Natural resources:
Should property be public or private?

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1 Introduction

Should natural resources be publicly or privately owned? Let us take the example of the ocean.

The ocean is the only renewable natural resource which is a global public good and provides productive services to the global economy. As a public good, the ocean suffers the tragedy of commons. A typical solution suggested by the economic theory to the tragedy of commons is to establish property rights. But, unlike land, the ocean cannot be split into individual plots. The failure is primarily a governance failure: there does not exist a global government able to manage the ocean efficiently. What does economic theory suggests as solutions? Either to create a global government, so that the ocean becomes public property (where the resource belongs to the government) or to delegate the management of the ocean to a private company (owned by individuals). In both cases one expects the outcome to be efficient because the governance failure have vanished.

In this paper we compare these two solutions. We develop a dynamic general equilibrium model of a global economy endowed with a renewable natural resource.

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The resource is used by a competitive production sector to produce a final consumption good. Agents are heterogeneous in their time preference.

In the case of public property, *ecological democracy* applies: one individual, one vote. Individuals vote on the extraction rate and receive the natural resource rent as a lump sum. In the case of private property, the natural resource is managed by a private company owned by shareholders. The shares can be bought and sold by individuals (households). The shareholders then vote on the extraction rate (one share, one vote) and they receive the rent. This is *ecological capitalism*.

We propose a definition of voting equilibrium in a dynamic setting. We prove the existence of voting equilibria and describe their asymptotic properties. In voting equilibria the desired extraction rate of each individual is determined by her own discount factor, and the desired extraction rate is decreasing in the individual's discount factor. A key result is that the equilibrium extraction rates will be fully determined by the discount factors.

In ecological democracy the equilibrium extraction rate is determined by the individuals with the median discount factor, that is, the discount factor that splits the whole population of individuals into two groups of same size. In ecological capitalism the outcome is different because it depends on the set of shareholders and the distribution of shares among them. Moreover, both can change over time. The extraction rate is thus determined by the individuals who have the median discount factor within the set of shareholders. Importantly, each agent's vote is weighted by the number of shares she owns. We study the equilibrium under two simple cases : the case where only patient agents hold shares at the initial period and the case where shares are distributed equally among the two groups of households. It is important to stress that the outcome of these two property regimes would not differ in the absence of heterogeneity.

Both in ecological democracy and ecological capitalism the macroeconomic variables converge to steady state levels. These steady states depend on the median discount factor in the case of public property, and on the discount factor of the most patient individuals in the case of private property.

We compare the aggregate and distributional properties of steady states under the two regimes and discuss the transition from public to private properties and (forthcioming) from private to public.

The paper is organized as follows. Section 2 presents the basic building blocks of the model, the description of technology, households and property regimes. In section 3 we concentrate on the ecological democracy. We study intertemporal voting equilibrium in the case of public property and characterize the steady state of economy. Section 4 is devoted to the ecological capitalism, which is a much more difficult problem. We give the notion of competitive equilibrium under given extraction rates, describe our voting procedure, and then turn to intertemporal voting equilibrium and the steady state in the case of private property. In section 5 we make the comparison between the two steady states and two regimes of property. Section 6 concludes.

2 The model

Consider a global economy, endowed with a renewable natural resource. The economy is populated with N agents, divided into two groups. First group consists of patient agents, with relatively high discount factor, and second group consists of impatient agents, with relatively low discount factor. That is, patient agents care more about their future consumption than impatient agents.

2.1 Technology and the natural resource extraction

Firms use labor and the extracted resource to produce the homogeneous good, which is the numeraire in the model. Extraction is costless and all markets are competitive. There is no physical capital - introducing capital would make the resolution much more complicated without adding appealing properties to the model.

The technology is Cobb-Douglas:

$$Y_t = F(N, E) = N^{1-\alpha} E_t^{\alpha}, \tag{1}$$

where N is constant over time labor supply and E_t is the volume of extraction.

The dynamics of the natural resource stock R_t are given by

$$R_{t+1} = \chi (R_t - E_t)^{\gamma}, \tag{2}$$

with $\chi > 0$ and $\gamma \in (0, 1)$. This dynamics describe renewable natural resource, parameter γ being the elasticity of reproduction.

We denote the extraction rate by ε_t $(0 < \varepsilon_t < 1)$. The volume of extraction is thus

$$E_t = \varepsilon_t R_t. \tag{3}$$

We rewrite the production function in intensive form:

$$y_t = e_t^{\alpha}, \tag{4}$$

where $y_t = Y_t/N$ is the per capita output and $e_t = E_t/N$ is the per capita volume of extraction.

The wage rate (w_t) and the price of the extracted natural resource (q_t) are determined in competitive markets and hence coincide with their respective marginal productivity, $w_t = (1-\alpha)e_t^{\alpha}$ and $q_t = \alpha e_t^{\alpha-1}$.

2.2 Households

The economy is populated with an odd number N of agents with discount factors $\beta_j,\ j = 1, \ldots, N$. Each agent is endowed with one unit of labor.

Agents differ in their discount factors, which can be either high (β_H) or low (β_L), where $1 > \beta_H > \beta_L > 0$. We denote the number of agents with discount factor β_H by N_H (patient agents) and the number of agents with discount factor β_L by $N_L = N - N_H$ (impatient agents).

Let H denote the set of agents who share discount factor β_H: $H = \{j \mid \beta_j = \beta_H\}$, and L denote the set of agents, who share discount factor β_L: $L = \{j \mid \beta_j = \beta_L\}$.

Consumers obtain utility from their consumption over infinite time horizon. The utility consumer $j = 1, \ldots, N$ derives from her consumption stream starting at time τ, $\{c^j_\tau, c^j_{\tau+1}, c^j_{\tau+2}, \ldots\}$, is given by

$$U^j_\tau = \ln c^j_\tau + \beta_j \ln c^j_{\tau+1} + \beta_j^2 \ln c^j_{\tau+2} + \ldots . \tag{5}$$

2.3 The natural resource property regimes

We consider two different property regimes on the natural resource, public and private. Our objectives are to compare these regimes, analyze their properties and study the transition from one regime to the other.

We call the regime with public property on natural resource "ecological democracy". We assume that the the stock of resource is owned by all agents and the revenue from this stock is distributed equally among agents, who vote on the extraction rate. Every agent has one vote.

The alternative regime with private property on natural resource is called "ecological capitalism". In the private property framework the financial market exists, as we assume that the stock of resource belongs to a private company (Natural Resource Corporation), which is owned by shareholders.

The process of the resource management is delegated to the shareholders, with the one-share-one-vote rule. The weight of each shareholder in the voting thus equals the number of shares she owns. The shareholders receive the resource income in proportion of their shares.

We acknowledge that our wording is probably misleadingly manichaean, but our purpose is not to oppose democracy to capitalism. It seems to us that it is just a good way to name and to contrast two different property right regimes on a public good.

In both regimes the resource income is given by:

$$V_t = F'_E(N, E_t)E_t = \alpha Y_t \tag{6}$$

3 Ecological democracy

3.1 Dynamics

The theoretical model of ecological democracy is rather simple to solve and to analyze because all decisions are time-independent (this is due to the absence of physical capital). Assume that we start from the arbitrary chosen moment of time τ. The volume of extraction is given by (2)-(3), and the output is given by (4). Thus under given sequence of extraction rates, for each $t = \tau, \tau+1, \ldots$, extraction and output are also given.

At every time period $t = \tau, \tau+1, \ldots$, each agent $j = 1, \ldots, N$, gets the wage w_t and receives the equal share of the resource income, V_t/N. Total income is spent only on consumption, which is thus equal for all agents and is given by

$$c_t = w_t + \frac{V_t}{N} = y_t = e_t^\alpha, \; t = \tau, \tau+1, \ldots.$$

3.2 Time τ voting equilibrium

Consider the process of resource management. We want to make the extraction rate endogenous. At each time τ agents are asked to vote on the time τ extraction rate, ε_τ. One agent has one vote. When voting they have some expectations about the future extraction rates, $\varepsilon_t^e, \; t = \tau+1, \tau+2, \ldots$.

It is clear from (2)-(3), that e_t for $t = \tau, \tau+1, \ldots$, depends on the extraction rate ε_τ. Thus the whole consumption stream of every agent in "ecological democracy", $\{c_\tau, c_{\tau+1}, c_{\tau+2}, \ldots\}$, also depends on the ε_τ. We denote by $c_t(\varepsilon_\tau)$ the consumption of agent at time t ($t = \tau, \tau+1, \ldots$), provided the extraction rate at initial time τ is ε_τ.

Agents' preferences over extraction rate are represented by their utility, which in turn naturally depends on ε_τ:

$$\mathcal{U}_\tau^j(\varepsilon_\tau) := \ln c_\tau(\varepsilon_\tau) + \beta_j \ln c_{\tau+1}(\varepsilon_\tau) + \ldots, \; j = 1, \ldots, N. \tag{7}$$

The consumption stream and the utility depend on $\{\varepsilon_t^e\}_{t=\tau+1}^\infty$ and on other parameters of the model as well, but here we underline only the dependence on the variable, on which voting takes place at the moment. It can be seen that the indirect utility functions $(\mathcal{U}^j(\varepsilon_\tau))_{j=1}^N$ are strictly concave.

Let us introduce the notion of time τ voting equilibrium.

Definition 1 *A Condorcet winner ε_τ^* in voting on the time τ extraction rate is called a time τ voting equilibrium.*

As N is odd and the functions $(\mathcal{U}^j(\varepsilon_\tau))_{j=1}^N$ are strictly concave, the median voter theorem is applied and a Condorcet winner always exists.

The question is, which extraction rate is the winner in this vote. In order to answer this question, we obtain from the indirect utility functions (7) the most preferred values of time τ extraction rate for our agents, ε_τ^j.

Proposition 1 *The most preferred value of extraction rate at time τ for agent j is time-independent and is given by*

$$\varepsilon_\tau^j = \varepsilon^j := 1 - \gamma\beta_j, \;\; j = 1, \ldots, N.$$

Therefore, patient agents vote in favor of $1 - \gamma\beta_H$ and impatient agents vote in favor of $1 - \gamma\beta_L$. The winner is one of these values, depending on the median voter, which in our case simply belongs to the majority. We use the notion $1 - \gamma\beta_{med}$ for the winner:

if the median voter is patient ($N_H > N_L$), then $\varepsilon_\tau^* = 1 - \gamma\beta_H$;

if the median voter is impatient ($N_L > N_H$), then $\varepsilon_\tau^* = 1 - \gamma\beta_L$.

The following theorem is a corollary of Proposition 1.

Theorem 1 *For any resource stock $R_\tau > 0$ and any expectations about future extraction rates $\{\varepsilon_t^e\}_{t=\tau+1}^{\infty}$, there exists a unique time τ voting equilibrium, ε_τ^*. It is given by*

$$\varepsilon_\tau^* = \varepsilon^* := 1 - \gamma\beta_{med}.$$

The above result provides an explicit solution for the equilibrium extraction rate. It shows that the extraction rate is fully determined by two parameters: the median discount factor, β_{med} (which is either β_H or β_L, depending on majority), and the elasticity of reproduction, γ.

It is important, that the equilibrium extraction rate is independent of τ, the current state of the economy and the expectations about future voting outcomes. This result eliminates a strategic motive to influence the outcomes of future votes, hence they can be taken as given.

It is easily seen that the higher the median discount factor, the lower the extraction rate. This observation is quite intuitive. An increase in the median discount factor means a more patient population, whose members care about future consumption and want today to extract less natural resource.

3.3 Intertemporal voting equilibrium

Let us now turn to the intertemporal voting equilibrium, which is simply the sequence of time τ voting equilibria extraction rates under perfect foresight.

Suppose we are given the initial stock of natural resources, $R_0^* > 0$, and a sequence of extraction rates, $\mathbb{E}^* = \{\varepsilon_t^*\}_{t=0}^{\infty}$.

The associated volumes of extraction, dynamics of the resource stock and consumption levels are given by:

$$e_t^* = \frac{\varepsilon_t^* R_t^*}{N},\ R_{t+1}^* = \chi[(1-\varepsilon_t^*)R_t^*]^\gamma,\ c_t^* = (e_t^*)^\alpha,\ t = 0, 1, \ldots.$$

Denote $\mathcal{P}^* := \{c_t^*, e_t^*, R_{t+1}^*\}_{t=0,1,\ldots}$.

Definition 2 *We call the couple $\{\mathbb{E}^*, \mathcal{P}^*\}$ an intertemporal voting equilibrium if for each time $\tau = 0, 1, \ldots$, ε_τ^* is a time τ voting equilibrium at $R_\tau = R_\tau^*$ and $\varepsilon_t^e = \varepsilon_t^*,\ t = \tau+1, \tau+2, \ldots$.*

It may be pointed out that all agents have perfect foresight (in particular about future extraction rates). But because the time τ voting equilibrium is fully determined by the discount factors, expectations actually play no role.

It follows from Theorem 1 that there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^*, \mathcal{P}^*\}$.

Theorem 2 *For any initial resource stock $R_0^* > 0$, there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^*, \mathcal{P}^*\}$, determined as follows:*

$$\mathbb{E}^* = \{\varepsilon^*, \varepsilon^*, \ldots\},$$

$$\mathcal{P}^* = \{c_t^*, e_t^*, R_{t+1}^*\}_{t=0,1,\ldots},$$

where

$$\varepsilon^* = 1 - \gamma\beta_{med},$$

and

$$e_t^* = \frac{\varepsilon^* R_t^*}{N},\ R_{t+1}^* = \chi[(1-\varepsilon^*)R_t^*]^\gamma,\ c_t^* = (e_t^*)^\alpha,\ t = 0, 1, \ldots.$$

3.4 Steady state equilibrium

Since $\gamma \in (0,1)$, it follows that the economy converges to the steady state equilibrium. The steady state extraction rate is clearly $\varepsilon_{ED} = 1 - \gamma\beta_{med}$. The steady state resource stock, R_{ED}, and volume of extraction per capita, e_{ED}, are thus given by

$$R_{ED} = \chi^{\frac{1}{1-\gamma}}(\gamma\beta_{med})^{\frac{\gamma}{1-\gamma}}, \tag{8}$$

$$e_{ED} = \frac{1}{N}\chi^{\frac{1}{1-\gamma}}(\gamma\beta_{med})^{\frac{\gamma}{1-\gamma}}(1-\gamma\beta_{med}).$$

The steady state per capita output, y_{ED}, which is equal to the consumption of every agent, c_{ED}, is therefore

$$y_{ED} = c_{ED} = \frac{1}{N^{\alpha}} \chi^{\frac{\alpha}{1-\gamma}} (\gamma\beta_{med})^{\frac{\alpha\gamma}{1-\gamma}} (1-\gamma\beta_{med})^{\alpha}. \tag{9}$$

These values are uniquely determined by model's parameters, in particular the median discount factor (or the discount factor of the majority). The steady state equilibrium extraction rate ε_{ED} is obviously decreasing in β_{med}.

Moreover, it is straightforward to show that the steady state volume of extraction of the natural resource and therefore the output and wage rate are increasing in β_{med}. So, unambiguously, the utility level of every agent in the steady state equilibrium is increasing in the median discount factor.

4 Ecological capitalism

4.1 Preliminary notes

We now turn to the analysis of "ecological capitalism". The management of the natural resource is delegated to a private company called the Natural Resource Corporation (NRC).

Again suppose that we start at an arbitrary chosen moment of time τ. Suppose further that there is an initial distribution of shares. Agents are then allowed to buy or to sell their shares on a competitive financial market. There is a big difference between two regimes. The agents can now make savings (through their positions on the financial market), and they face an intertemporal maximization problem. We assume that agents cannot borrow against future incomes and their savings must be non-negative.

Given the resource income V_t by (6) and gross interest rate, $1 + r_t$, for $t = \tau, \tau + 1, \ldots$, the value of the NRC at the initial time τ is given by the present value of its cumulated profits over the infinite time horizon. It coincides with the present value of the cumulated sales of the natural resource to the production sector, because extraction costs are zero. Formally

$$\Pi_{\tau} = \frac{V_{\tau}}{1+r_{\tau}} + \frac{V_{t+1}}{(1+r_{\tau})(1+r_{\tau+1})} + \ldots, \tag{10}$$

At every time τ, consumer j solves the following problem, given the savings made in the previous period $\tau - 1$:

$$\max(\ln c_{\tau}^{j} + \beta_j \ln c_{\tau+1}^{j} + \beta_j^2 \ln c_{\tau+2}^{j} + \ldots) = \max \sum_{t=\tau}^{\infty} \beta_j^{t-\tau} \ln c_t^{j},$$

subject to

$$c_t^j + s_t^j = (1+r_t)s_{t-1}^j + w_t,\ t = \tau, \tau+1, \ldots,$$

$$s_t^j \geq 0,\ t = \tau, \tau+1, \ldots.$$

Our goal is again to define an intertemporal voting equilibrium. Here we do this in two steps. First, a competitive equilibrium at given extraction rates is introduced. Second, agents are asked to vote on the extraction rate at each time period.

4.2 Competitive equilibrium at given extraction rates

In this subsection we suppose that extraction rates are given and define the associated equilibrium, which we call competitive equilibrium. Further we will make extraction endogenous.

Let $\mathbb{E}_\tau = \{\varepsilon_t\}_{t=\tau}^\infty$, $(0 < \varepsilon_t < 1)$ be a sequence of given extraction rates. Suppose that the resource stock, $R_\tau > 0$, and the shares of agents in NRC, $\pi_\tau^{j*} \geq 0$, $(\sum_{j=1}^N \pi_\tau^{j*} = 1)$, are given at time τ. The sequences $\{R_{t+1}\}_{t=\tau}^\infty$, $\{e_t\}_{t=\tau}^\infty$, $\{w_t\}_{t=\tau}^\infty$ and $\{V_t\}_{t=\tau}^\infty$ are thus also known and are determined as follows:

$$R_{t+1} = \chi[(1-\varepsilon_t)R_t]^\gamma,\ \chi > 0,\ 0 < \gamma < 1,$$

$$e_t = \frac{\varepsilon_t R_t}{N},$$

$$w_t = (1-\alpha)e_t^\alpha,$$

$$V_t = \alpha e_t^\alpha N.$$

We define a competitive equilibrium at given extraction rates starting at time τ as follows.

Definition 3 *A sequence $\mathcal{E}^* = \{(c_t^{j*})_{j=1}^N, (s_{t-1}^{j*})_{j=1}^N, (\pi_{t+1}^{j*})_{j=1}^N, r_t^*, \Pi_t^*\}_{t=\tau,\tau+1,\ldots}$ is called a competitive $\mathbb{E}_\tau$-equilibrium starting from $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$, if*

C1) *for each $j = 1, \ldots, N$, the sequence $\{(c_t^{j*}), (s_t^{j*})\}_{t=\tau,\tau+1,\ldots}$ is a solution to the following problem:*

$$\max \sum_{t=\tau}^\infty \beta_j^{t-\tau} \ln c_t^j, \quad s.\ t. \tag{11}$$

$$c_\tau^j + s_\tau^j = (1+r_\tau^*)s_{\tau-1}^j + w_\tau,$$

$$c_t^j + s_t^j = (1+r_t^*)s_{t-1}^j + w_t,\ t = \tau+1, \tau+2, \ldots, \tag{12}$$

$$s_t^j \geq 0,\ t = \tau, \tau+1, \ldots;$$

C2)

$$\Pi_t^* = \sum_{j=1}^{N} s_{t-1}^{j*}, \ \ t = \tau+1, \tau+2, \ldots; \tag{13}$$

C3)

$$s_{t-1}^{j*}(1+r_t^*) = \pi_t^{j*}(V_t + \Pi_{t+1}^*), \ \ j = 1, \ldots, N, \ \ t = \tau, \tau+1, \ldots, \tag{14}$$

$$\Pi_t^*(1+r_t^*) = (V_t + \Pi_{t+1}^*), \ \ t = \tau, \tau+1, \ldots. \tag{15}$$

This definition looks cumbersome, but in fact it is quite simple and intuitive.

Condition **C1** means that agents maximize their utilities subject to budget constraints. Every time moment they divide their income (wages plus interest on previous period savings) into consumption and savings.

Condition **C2** is the equilibrium on the financial market: aggregate savings are equal to the value of the NRC.

Conditions **C3** for $t = \tau$ define the initial values $s_{\tau-1}^{j*}(1+r_\tau^*)$ and $\Pi_\tau^*(1+r_\tau^*)$. The arrangement of our equilibrium is such that we cannot determine simultaneously at the initial moment τ the values of r_τ^*, Π_τ^* and $s_{\tau-1}^{j*}$. This is not a problem, but one have to keep in mind that at time τ term $s_{\tau-1}^{j*}(1+r_\tau^*)$ is not separable and its components, $s_{\tau-1}^{j*}$ and $(1+r_\tau^*)$, cannot be identified separately at this stage. The same is true for the term $\Pi_\tau^*(1+r_\tau^*)$, for which the components Π_τ^* and $(1+r_\tau^*)$ are unknown separately.

Conditions **C3** for $t = \tau+1, \tau+2, \ldots$ define the shares of agents in NRC, $(\pi_t^{j*})_{j=1}^N$, and the value of NRC, Π_t^*.

Indeed, combining (14) and (15) we get

$$\pi_t^{j*} = \frac{s_{t-1}^{j*}}{\Pi_t^*}, \ \ t = \tau+1, \tau+2, \ldots.$$

Thus, $\sum_{j=1}^N \pi_t^{j*} = 1$.

Moreover, iterating (15) we get

$$\Pi_t^* = \frac{V_t}{1+r_t^*} + \frac{V_{t+1}}{(1+r_{t+1}^*)(1+r_t^*)} + \ldots, \ \ t = \tau+1, \tau+2, \ldots. \tag{16}$$

The use of the shares $(\pi_\tau^{j*})_{j=1}^N$ as initial conditions must also be made clear. Actually, agents' savings $(s_{\tau-1}^{j*})_{j=1}^N$ cannot be used as initial conditions because they are related to the value of the NRC at time τ. It appears clearly that Π_τ^* and individual savings $s_{\tau-1}^{j*}$ are linked:

$$\sum_{j=1}^{N} s_{\tau-1}^{j*}(1+r_\tau^*) = \Pi_\tau^*(1+r_\tau^*).$$

We know that the value of NRC at time τ is given by the present value of its revenue, $\{V_t\}_{t=\tau}^{\infty}$, discounted by the interest rates, $\{r_t\}_{t=\tau}^{\infty}$. Given the sequence of extraction rates $\mathbb{E}_\tau$, the sequence of revenue is known, but the sequence of interest rates is still undetermined because it is endogenous. Thus, the value of NRC is known only in equilibrium. As a consequence, the initial savings $(s_{\tau-1}^{j*})_{j=1}^{N}$ are indeterminate and we take the shares in NRC $(\pi_\tau^j)_{j=1}^N$ as the initial conditions.

Note also, that for each $t = \tau, \tau+1, \ldots$, our definition implies the output market balance (the total output at each time t is consumed):

$$\sum_{j=1}^{N} c_t^{j*} = Nw_t + V_t = Ne_t^{\alpha} = Y_t.$$

Remark 1 *It is important to note that if* $\{(c_t^{j*})_{j=1}^N, (s_{t-1}^{j*})_{j=1}^N, (\pi_{t+1}^{j*})_{j=1}^N, r_t^*, \Pi_t^*\}_{t=0,1,\ldots}$ *is a competitive* $\mathbb{E}_0$*-equilibrium starting from some initial conditions* $\{(\pi_0^{j*})_{j=1}^N, R_0\}$*, then for each* $\tau = 1, 2, \ldots$ *the sequence* $\{(c_t^{j*})_{j=1}^N, (s_{t-1}^{j*})_{j=1}^N, (\pi_{t+1}^{j*})_{j=1}^N, r_t^*, \Pi_t^*\}_{t=\tau,\tau+1,\ldots}$ *is a competitive* $\mathbb{E}_\tau$*-equilibrium starting from* $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$*. Thus, competitive equilibria are time consistent.*

The important points arising here are the questions of the existence and uniqueness of a competitive equilibrium.

We can prove the existence and uniqueness of a competitive equilibrium for two important special cases. This is sufficient for us, as these cases are particularly interesting for our analysis and allow us to study the steady states and the transition between "ecological capitalism" and "ecological democracy"[1].

In the generic case we cannot guarantee the uniqueness of competitive equilibrium for arbitrary initial distributions of shares $(\pi_\tau^{j*})_{j=1}^N$. Though still the proof of the uniqueness in general case is missing, we conjecture that the competitive equilibrium is actually unique for every initial distributions of shares. Two substantial cases considered below provide some basis for this conjecture.

The following proposition maintains that if at the initial moment of time all shares of NRC belong to the most patient agents ($\pi_\tau^{j*} = 0,\ j \in L$), the competitive equilibrium is unique. The initial owners in this equilibrium continue to possess the shares.

Proposition 2 *Suppose that* $\pi_\tau^{j*} = 0,\ j \in L$*. Then there exists a unique competitive* $\mathbb{E}_\tau$*-equilibrium*

$$\{(c_t^{j*})_{j=1}^N, (s_{t-1}^{j*})_{j=1}^N, (\pi_{t+1}^{j*})_{j=1}^N, r_t^*, \Pi_t^*\}_{t=\tau,\tau+1,\ldots},$$

[1]There is no actual problem with the existence. It can be proved as a modification of the extended existence theorem for the Ramsey model with heterogeneous agents, delivered in Becker, Boyd III and Foias (1991). However, we don't want to stop at this point, as for our purposes simple versions of the existence and uniqueness theorems will be enough. They are formulated further.

starting from $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$*, which is given for* $t = \tau, \tau+1, \ldots,$ *by*

$$\pi_{t+1}^{j*} = \pi_\tau^{j*},\ j \in H;\ \pi_{t+1}^{j*} = 0,\ j \in L,$$

$$\Pi_{t+1}^* = \beta_H (1 + r_t^*) \Pi_t^*,$$

$$s_t^{j*} = \beta_H (1 + r_t^*) s_{t-1}^{j*},\ j \in H;\ s_t^{j*} = 0,\ j \in L,$$

$$c_t^{j*} = (1 - \beta_H)(1 + r_t^*) s_{t-1}^{j*} + w_t,\ j \in H;\ c_t^{j*} = w_t,\ j \in L,$$

$$1 + r_{t+1}^* = \frac{1}{\beta_H} \left(\frac{e_{t+1}}{e_t} \right)^\alpha .$$

This case is important since in any competitive equilibrium with an arbitrary distribution of initial shares, the NRC from some time T^* will be owned only by the patient agents. The impatient agents prefer to sell all their shares. This also means that from some time we exactly know who owns the corporation and who decides about the resource management.

Formally,

Proposition 3 *In any competitive* $\mathbb{E}_\tau$*-equilibrium the shares of patient agents are non-decreasing,*

$$\pi_{t+1}^{j*} \geq \pi_t^{j*},\ j \in H,\ t = \tau, \tau+1, \ldots.$$

Moreover, there exists a T^* *such that for all* $t > T^*$

$$\pi_t^{j*} = 0,\ j \in L,\ \textit{and hence}\ s_{t-1}^{j*} = 0,\ j \in L;\ \sum_{j \in H} s_{t-1}^{j*} = \Pi_t^*.$$

The second important case where the competitive equilibrium is unique, is the case of equal distribution of shares within group of patient agents and within group of impatient agents.

Proposition 4 *Suppose that* $\pi_\tau^{j*} = p,\ j \in H$ *and* $\pi_\tau^{j*} = q,\ j \in L,$ *where* $p, q \geq 0,\ pN_H + qN_L = 1$*. Then there exists a unique competitive* $\mathbb{E}_\tau$*-equilibrium starting from* $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$*.*

Proposition 3 maintains that in this unique equilibrium the shares of patient agents are non-decreasing. Moreover, it is clear, that the shares of impatient agents here are non-increasing, as under the conditions of Proposition 4 impatient agents have not only the same discount factor, but also the same endowments.

The special case where at the initial moment of time the shares are distributed equally among the whole population, $\pi_\tau^{j*} = 1/N$, fits into this framework (with $p = q = 1/N$) and is particularly interesting for our analysis. We consider it later.

4.3 The voting procedure

Recall that voting procedure in the case of "ecological democracy" is quite simple and is based on two ideas. First, each agent has one vote. Second, we know the indirect utility functions of agents when they vote.

But in the "ecological capitalism" the situation is a bit complicated. Here voting is carried out in proportion to agents' shares in NRC. These shares are not necessarily integer numbers. So we should define the winner in this vote accurately.

Suppose that at some time the share of agent j in the NRC is $\pi^j \geq 0$. Then agent j has π^j votes. Naturally, $\sum_{j=1}^N \pi^j = 1$. Note also that $\pi^{j'} = 0$ means that agent j' has no voting power (no votes at all).

We assume that agents have some preferences over extraction rate ε, represented by the strictly concave functions $(\mathcal{U}^j(\varepsilon))_{j=1}^N$, with $\varepsilon^j = \arg\max_{\varepsilon\in(0,1)} \mathcal{U}^j(\varepsilon)$, $j = 1, \ldots, N$, being their most preferred values of extraction rate.

We need to define the winner, ε^*, in this vote. To do this, we impose on ε^* the natural requirement that no other extraction rate ε defeats ε^* in pairwise majority voting:

$$\sum_{j:\mathcal{U}^j(\varepsilon^*)\geq\mathcal{U}^j(\varepsilon)} \pi^j \geq \frac{1}{2}, \ \forall\varepsilon \in (0,1). \tag{17}$$

Depending on the distribution of shares, $(\pi^j)_{j=1}^N$, two cases are possible:

i) there is an agent k such that her preferred value ε^k satisfies $\sum_{j:\varepsilon^j<\varepsilon^k} \pi^j < 1/2$ and $\sum_{j:\varepsilon^j>\varepsilon^k} \pi^j < 1/2$;

ii) there are agents l and m such that their preferred values ε^l and $\varepsilon^m > \varepsilon^l$ satisfies $\sum_{j:\varepsilon^j\leq\varepsilon^l} \pi^j = 1/2$ and $\sum_{j:\varepsilon^j\geq\varepsilon^m} \pi^j = 1/2$.

The first case is generic. The winner ε^* is uniquely determined by inequality (17) and coincides with ε^k. Here we have the analogy with the median voter in the "ecological democracy" case.

In the second case, which is non-generic, any ε from the segment $[\varepsilon^l, \varepsilon^m]$ satisfies the requirement that no other extraction rate defeats it in pairwise majority voting. Here we have to specify the winner, just for determinacy. In this case we define the winner as the greatest value among them, ε^m.

In both cases the winner in voting represents one of the preferred extraction rates, that is, belongs to the set $\{\varepsilon^1, \ldots, \varepsilon^N\}$. So it is clear, that the winner naturally depends on the agents' preferred extraction rates and on their shares. Thus for what follows we denote the winner in voting by $\varepsilon(\varepsilon^1, \ldots, \varepsilon^N, \pi^1, \ldots, \pi^N)$.

4.4 Time τ voting equilibrium

Again our goal is to make the extraction rate endogenous. As in "ecological democracy", we start with voting on time τ extraction rate.

In order to apply our definition of voting, we have to specify the preferences over extraction rates of agents by time τ, or the functions $(\mathcal{U}_\tau^j(\varepsilon))_{j=1}^N$. They could be defined unambiguously assuming the uniqueness of competitive equilibrium. This case is relatively simple, since we specify $(\mathcal{U}_\tau^j(\varepsilon))_{j=1}^N$ just as indirect utility functions.

So we give definitions of time τ voting equilibrium and intertemporal voting equilibrium assuming the uniqueness of competitive equilibrium.

Suppose we are given expectations $\{\varepsilon_t^e\}_{t=\tau+1}^\infty$. For $\varepsilon_\tau \in (0,1)$ denote

$$\mathbb{E}_\tau(\varepsilon_\tau) := \{\varepsilon_\tau, \varepsilon_{\tau+1}^e, \varepsilon_{\tau+2}^e, \ldots\}. \tag{18}$$

Assume that for any ε_τ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$:

$$\mathcal{E}^*(\varepsilon_\tau) = \{(c_t^{j*}(\varepsilon_\tau))_{j=1}^N, (s_{t-1}^{j*}(\varepsilon_\tau))_{j=1}^N, (\pi_{t+1}^{j*}(\varepsilon_\tau))_{j=1}^N, r_t^*(\varepsilon_\tau), \Pi_t^*(\varepsilon_\tau)\}_{t=\tau,\tau+1,\ldots}. \tag{19}$$

In this case agents' preferences over extraction rates are given by the indirect utility functions

$$\mathcal{U}_\tau^j(\varepsilon_\tau) = \ln c_\tau^{j*}(\varepsilon_\tau) + \beta_j \ln c_{\tau+1}^{j*}(\varepsilon_\tau) + \ldots,\ j = 1, \ldots, N. \tag{20}$$

Suppose we are given the stock of natural resources at time τ, R_τ, and the distribution of shares of NRC, $(\pi_\tau^{j*})_{j=1}^N$. Suppose that agents have some expectations $\varepsilon_t^e,\ t = \tau+1, \tau+2, \ldots$ about future extraction rates and they vote on the time τ extraction rate, ε_τ.

The definition of time τ voting equilibrium is as follows.

Definition 4 *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$, and hence agents' preferences over extraction rates are represented by functions (20).*
We call a couple $\{\varepsilon_\tau^*, \mathcal{E}^*\}$ a time τ voting equilibrium if ε_τ^* is a winner in the vote on time τ extraction rate, and $\mathcal{E}^* = \mathcal{E}^*(\varepsilon_\tau^*)$.

As in the case of "ecological democracy", to describe the outcome of voting about time τ extraction rate we have to obtain the most preferred values of extraction rate at time τ for agents, ε_τ^j, from the indirect utility functions (20).

Proposition 5 *The most preferred value of extraction rate for agent j is time-independent and is given by*

$$\varepsilon_\tau^j = \varepsilon^j = 1 - \gamma\beta_j,\ j = 1, \ldots, N. \tag{21}$$

This proposition is quite similar to the Proposition 1 in the case of "ecological democracy", though the proof is more complicated. It turns out that the most preferred values are just the same as in the case of "ecological democracy".

Voting decisions of agents are not influenced by their expectations or the number of votes they have. The preferred value of extraction rate depends only on agents' discount factor.

The situation here appears to be quite the same as in "ecological democracy" with one minor change: in order for $1-\gamma\beta_H$ to be the winner in voting, patient agents must have the majority of shares (instead of majority in population). Alternatively, if impatient agents have the majority of shares, then their choice $1-\gamma\beta_L$ is the winner. There could be the case of equal distribution of shares. Our definition covers this opportunity and assigns the winner to be $1-\gamma\beta_L$.

However, this minor change can make the equilibrium extraction rate time-dependent, because here the winner in voting on time τ extraction rate depends on the distribution of shares at time τ. More precisely,

$$\begin{cases} \varepsilon_\tau^* & = 1-\gamma\beta_H, \text{if } \sum_{j\in H}\pi_\tau^{j*} > \sum_{j\in L}\pi_\tau^{j*}, \\ \varepsilon_\tau^* & = 1-\gamma\beta_L, \text{if } \sum_{j\in L}\pi_\tau^{j*} \geq \sum_{j\in H}\pi_\tau^{j*}. \end{cases}$$

The following theorem is quite similar to the Theorem 1.

Theorem 3 *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$, where $\mathbb{E}_\tau(\varepsilon_\tau)$ is given by (18). Then there exists a unique time τ voting equilibrium, $\{\varepsilon_\tau^*, \mathcal{E}^*\}$ and the equilibrium extraction rate is*

$$\varepsilon_\tau^* = \varepsilon(1-\gamma\beta_H, 1-\gamma\beta_L, \pi_\tau^{1*}, \ldots, \pi_\tau^{N*}).$$

4.5 Intertemporal voting equilibrium

In the previous subsection agents have voted on the current extraction rate (under some expectations about future extraction rates). Now we assume the perfect foresight about future extraction rates and define the sequence of time τ voting equilibria for $\tau = 0, 1, \ldots$ as an intertemporal voting equilibrium.

Formally, suppose we are given the initial stock of natural resources, $R_0^* > 0$, the initial distribution of shares $(\pi_0^{j*})_{j=1}^N$, and a sequence of extraction rates, $\mathbb{E}^* = \mathbb{E}_0^* = \{\varepsilon_t^*\}_{t=0}^\infty$. The volume of extraction and the resource stock are thus given by:

$$e_t^* = \frac{\varepsilon_t^* R_t^*}{N},\ R_{t+1}^* = \chi[(1-\varepsilon_t^*)R_t^*]^\gamma,\ t = 0, 1, \ldots.$$

Let $\mathcal{E}^*$ be a competitive $\mathbb{E}^*$-equilibrium starting from $\{(\pi_0^{j*})_{j=1}^N, R_0^*\}$:

$$\mathcal{E}^* = \mathcal{E}_0^* = \{(c_t^{j*})_{j=1}^N, (s_{t-1}^{j*})_{j=1}^N, (\pi_{t+1}^{j*})_{j=1}^N, r_t^*, \Pi_t^*\}_{t=0,1,\ldots}.$$

Remark 1 allows us to define for every $\tau = 1, 2, \ldots$, the competitive equilibrium $\mathcal{E}_\tau^*$, which is the tail of $\mathcal{E}^*$.

Denote $\mathcal{P}^* := \{\mathcal{E}^*, e_t^*, R_{t+1}^*\}_{t=0,1,\ldots}$. Now the intertemporal voting equilibrium is defined as follows.

Definition 5 *We call a couple $\{\mathbb{E}^*, \mathcal{P}^*\}$ an intertemporal voting equilibrium if for each time $\tau = 0, 1, \ldots$, a couple $\{\varepsilon_\tau^*, \mathcal{E}_\tau^*\}$ is a time τ voting equilibrium.*

Let us make this definition clear. The couple consisted of $\mathbb{E}^*$ and $\mathcal{P}^*$ is an intertemporal equilibrium if it is organized as follows. Every time τ extraction rate, ε_τ^*, and the corresponding competitive equilibrium, $\mathcal{E}_\tau^*$, should be time τ voting equilibrium.

Theorem 3 determines equilibrium time τ extraction rate, ε_τ^*, as a function of the agents' preferred values of time τ extraction rates and of the structure of property at time τ. The preferred values are independent of the time and of the expectations, which follows from Proposition 5.

As for the distribution of shares and its dynamics, the following proposition maintains that they are also consistent and don't depend neither on ε_τ, nor on expectations.

Proposition 6 *Let $\mathbb{E}_\tau$ and $\tilde{\mathbb{E}}_\tau$ be two arbitrary sequences of extraction rates. Suppose that there is a unique competitive $\mathbb{E}_\tau$-equilibrium starting from $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$:*

$$\{(c_t^{j*})_{j=1}^N, (s_{t-1}^{j*})_{j=1}^N, (\pi_{t+1}^{j*})_{j=1}^N, r_t^*, \Pi_t^*\}_{t=\tau,\tau+1,\ldots},$$

and there is a unique competitive $\tilde{\mathbb{E}}_\tau$-equilibrium starting from the same initial state $\{(\pi_\tau^{j*})_{j=1}^N, R_\tau\}$:

$$\{(\tilde{c}_t^{j*})_{j=1}^N, (\tilde{s}_{t-1}^{j*})_{j=1}^N, (\tilde{\pi}_{t+1}^{j*})_{j=1}^N, \tilde{r}_t^*, \tilde{\Pi}_t^*\}_{t=\tau,\tau+1,\ldots}.$$

Then for every $j = 1, \ldots, N$ and $t = \tau, \tau+1, \ldots$,

$$\pi_{t+1}^{j*} = \tilde{\pi}_{t+1}^{j*}.$$

This proposition states that for the competitive equilibria based on different sequences of extraction rates, but starting from the same initial distribution of shares, the dynamics of shares are the same. Thus the dynamics of the shares depend only on the initial distribution, but do not depend on the extraction rates. As we assume the uniqueness of the competitive equilibrium, the dynamics of distribution are also unique.

For two important cases where we have proven the uniqueness of the competitive equilibrium, we can characterize intertemporal voting equilibrium[2].

Theorem 4 *Suppose the initial state $\{(\pi_0^{j*})_{j=1}^N, R_0^*\}$ is such that $\pi_0^{j*} = 0,\ j \in L$. Then there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^*, \mathcal{P}^*\}$, which is determined as follows.*

$$\mathbb{E}^* = \{\varepsilon^*, \varepsilon^*, \ldots\},$$

with

$$\varepsilon^* = 1 - \gamma\beta_H, \tag{22}$$

and

$$\mathcal{P}^* = \{\mathcal{E}^*, e_t^*, R_{t+1}^*\}_{t=0,1,\ldots},$$

where $\mathcal{E}^*$ is a unique competitive $\mathbb{E}^*$-equilibrium starting from $\{(\pi_0^{j*})_{j=1}^N, R_0^*\}$, characterized in Proposition 2, and

$$e_t^* = \frac{\varepsilon^* R_t^*}{N},\ R_{t+1}^* = \chi[(1-\varepsilon^*)R_t^*]^\gamma,\ t = 0, 1, \ldots.$$

Theorem 4 states that if at the initial moment the NRC is owned by the patient agents, then the intertemporal voting equilibrium exists, is unique and the equilibrium extraction rate, ε^*, is time-independent and is fully determined by the parameters of the model (β_H and γ).

Theorem 5 *Suppose the initial state $\{(\pi_0^{j*})_{j=1}^N, R_0^*\}$ is such that $\pi_0^{j*} = p,\ j \in H$ and $\pi_0^{j*} = q,\ j \in L$, where $p, q \geq 0,\ pN_H + qN_L = 1$. Then there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^*, \mathcal{P}^*\}$, which is determined as follows.*

$$\mathbb{E}^* = \{\varepsilon_t^*\}_{t=0}^\infty,$$

with

$$\varepsilon_0^* = \varepsilon(1-\gamma\beta_H, 1-\gamma\beta_L, \pi_0^{1*}, \ldots, \pi_0^{N*}), \tag{23}$$

[2]For the sake of clear statement we limit ourselves to these two special cases. The problem with possible non-uniqueness of competitive equilibria is that we cannot explicitly use indirect utility functions, since we cannot specify the consumption stream, on which these utilities depend. However, the extensions of the definitions of voting equilibria for the general case of non-unique competitive equilibria also exist. We can define the voting equilibrium without the rigorous assumption about uniqueness. This could be done in the way developed in Borissov, Hanna and Lambrecht (2014). It is important to note, that in the general case all the results are the same, but with more tedious and nontransparent wording. Moreover, even all the formulas are valid, especially the expression (21) for the preferred value of the extraction rate. But, for ease of the exposition we don't want to consider the general case and concentrate on the two unique cases, which are sufficient for our purposes.

$$\varepsilon_t^* = \varepsilon(1-\gamma\beta_H, 1-\gamma\beta_L, \pi_t^{1*}, \dots, \pi_t^{N*}),\ t = 1, \dots, T^* - 1, \tag{24}$$

$$\varepsilon_t^* = 1-\gamma\beta_H,\ t = T^*, T^*+1, \dots, \tag{25}$$

where T^* is defined in Proposition 3.

Furthermore,

$$\mathcal{P}^* = \{\mathcal{E}^*, e_t^*, R_{t+1}^*\}_{t=0,1,\dots},$$

where $\mathcal{E}^*$ is a unique competitive $\mathbb{E}^*$-equilibrium starting from $\{(\pi_0^{j*})_{j=1}^N, R_0^*\}$, and

$$e_t^* = \frac{\varepsilon_t^* R_t^*}{N},\ R_{t+1}^* = \chi[(1-\varepsilon_t^*)R_t^*]^\gamma,\ t = 0, 1, \dots.$$

Theorem 5 maintains that if the initial shares of NRC are the same within two groups of agents, then the intertemporal voting equilibrium exists and is unique. In this intertemporal voting equilibrium the equilibrium extraction rate at the initial moment, ε_0^*, is fully specified by exogenous parameters of the model and predetermined values:

$$\begin{cases} \varepsilon_0^* & = 1-\gamma\beta_H, \text{if } pN_H > qN_L, \\ \varepsilon_0^* & = 1-\gamma\beta_L, \text{if } qN_L \geq pN_H. \end{cases}$$

Further values of extraction rates, $t = 1, 2, \dots$, depend on the dynamics of the distribution of shares. This dynamics are unique in our assumptions, but unknown, as we don't provide the explicit formulas for the competitive equilibrium.

Finally, Proposition 3 states that from some time T^* the whole NRC belongs to the patient agents. This means that for all $t \geq T^*$, $\pi_t^{j*} = 0,\ j \in L$, and only patient consumers can vote. They vote unanimously and thus for $t \geq T^*$, the extraction rate is given by (25).

4.6 Steady state equilibrium

In "ecological capitalism" the economy also converges to a steady state equilibrium. The steady state extraction rate is clearly $\varepsilon_{EC} = 1-\gamma\beta_H$. The steady state resource stock, per capita volume of extraction and per capita output are given by almost the same formulas as in the "ecological democracy", but here they depend on the discount factor of the most patient agents:

$$R_{EC} = \chi^{\frac{1}{1-\gamma}}(\gamma\beta_H)^{\frac{\gamma}{1-\gamma}},$$

$$e_{EC} = \frac{1}{N}\chi^{\frac{1}{1-\gamma}}(\gamma\beta_H)^{\frac{\gamma}{1-\gamma}}(1-\gamma\beta_H).$$

$$y_{EC} = \frac{1}{N^\alpha}\chi^{\frac{\alpha}{1-\gamma}}(\gamma\beta_H)^{\frac{\alpha\gamma}{1-\gamma}}(1-\gamma\beta_H)^\alpha. \tag{26}$$

By the steady state equilibrium we apparently mean the sequence consisted of the constant consumption for all agents, $(c^j_{EC})^N_{j=1}$, constant agents' savings, $(s^j_{EC})^N_{j=1}$, and shares, $(\pi^j_{EC})^N_{j=1}$, as well as constant interest rate, r_{EC}, and firm value, Π_{EC}.

It can be derived from Propositions 2 and 3 that the steady state equilibrium is given by

$$\Pi_{EC} = \frac{\beta_H}{1-\beta_H}\alpha N y_{EC},$$

$$s^j_{EC} = \pi^j_{EC}\Pi_{EC},\ j \in H;\ s^j_{EC} = 0,\ j \in L,$$

$$c^j_{EC} = (1-\alpha)y_{EC} + \alpha\pi^j_{EC}Ny_{EC},\ j \in H;\ c^j_{EC} = (1-\alpha)y_{EC},\ j \in L,$$

$$1 + r_{EC} = \frac{1}{\beta_H},$$

with $(\pi^j_{EC})^N_{j=1}$ being the final distribution of shares formed at the moment T^*, when impatient agents lose their last shares. It is clear that $\pi^j_{EC} = 0,\ j \in L$, but π^j_{EC} for $j \in H$, as well as T^*, are unknown. However, Proposition 6 maintains that they depend only on the initial distribution of shares, $(\pi^{j*}_0)^N_{j=1}$.

As in the case of "ecological democracy", these values are determined by model's parameters. The steady state equilibrium extraction rate ε_{EC} is obviously decreasing in β_H, while the steady state extraction level of the natural resource, the output level and wage rate are increasing in β_H.

5 Comparison

5.1 Comparison of steady states

Every equilibrium converges to the steady state, which allows us to compare different steady states, in "ecological democracy" and "ecological capitalism", and to study the transition between two property regimes.

We have seen that the steady state extraction rate in the case of public property, ε_{ED}, depends on the ratio of patient and impatient agents. If in the population $N_H > N_L$, patient agents constitute the majority, then $\varepsilon_{ED} = 1-\gamma\beta_H$, which is just the same as ε_{EC} in the case of private property. Therefore the steady state output, volume of extraction and resource stock in "ecological democracy" and "ecological capitalism" are the same. Note, that this is, in particular, the case of homogeneous agents ($\beta_H = \beta_L$).

If, on the contrary, $N_L > N_H$, the majority in population belongs to the impatient agents, then the steady state extraction rate in "ecological democracy" is greater than in "ecological capitalism", $\varepsilon_{ED} > \varepsilon_{EC}$. The impatient agents tend to extract more resource, as they don't care much about future, relative to the patient ones. It can be shown, that the steady state per capita volume of extraction, e_{ED}, is increasing in β_{med}. In particular, this leads to the higher steady state volume of extraction, higher output ($y_{ED} < y_{EC}$) and higher wage rates for private property.

However, the consumption of different agents could increase as well as decrease. The question is, does this increase in wage rate compensate the loss of resource income for impatient agents.

Recall that in "ecological democracy" steady state consumption of agents is given by

$$c_{ED} = \frac{1}{N^{\alpha}} \chi^{\frac{\alpha}{1-\gamma}} (\gamma \beta_{med})^{\frac{\alpha\gamma}{1-\gamma}} (1 - \gamma \beta_{med})^{\alpha}.$$

In "ecological capitalism", steady state consumption of agent j is

$$c^{j}_{EC} = \xi_j \frac{1}{N^{\alpha}} \chi^{\frac{\alpha}{1-\gamma}} (\gamma \beta_H)^{\frac{\gamma\alpha}{1-\gamma}} (1 - \gamma \beta_H)^{\alpha},$$

where $\xi_j = (1-\alpha) + \alpha \pi^{j}_{EC} N$ for patient agents ($j \in H$) and $\xi_j = 1 - \alpha$ for impatient agents ($j \in L$).

It is clear that for patient agents if only $\xi_j > 1$, that is, $\pi^{j}_{EC} > \frac{1}{N}$ (final share in corporation is greater than 1 % per capita), then the steady state consumption in "ecological capitalism" is always greater than "ecological democracy" steady state consumption, no matter who has the majority.

As for impatient agents, if $N_H > N_L$, then the steady state consumption of impatient agents in "ecological capitalism" is always lower than "ecological democracy" steady state consumption, since they consume only fraction $(1 - \alpha)$ instead of the whole output.

If on the contrary $N_L > N_H$, then it can be shown that only when β_L is low enough,

$$\left(\frac{\beta_L}{\beta_H}\right)^{\gamma} \left(\frac{1 - \gamma \beta_L}{1 - \gamma \beta_H}\right)^{1-\gamma} < (1 - \alpha)^{\frac{1-\gamma}{\alpha}},$$

the steady state consumption in "ecological capitalism" is always greater for impatient agents. Therefore, all agents in "ecological capitalism" steady state consume more and thus in this case private property on natural resources can be considered as preferable to public property.

5.2 Transition from private property to public property

We are particularly interested in the transition between two steady states. However, it is not clear, how to model the transition from "ecological capitalism" to "ecological democracy". This means a nationalization of resource, but on what potential grounds could this happen?

(TO BE DEVELOPED)

5.3 Transition from public property to private property

Consider the transition from steady state of “ecological democracy” to the steady state of “ecological capitalism”. Suppose that for a long time there have been public property on natural resource, when all agents, patient and impatient, received equal rent. But once agents are offered a referendum on the transition to the private property regime. There are two alternatives. First, agents can choose to stay in the “ecological democracy” and continue receiving equal resource rent. Second, agents can choose “ecological capitalism”. In this case Natural Resource Corporation emerges, agents are initially endowed with equal shares in the NRC and are allowed to trade them on the financial market.

We want to know, how agents will vote in this referendum about the transition assuming perfect forecast. This is the question of whether the transition from “ecological democracy” to “ecological capitalism” is Pareto-improving or not.

Note that by Pareto-improving here we mean the increase in the utilities of agents on the whole intertemporal voting equilibria, not just in the steady state. The intertemporal voting equilibrium without transition starting from the steady state of “ecological democracy” is clearly $\{\mathbb{E}^{ED}, \mathcal{P}^{ED}\}$, where

$$\mathbb{E}^{ED} = \{1 - \gamma\beta_{med}, 1 - \gamma\beta_{med}, \ldots\},$$

and

$$\mathcal{P}^{ED} = \{c_{ED}, e_{ED}, R_{ED}\}.$$

We refer to it further as “ecological democracy” intertemporal voting equilibrium.

Consider, on the contrary, “ecological capitalism” intertemporal voting equilibrium, which is starting from the same steady state of “ecological democracy”, with $R_0^* = R_{ED}$ and the equal initial distribution of shares, $\pi_0^{j*} = 1/N$. This is the special case, for which Proposition 4 states that the competitive $\mathbb{E}_\tau$-equilibrium is unique and Theorem 4 maintains that there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^*, \mathcal{P}^*\}$.

It is clear that since agents have equal initial shares, patient agents have the same consumption levels, make the same savings and obtain the same utilities. This is also true for the group of impatient agents. We denote their common values of consumption by $c_t^{H*},\ j \in H$ and $c_t^{L*},\ j \in L$.

Thus in order to understand the outcome of referendum, we have to compare utilities of patient and impatient agents from two consumption streams, the constant consumption stream in the “ecological democracy” intertemporal voting equilibrium and some new consumption stream in “ecological capitalism” intertemporal voting equilibrium.

Two cases are logically possible.

1) If initially most agents are patient ($N_H > N_L$), then the extraction rate in “ecological democracy” is $\varepsilon_{ED} = 1 - \gamma\beta_H$. After the change of the regime, the

extraction rate doesn't change - the patient majority continue to vote in favor of $\varepsilon_{EC} = 1 - \gamma\beta_H$. That is, the output and wage rates also don't change ($y_{EC} = y_{ED}$).

This kind of transition is always Pareto-improving. The reason is that in the "ecological capitalism" intertemporal voting equilibrium the possibility of consuming each period the output per capita (as in "ecological democracy" intertemporal voting equilibrium) is feasible - it satisfies budget constraints. However, individuals maximize their utilities, and from the Proposition 3 it is clear that impatient agents sell all their shares. They consume more today at the expense of future consumption and their utilities on the whole "ecological capitalism" intertemporal voting equilibrium is higher than on the "ecological democracy" intertemporal voting equilibrium. Thus in this case the transition to private property is strictly Pareto-improving and all agents vote unanimously in favor of the "ecological capitalism".

Note that here we compare the utilities on the intertemporal voting equilibria. Recall that in the steady state of "ecological capitalism" the consumption of patient agents consists of wage plus the resource income, while impatient agents consume only their wages. They voted unanimously for the transition, and they got the higher utility during the process of transition, but this doesn't mean that in the new steady state they don't regret about their previous "myopic" decision.

We can use numerical modeling for studying the dynamics of the consumption of both groups of agents in "ecological capitalism" intertemporal voting equilibrium in comparison with their constant consumption in "ecological democracy" intertemporal voting equilibrium. It is clear that in this case the steady state consumption of impatient agents is always lower for "ecological capitalism" steady state.

For instance, for parameters

$$\alpha = 0.6;\ \frac{N_H}{N} = 0.54;\ \frac{N_L}{N} = 0.46;\ \beta_H = 0.9;\ \beta_L = 0.8; \gamma = 0.9; \chi = 1.3;$$

the consumption of patient and impatient agents in "ecological capitalism" and "ecological democracy" steady states is shown in the Figure 1.

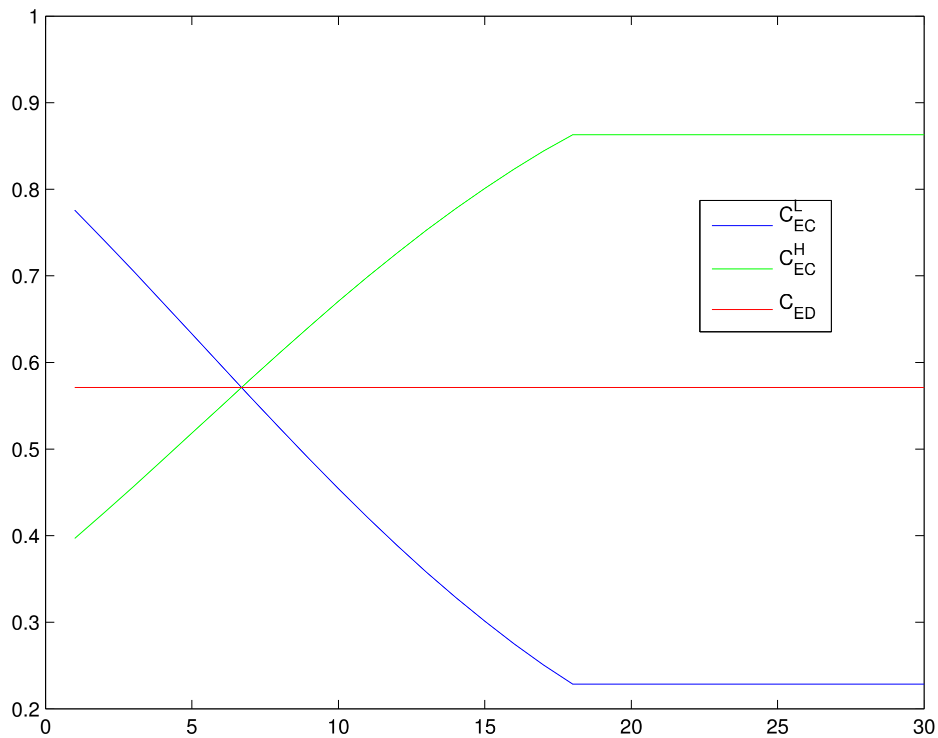


Figure 1: Dynamics of consumption, patient and impatient agents, "ecological democracy" and "ecological capitalism"

2) If initially most agents are impatient ($N_L > N_H$), then the extraction rate in democracy is $\varepsilon_{ED} = 1 - \gamma\beta_L$. At the first moment, since the shares of NRC are equally distributed among agents, the extraction rate doesn't change. Indeed, $\pi_0^{j*} = 1/N$ and therefore $\sum_{j\in L} \pi_0^{j*} = N_L/N > N_H/N = \sum_{j\in H} \pi_0^{j*}$. Thus, $\varepsilon_0 = \varepsilon_{ED}$.

But further, as a result of financial market, impatient agents sell their shares for the immediate consumption, the shares are redistributed in favor of most patient agents and at some moment of time T' there is a change in the extraction rate.

This moment is characterized by

$$\sum_{j\in L} \pi_{T'-1}^{j*} \geq \sum_{j\in H} \pi_{T'-1}^{j*}, \quad \sum_{j\in L} \pi_{T'}^{j*} < \sum_{j\in H} \pi_{T'}^{j*}.$$

T' is the moment, when patient agents obtain majority in the NRC. It is clear that $T' \leq T^*$, where T^* is described in Proposition 3. Impatient agents couldn't lose the majority in shares after their shares become zero.

From T', the economy begins to converge to the steady state with $\varepsilon_{EC} = 1 - \gamma\beta_H$.

So we have to compare the following utilities. In"ecological democracy" the consumption streams of patient and impatient agents remain constant and is given by (9). Thus

$$
\begin{aligned}
U^H_{ED} &= \ln c_{ED} + \beta_H \ln c_{ED} + \beta_H^2 \ln c_{ED} + \dots, \\
U^L_{ED} &= \ln c_{ED} + \beta_L \ln c_{ED} + \beta_L^2 \ln c_{ED} + \dots.
\end{aligned} \tag{27}
$$

In "ecological capitalism" their consumption streams are the part of the competitive $\mathbb{E}^{EC}$-equilibrium, $\{c_t^{H*}, c_t^{L*}, s_{t-1}^{H*}, s_{t-1}^{L*}, \pi_{t+1}^{H*}, \pi_{t+1}^{L*}, r_t^*, \Pi_t^*\}_{t=0,1,\dots}$, starting from $\{\pi_0^{H*}, \pi_0^{L*}, R_{ED}\}$, where

$$
\mathbb{E}^{EC} = \{\underbrace{\varepsilon_{ED}, \dots, \varepsilon_{ED}}_{0,\dots,T'-1}, \underbrace{\varepsilon_{EC}, \varepsilon_{EC}, \dots}_{T',T'+1,\dots}\},
$$

$\pi_0^{H*} = N_H/N$, $\pi_0^{L*} = N_L/N$, and R_{ED} is given by (8).

Namely,

$$
\begin{aligned}
U^H_{EC} &= \ln c_0^{H*} + \beta_H \ln c_1^{H*} + \beta_H^2 \ln c_2^{H*} + \dots, \\
U^L_{EC} &= \ln c_0^{L*} + \beta_L \ln c_1^{L*} + \beta_L^2 \ln c_2^{L*} + \dots.
\end{aligned}
$$

The outcome of the referendum about the transition to the "ecological capitalism" in this case is ambiguous. There are numerical examples of situations when this transition is Pareto-improving as well as when it is not Pareto-improving.

For instance, for parameters

$$
\alpha = 0.6;\ \frac{N_H}{N} = 0.46;\ \frac{N_L}{N} = 0.54;\ \beta_H = 0.9;\ \beta_L = 0.8; \gamma = 0.99; \chi = 1.3,
$$

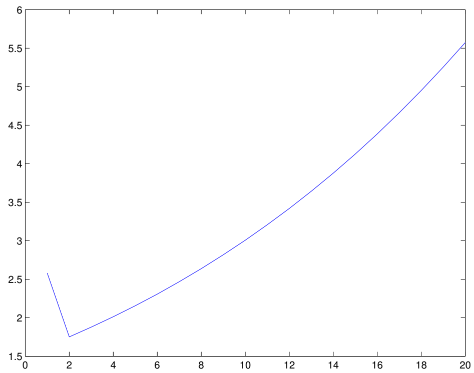
the dynamics of the shares and the dynamics of output in "ecological capitalism" are shown in the Figure 2.

The utilities are

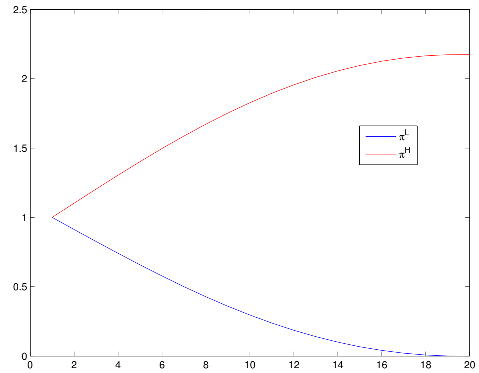
$$
U^H_{ED} = 9.4801; U^H_{EC} = 11.7111; U^L_{ED} = 4.7400; U^L_{EC} = 4.5820.
$$

It is clear that after the transition patient agents obtain higher utility ($U^H_{EC} > U^H_{ED}$), but this is not true for impatient agents ($U^L_{EC} < U^L_{ED}$). In this example the voting equilibrium extraction rate falls immediately ($T' = 1$), which is accompanied by the fall in output. Since impatient agents appreciate today's consumption more than future consumption, such fall adversely affect their utility, and larger steady state output can't help.

Thus, on the referendum, impatient agents will vote against the transition.



(a)



(b)

Figure 2: Ecological capitalism: (a) Dynamics of output; (b) Dynamics of shares

While for parameters

$$\alpha = 0.6;\ \frac{N_H}{N} = 0.46;\ \frac{N_L}{N} = 0.54;\ \beta_H = 0.9;\ \beta_L = 0.8; \gamma = 0.6; \chi = 5,$$

the utilities are

$$U_{ED}^{H} = 7.0893; U_{EC}^{H} = 14.1189; U_{ED}^{L} = 6.8061; U_{EC}^{L} = 13.5491.$$

In this case after the transition both groups of agents has higher utility and thus they vote unanimously for the transition to the "ecological capitalism".

6 Income inequality effects

So far our results can be interpreted in a sense that private property on natural resource is preferred for the economy to the public property, as the steady state output and wage rates in "ecological capitalism" are higher than in "ecological democracy". However, this conclusion is somewhat hasty. In the public property regime the income of all agents is the same, while in the private property framework, there is a rising income inequality, since only patient agents eventually take possession of the resource rent.

It is well recognized that high level of inequality adversely affects the society and the economy, by increasing social tension or political instability. This effect can be modeled in different ways. We can assume that discount factors of agents are endogenous. Due to the insecurity of property rights, the inequality reduces confidence about future, resulting in lower discount factors of population, higher extraction rate and lower steady state output for private property (see, e. g., Borissov, Lambrecht, 2009).

Instead, we can assume that depending on level of inequality some fraction of the output is wasted. Owing to increasing inequality, part of the output could be spent on the army, police or special forces, as well as on extension of social programs, to reduce the possibility of revolution.

Formally speaking, we have to take these distributional effects into account and again analyze the whole dynamics of "ecological capitalism" intertemporal voting equilibrium. However, for ease of exposition we limit ourselves to the comparison of steady states.

Consider first the case where in the steady state of "ecological capitalism" the shares of NRC are equally distributed among patient agents, ($\pi^j_{EC} = 1/N_H,\ j \in H$). Further we make some comments on this distribution of shares.

Assume that some part of the steady state output is wasted, according to the level of inequality. As a measure of inequality we use the Gini coefficient, based on the steady state consumption levels.

We can explicitly calculate the Gini coefficient, G, for the population in both regimes, public and private property:

$$G_{ED} = 0; \quad G_{EC} = \alpha \left(1 - \frac{N_H}{N}\right).$$

It is clear, that in "ecological democracy" there is no inequality and the Gini coefficient is zero. While in "ecological capitalism" the Gini coefficient depends on the fraction of patient agents.

Suppose that the steady state per capita output in both cases is determined as

$$y^*_k = (1-p)y_k,\ k = ED, EC,$$

where y_{ED} is given by (9), y_{EC} is given by (26) and p is the share of global GDP which is drawn away to maintain public order and to prevent possible dissatisfaction of population about the distribution of revenue. This share can be thought as a maintenance costs for army, policy and other authority institutions.

It is natural to assume that $p = \psi(G)$, where function $\psi : [0,1] \to [0,1]$ is continuous, $\psi(0) = 0$ and increasing for $G > 0$.

Thus in the steady states we have to compare the following values:

$$y^*_{EC} = \left(1 - \psi(\alpha(1 - \frac{N_H}{N}))\right) \frac{1}{N^\alpha} \chi^{\frac{\alpha}{1-\gamma}} (\gamma\beta_H)^{\frac{\alpha\gamma}{1-\gamma}} (1-\gamma\beta_H)^\alpha$$

and

$$y^*_{ED} = \frac{1}{N^\alpha} \chi^{\frac{\alpha}{1-\gamma}} (\gamma\beta_{med})^{\frac{\alpha\gamma}{1-\gamma}} (1-\gamma\beta_{med})^\alpha.$$

It follows that

$$y^*_{EC} \lesseqgtr y^*_{ED} \Leftrightarrow \left(1 - \psi(\alpha(1 - \frac{N_H}{N}))\right) (\gamma\beta_H)^{\frac{\alpha\gamma}{1-\gamma}} (1-\gamma\beta_H)^\alpha \lesseqgtr (\gamma\beta_{med})^{\frac{\alpha\gamma}{1-\gamma}} (1-\gamma\beta_{med})^\alpha.$$

It turns out, that if the majority of population is patient ($N_H > N_L$), then we have unambiguously $y^*_{EC} < y^*_{ED}$.

If impatient agents constitute the majority ($N_L \geq N_H > 0$), then for sufficiently low fraction N_H/N, when resource rent belongs to a few agents, we also have $y^*_{EC} < y^*_{ED}$. Therefore, for public property on natural resource to be preferable to the private property, the following should hold.

$$\psi\left(\alpha(1-\frac{N_H}{N})\right) > 1 - \left(\frac{\beta_L}{\beta_H}\right)^{\frac{\alpha\gamma}{1-\gamma}} \left(\frac{1-\gamma\beta_L}{1-\gamma\beta_H}\right)^{\alpha}.$$

$$\frac{N_H}{N} < 1 - \frac{1}{\alpha}\,\psi^{-1}\left(1 - \left(\frac{\beta_L}{\beta_H}\right)^{\frac{\alpha\gamma}{1-\gamma}} \left(\frac{1-\gamma\beta_L}{1-\gamma\beta_H}\right)^{\alpha}\right).$$

The account of the income inequality effects reduces the steady state output in the case of "ecological capitalism" and the comparison of steady states in "ecological democracy" and "ecological capitalism" is very ambiguous.

Note, that the final shares of NRC in the "ecological capitalism" within group of patient agents could be arbitrary. In this case the Gini coefficient couldn't be simply expressed in terms of the patient to impatient ratio. However, it is clear, that any other distribution of final shares among patient agents, different from equal, results in the higher Gini coefficient. Thus the value $G_{EC} = \alpha\left(1 - \frac{N_H}{N}\right)$ is a lower bound of inequality measure.

7 Appendix

7.1 Proof of Proposition 1

Let ε^0 be an arbitrary chosen extraction rate. And let

$$e^\circ_\tau = \varepsilon^0 R_\tau,\ R^\circ_{\tau+1} = \chi[(1-\varepsilon^0)R_\tau]^\gamma,$$

$$e^\circ_t = \varepsilon^e_t R^\circ_t,\ R^\circ_{t+1} = \chi[(1-\varepsilon^e_t)R^\circ_t]^\gamma,\ t = \tau+1, \tau+2, \dots.$$

$$c^\circ_t = [e^\circ_t]^\alpha,\ t = \tau, \tau+1, \dots.$$

We have

$$R_{\tau+1}(\varepsilon_\tau) = \left(\frac{1-\varepsilon_\tau}{1-\varepsilon^0}\right)^\gamma R^\circ_{\tau+1},\ R_{\tau+2}(\varepsilon_\tau) = \left(\frac{1-\varepsilon_\tau}{1-\varepsilon^0}\right)^{\gamma^2} R^\circ_{\tau+2}, \dots,$$

hence

$$e_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0}\right) e^\circ_\tau,\ e_{\tau+1}(\varepsilon_\tau) = \left(\frac{1-\varepsilon_\tau}{1-\varepsilon^0}\right)^\gamma e^\circ_{\tau+1},\ e_{\tau+2}(\varepsilon_\tau) = \left(\frac{1-\varepsilon_\tau}{1-\varepsilon^0}\right)^{\gamma^2} e^\circ_{\tau+2}, \dots,$$

and

$$c_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0}\right)^\alpha c_\tau^{j\circ},\ c_{\tau+1}(\varepsilon_\tau) = \left(\frac{1-\varepsilon_\tau}{1-\varepsilon^0}\right)^{\alpha\gamma} c_{\tau+1}^{j\circ},\ c_{\tau+2}(\varepsilon_\tau) = \left(\frac{1-\varepsilon_\tau}{1-\varepsilon^0}\right)^{\alpha\gamma^2} c_{\tau+2}^{j\circ}, \dots.$$

Therefore,

$$\begin{aligned}\mathcal{U}_\tau^j(\varepsilon_\tau) = \Gamma^j + \alpha\left(\ln\varepsilon_\tau + \gamma\beta_j\ln(1-\varepsilon_\tau) + \gamma^2\beta_j^2\ln(1-\varepsilon_\tau) + \dots\right) \\ = \Gamma^j + \alpha\ln\varepsilon_\tau + \frac{\alpha\gamma\beta_j}{1-\gamma\beta_j}\ln(1-\varepsilon_\tau),\end{aligned}$$

where

$$\Gamma^j = \ln\left(\frac{1}{\varepsilon^0}\right)^\alpha c_\tau^\circ + \beta_j\ln\left(\frac{1}{1-\varepsilon^0}\right)^{\alpha\gamma} c_1^\circ + \dots,\ j = 1,\dots,N.$$

When voting on ε_τ, agent j maximizes $\mathcal{U}_\tau^j(\varepsilon_\tau)$, that is, solves $\frac{\partial\mathcal{U}_\tau^j(\varepsilon_\tau)}{\partial\varepsilon_\tau} = 0$. It is clear that the solution to this equation is

$$\varepsilon_\tau^j = 1 - \gamma\beta_j,$$

which completes the proof.

7.2 Proof of Proposition 3

7.3 Proof of Proposition 4

Since $\pi_\tau^{j*} = 1/N$, patient and impatient agents differ only in their discount factor. Thus the whole population could be treated as consisted of only two agents, patient H and impatient L (with different "weights" $\frac{N_H}{N}$ and $\frac{N_L}{N}$). That is, within the two types agents are homogeneous and indistinguishable.

In particular, all agents with discount factor β_H share the same consumption stream, savings, etc. We denote these common values by c_t^{H*}, s_t^{H*}, etc.

The same is true for the all agents with discount factor β_L. Their corresponding values are c_t^{L*}, s_t^{L*}, etc.

Consider a competitive equilibrium path $\{c_t^{H*}, c_t^{L*}, s_{t-1}^{H*}, s_{t-1}^{L*}, \pi_{t+1}^{H*}, \pi_{t+1}^{L*}, r_t^*, \Pi_t^*\}_{t=0,1,\dots}$ starting from $\{\pi_0^H, \pi_0^L\}$, where $\pi_0^H = N_H/N$ and $\pi_0^L = N_L/N$.

We have

$$c_{t+1}^{H*} = \beta_H(1 + r_{t+1}^*)c_t^{H*},\ \forall t,$$

and

$$\begin{aligned}c_{t+1}^{L*} = \beta_L(1 + r_{t+1}^*)c_t^{L*},\ \text{for } t \le T^* - 1, \\ c_{t+1}^{L*} > \beta_L(1 + r_{t+1}^*)c_t^{L*},\ \text{for } t \ge T^*.\end{aligned}$$

Also we have

$$N_H c_t^{H*} + N_L c_t^{L*} = Y_t,\ \forall t.$$

Therefore, for $t = 1, \ldots, T^*$,

$$c_t^{H*} = \frac{\beta_H c_{t-1}^{H*} Y_t}{N_H \beta_H c_{t-1}^{H*} + N_L \beta_L c_{t-1}^{L*}},\ c_t^{L*} = \frac{\beta_L c_{t-1}^{L*} Y_t}{N_H \beta_H c_{t-1}^{H*} + N_L \beta_L c_{t-1}^{L*}}. \tag{28}$$

and hence

$$(N_H\beta_H c_0^{H*} + N_L\beta_L c_0^{L*})(N_H\beta_H c_1^{H*} + N_L\beta_L c_1^{L*}) = (N_H\beta_H^2 c_0^{H*} + N_L\beta_L^2 c_0^{L*})Y_1, \ldots,$$
$$(N_H\beta_H c_0^{H*}+N_L\beta_L c_0^{L*}) \ldots (N_H\beta_H c_{t-1}^{H*}+N_L\beta_L c_{t-1}^{L*}) = (N_H\beta_H^t c_0^{H*}+N_L\beta_L^t c_0^{L*})Y_1 \ldots Y_{t-1}, \tag{29}$$

and therefore,

$$c_t^{H*} = \frac{\beta_H^t c_0^{H*} Y_t}{N_H \beta_H^t c_0^{H*} + N_L \beta_L^t c_0^{L*}},\ c_t^{L*} = \frac{\beta_L^t c_0^{L*} Y_t}{N_H \beta_H^t c_0^{H*} + N_L \beta_L^t c_0^{L*}}. \tag{30}$$

Clearly, $\{c_t^{H*}\}_{t=0}^{T^*}$ is an increasing sequence and $\{c_t^{L*}\}_{t=0}^{T^*}$ is a decreasing sequence. Thus, if we know c_0^{L*} and $c_0^{H*} = \frac{Y_0 - N_L c_0^{L*}}{N_H}$, we know everything. Namely:

$$c_1^{L*} = \min\{\frac{1-\alpha}{N} Y_1,\ \frac{\beta_L c_0^{L*} Y_1}{N_H\beta_H c_0^{H*} + N_L\beta_L c_0^{L*}}\}, \ldots, c_t^{L*} = \min\{\frac{1-\alpha}{L} Y_t,\ \frac{\beta_L^t c_0^{L*} Y_t}{N_H\beta_H^t c_0^{H*} + N_L\beta_L^t c_0^{L*}}\}$$

$$c_1^{H*} = \frac{Y_1 - N_L c_1^{L*}}{N_H} = \min\{\frac{1+\alpha\frac{N_L}{N_H}}{N} Y_1,\ \frac{\beta_H c_0^{H*} Y_1}{N_H\beta_H c_0^{H*} + N_L\beta_L c_0^{L*}}\}, \ldots,$$
$$c_t^{H*} = \min\{\frac{1+\alpha\frac{N_L}{N_H}}{L} Y_t,\ \frac{\beta_H^t c_0^{H*} Y_t}{N_H\beta_H^t c_0^{H*} + N_L\beta_L^t c_0^{L*}}\}.$$

In particular, we can calculate T^* as a function of c_0^{L*}, defined on the interval $[\frac{1-\alpha}{N} Y_0, \frac{1}{N_L} Y_0)$.

$$T^* = T(c_0^{L*}) := \max\{t \mid \frac{\beta_L^t c_0^{L*}}{N_H\beta_H^t c_0^{H*} + N_L\beta_L^t c_0^{L*}} \geq \frac{1-\alpha}{N}\}.$$

The structure of this function is as follows. Let δ_t be the solution to the following equation in x:

$$\frac{\beta_L^t x}{\beta_H^t(1 - N_L x) + N_L\beta_L^t x} = \frac{1-\alpha}{N}.$$

It is clear that $\delta_0 = \frac{1-\alpha}{N}$ and that the sequence $(\delta_0, \delta_1, \ldots)$ is increasing and converge to $\frac{1}{N_L}$. Also it is easy to check that

if $\delta_0 Y_0 < c_0^{L*} \leq \delta_1 Y_0$, then $T(c_0^{H*}) = 0$,

if $\delta_1 Y_0 < c_0^{L*} \le \delta_2 Y_0$, then $T(c_0^{H*}) = 1$,
$\ldots$,
if $\delta_t Y_0 < c_0^{L*} \le \delta_{t+1} Y_0$, then $T(c_0^{H*}) = t$,
$\ldots$.

Also we can calculate $r_t^*,\ t = 0, 1, \ldots$:

$$1 + r_{t+1}^* = \frac{Y_{t+1}}{N_H \beta_H c_t^{H*} + N_L \beta_L c_t^{L*}},\ t \le T^* - 1,$$

$$\frac{Y_{T^*+1}}{Y_{T^*}} \frac{1}{\beta_H} \le 1 + r_{T^*+1}^* = \frac{(1 + \alpha \frac{N_L}{N_H}) Y_{T^*+1}}{N \beta_H c_{T^*}^{H*}} \le \frac{Y_{T^*+1}}{N_H \beta_H c_{T^*}^{H*} + N_L \beta_L c_{T^*}^{L*}},$$

$$1 + r_{t+1}^* = \frac{1}{\beta_H} \frac{Y_{t+1}}{Y_t},\ t > T^*.$$

It is clear that the sequence $\{1 + r_{t+1}^*\}$ is non-increasing and decreasing for $t < T^*$.

Thus, our task is to find c_0^{L*} and $c_0^{H*} = \frac{Y_0 - N_L c_0^{L*}}{N_H}$.

It is clear that c_{t+1}^{L*} is increasing in c_t^{L*} and c_{t+1}^{H*} is decreasing in c_t^{L*} for $t < T^*$. Therefore, c_{t+1}^{L*} is increasing and c_{t+1}^{H*} is decreasing in c_0^{L*} for $t < T^*$.

We have

$$(1 + r_0^*) \Omega_0^* = w_0 + \frac{w_1}{1 + r_1^*} + \ldots,$$

and

$$(1 + r_0^*) \Pi_0^* = \lambda (1 + r_0^*) \Omega_0^*.$$

Thus,

$$(1 + r_0^*) s_{-1}^{H*,L*} + w_0 + \frac{w_1}{(1 + r_1^*)} + \frac{w_2}{(1 + r_1^*)(1 + r_2^*)} + \ldots$$
$$= (1 + \lambda \pi_0^{H*,L*})(w_0 + \frac{w_1}{(1 + r_1^*)} + \frac{w_2}{(1 + r_1^*)(1 + r_2^*)} + \ldots)$$
$$= (1 + \lambda \pi_0^{H*,L*}) \frac{1 - \alpha}{N} (Y_0 + \frac{Y_1}{(1 + r_1^*)} + \frac{Y_2}{(1 + r_1^*)(1 + r_2^*)} + \ldots).$$

We have

$$\frac{1}{(1 + r_1^*)} = \frac{N_H \beta_H c_0^{H*} + N_L \beta_L c_0^{L*}}{Y_1}, \frac{1}{(1 + r_1^*)(1 + r_2^*)} = \frac{N_H \beta_H^2 c_0^{H*} + N_L \beta_L^2 c_0^{L*}}{Y_2}, \ldots,$$

$$\frac{1}{(1 + r_1^*)(1 + r_2^*) \ldots (1 + r_t^*)} = \frac{N_H \beta_H^t c_0^{H*} + N_L \beta_L^t c_0^{L*}}{Y_t},\ t \le T^*.$$

Also we have

$$\frac{1}{(1 + r_1^*)(1 + r_2^*) \ldots (1 + r_{T^*+1}^*)} = \frac{N \beta_H^{T^*+1} c_0^{H*}}{(1 + \alpha \frac{N_L}{N_H}) Y_{T^*+1}}.$$

Hence

$$\frac{1}{(1+r_1^*)(1+r_2^*)\dots(1+r_t^*)} = \frac{N\beta_H^t c_0^{H*}}{(1+\alpha\frac{N_L}{N_H})Y_t},\ t > T^*.$$

and hence

$$\begin{aligned}
c_0^{H*}(1+\beta_H+\beta_H^2+\dots) &= c_0^{H*} + \frac{c_1^{H*}}{(1+r_1^*)} + \frac{c_2^{H*}}{(1+r_1^*)(1+r_2^*)} + \dots \\
&= (1+r_0^*)s_{-1}^{H*} + w_0 + \frac{w_1}{(1+r_1^*)} + \frac{w_2}{(1+r_1^*)(1+r_2^*)} + \dots \\
&= (1+\lambda\pi_0^{H*})\frac{1-\alpha}{N}(Y_0 + \frac{Y_1}{(1+r_1^*)} + \frac{Y_2}{(1+r_1^*)(1+r_2^*)} + \dots) \\
&= (1+\lambda\pi_0^{H*})\frac{1-\alpha}{N}(Y_0 + N_H\beta_H c_0^{H*} + N_L\beta_L c_0^{L*} + N_H\beta_H^2 c_0^{H*} + N_L\beta_L^2 c_0^{L*} + \dots \\
&\quad + N_H\beta_H^{T^*} c_0^{H*} + N_L\beta_L^{T^*} c_0^{L*} + \frac{N\beta_H^{T^*+1}c_0^{H*}}{(1+\alpha\frac{N_L}{N_H})} + \frac{N\beta_H^{T^*+2}c_0^{H*}}{(1+\alpha\frac{N_L}{N_H})} + \dots)
\end{aligned}$$

It follows that c_0^{L*} is a solution to the following equation in c_0^L:

$$\frac{1}{1-\beta_H}\frac{Y_0 - N_L c_0^L}{N_H} = \Phi(c_0^L), \tag{31}$$

where

$$\begin{aligned}
\Phi(c_0^L) &:= (1+\lambda\pi_0^H)\frac{(1-\alpha)}{N}Y_0(1+\beta_H+\dots+\beta_H^{T(c_0^L)} + \frac{N}{N_H+\alpha N_L}\frac{\beta_H^{T(c_0^L)+1}}{1-\beta_H}) \\
&\quad + (1+\lambda\pi_0^H)\frac{(1-\alpha)}{N}N_L c_0^L(\beta_L+\dots+\beta_L^{T(c_0^L)} - \beta_H - \dots - \beta_H^{T(c_0^L)}) \\
&\quad - (1+\lambda\pi_0^H)\frac{1-\alpha}{(N_H+\alpha N_L)}N_L\frac{\beta_H^{T(c_0^L)+1}}{1-\beta_H}c_0^L.
\end{aligned}$$

Taking account of the structure of the function $T(c_0^L)$, we know that the function $\Phi(c_0^L)$ satisfies the following conditions:

i) It is piecewise affine. Indeed, for $c_0^L \in (\delta_\tau Y_0, \delta_{\tau+1}Y_0]$, $T(c_0^L) = \tau$ and hence

$$\begin{aligned}
\Phi(c_0^L) &:= (1+\lambda\pi_0^H)\frac{(1-\alpha)}{N}Y_0(1+\beta_H+\dots+\beta_H^{\tau} + \frac{N}{N_H+\alpha N_L}\frac{\beta_H^{\tau+1}}{1-\beta_H}) \\
&\quad + (1+\lambda\pi_0^H)\frac{(1-\alpha)}{N}N_L c_0^L(\beta_L+\dots+\beta_L^{\tau} - \beta_H - \dots - \beta_H^{\tau}) \\
&\quad - (1+\lambda\pi_0^H)\frac{1-\alpha}{(N_H+\alpha N_L)}N_L\frac{\beta_H^{\tau+1}}{1-\beta_H}c_0^L.
\end{aligned}$$

ii) It is continuous. This follows from the choice of the sequence $(\delta_0, \delta_1, \ldots)$.

iii) It is convex. Indeed, for $c_0^L \in (\delta_\tau Y_0, \delta_{\tau+1} Y_0]$,

$$\Phi'(c_0^L) = (1 + \lambda\pi_0^H)\frac{(1-\alpha)}{N} N_L(\beta_L + \ldots + \beta_L^\tau - \beta_H - \ldots - \beta_H^\tau) \\ - (1 + \lambda\pi_0^H)\frac{1-\alpha}{(N_H + \alpha N_L)} N_L \frac{\beta_H^{\tau+1}}{1-\beta_H}.$$

It can be shown that $\beta_L^\tau - \beta_H^\tau + \frac{N_H+N_L}{N_H+\alpha N_L}\beta_H^\tau > 0$, the derivative $\Phi'(c_0^L)$ on the interval $(\delta_\tau Y_0, \delta_{\tau+1} Y_0]$ is greater than on the interval $(\delta_{\tau-1} Y_0, \delta_\tau Y_0]$.

Recall that c_0^L is defined on the interval $[\frac{1-\alpha}{N} Y_0, \frac{1}{N_L} Y_0)$.

Since clearly $\Phi(\frac{1}{N_L} Y_0) > 0$ and $\Phi(\frac{1-\alpha}{N} Y_0) < \frac{1+\alpha\frac{N_L}{N_H}}{N(1-\beta_H)} Y_0$, the solution to the (31) exists and is unique.