 0 \), we have \( \mu > \lambda > 0 \). □

Eq. (28)

\[ \dot{\lambda} = - \left[ \frac{\partial H}{\partial x_i} \right]^T \]

\[ = - \left[ 4S^{-1} [-N (x_i - x_j)] I + 4S^{-1} [-N (x_i - x_j) - f (|g - x_j|)] I - \lambda N \right] \]

\[ = 8INS^{-1} (x_i - x_j) + 4IS^{-1} f (|g - x_j|) + \lambda N \]

Eq. (29)

\[ \dot{\mu} = - \left[ \frac{\partial H}{\partial x_j} \right]^T \]

\[ = - \left[ -4S^{-1} [-N (x_i - x_j)] I - 4S^{-1} [-N (x_i - x_j) - f (|g - x_j|)] I + \mu (-N + 1) \right] \]

\[ = -8INS^{-1} (x_i - x_j) - 4IS^{-1} f (|g - x_j|) + \mu (N - 1) \]
Proof of Theorem 3. By Theorem 1, we know that the fully connected network is at the equilibrium. Through (27), (28) and (29), we have obtained the first-order necessary optimality conditions to set up the Hamiltonian coordinates. All the necessary and sufficient conditions for the fully connected network to evolve optimally are met.

Proof of Lemma 6. Given (34) and (36), we obtain

\[ (\mu - \lambda)(16I)^{-1} S = \frac{\mu(N - 1) - \lambda N}{2N - 1} (8I)^{-1} S \]
\[ \Leftrightarrow \frac{1}{2}(\mu - \lambda) = \frac{\mu(N - 1) - \lambda N}{2N - 1} \]
\[ \Leftrightarrow (2N - 1)(\mu - \lambda) = 2\mu(N - 1) - 2\lambda N \Leftrightarrow -\mu + \lambda = -2\mu \]

And thus \( \mu = -\lambda \) ■

Eq. (38)

\[ \dot{\lambda} = -\left[ \frac{\partial H}{\partial x_i} \right]^T = -\left[ 8S^{-1} [-N(x_i - x_j)] I + 8S^{-1} [-N(x_i - x_j) - f(|g - x_j|)] I - \lambda N + \mu N \right] \]
\[ = 16INS^{-1}(x_i - x_j) + 8IS^{-1} f(|g - x_j|) + \lambda N - \mu N \]

Eq. (39)

\[ \dot{\mu} = -\left[ \frac{\partial H}{\partial x_j} \right]^T = -\left[ -8S^{-1} [-N(x_i - x_j)] I - 8S^{-1} [-N(x_i - x_j) - f(|g - x_j|)] I + \lambda N + \mu (-N + 1) \right] \]
\[ = -16INS^{-1}(x_i - x_j) - 8IS^{-1} f(|g - x_j|) - \lambda N + \mu (N - 1) \]

Proof of Theorem 4. By Theorem 2, we know that the partially connected network is at the equilibrium. Through (37), (38) and (39), we have obtained the first-order necessary optimality conditions to set up the Hamiltonian coordinates. All the necessary and sufficient conditions for the partially connected network to evolve optimally are met. ■

Tables and Figures
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Table 1: Distance matrix

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Table 2: Variance-covariance Matrix

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Table 3: Mahalanobis coordinates
Figure 1: From forest mapping to graph theory in the complete graph case

Figure 2: From forest mapping to graph theory in the incomplete graph case
Figure 3: Mahalanobis coordinates (ordinates) as functions of time (abscissa) in a fully connected ecological network of bioreserves (B₁, B₂) and patches (P₃, P₄, P₅, P₆, P₇).

Figure 4: Comparison between the Mahalanobis coordinates of B₁, P₁, λ and μ with (ordinates) and without (abscissa) the fully connected network control for I = 1 Eur.
Figure 5: Comparison between the Mahalanobis coordinates of $B_1$, $P_1$, $\lambda$ and $\mu$ with (ordinates) and without (abscissa) the fully connected network control for $I = 10$ Eur.

Figure 6: Comparison between the Mahalanobis coordinates of $B_1$, $P_1$, $\lambda$ and $\mu$ with (ordinates) and without (abscissa) the fully connected network control for $I = 100$ Eur.
Figure 7: Comparison between the Mahalanobis coordinates of $B_1$, $P_1$, $\lambda$ and $\mu$ with (ordinates) and without (abscissa) the fully connected network control for $I = 1,000$ Eur.

Figure 8: Mean shadow values (ordinates) as functions of opportunity costs (abscissa) for the optimal control of a fully connected ecological network.
Figure 9: Mahalanobis coordinates (ordinates) as functions of time (abscissa) in a partially connected ecological network of bioreserves ($B_1, B_2$) and patches ($P_3, P_4, P_5, P_6, P_7$).

Figure 10: Comparison between the Mahalanobis coordinates of $B_1, P_1, \lambda$ and $\mu$ with (ordinates) and without (abscissa) the partially connected network control for $I = 1$ Eur.
Figure 11: Comparison between the Mahalanobis coordinates of $B_1$, $P_1$, $\lambda$ and $\mu$ with (ordinates) and without (abscissa) the partially connected network control for $I = 10$ Eur.

Figure 12: Comparison between the Mahalanobis coordinates of $B_1$, $P_1$, $\lambda$ and $\mu$ with (ordinates) and without (abscissa) the partially connected network control for $I = 100$ Eur.
Figure 13: Comparison between the Mahalanobis coordinates of $B_1, P_1, \lambda$ and $\mu$ with (ordinates) and without (abscissa) the partially connected network control for $I = 1,000$ Eur.

Figure 14: Mean shadow values (ordinates) as functions of opportunity costs (abscissa) for the optimal control of a partially connected ecological network.